行政院國家科學委員會補助專題研究計畫成果報告

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※ 零維 Gorenstein 環 ※

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計畫類別：☑個別型計畫　□整合型計畫

計畫編號：NSC90－2115－M－002－005

執行期間：90 年 08 月 01 日至 91 年 07 月 31 日

計畫主持人：朱樺

共同主持人：

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執行單位：台大數學系

中　華　民　國　91年　9　月2　日

行政院國家科學委員會專題研究計畫成果報告

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一、中文摘要

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Abstract

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摘　　要

令 $R = k[x_1, x_2, \ldots, x_n]$. 任意$R$中的零維 Gorenstein 理想必呈

$$((x_1, x_2, \ldots, x_n) : f)$$

之型式. 此理想之生成元個數變化範圍非常大. 在此報告中我們要證明以下的結果: 令

$$F = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_l^{\alpha_l} (x_1^{k_1} - x_2^{k_2} x_3^{k_3} \ldots x_l^{k_l}), k_1 = k_2 + \ldots + k_l, \alpha_j \geq 0,$$

且

$$P_i = x_1^{ik_1} + x_1^{(i-1)k_1} x_2^{k_2} \ldots x_l^{k_l} + x_1^{(i-2)k_1} x_2^{2k_2} \ldots x_l^{2k_l} + \ldots + x_2^{ik_2} \ldots x_l^{ik_l}.$$

令 $\theta(j) = j$, 若 $j > 0; = 0$, 若 $j \leq 0$. 對任意 $n \in \mathbb{N}$ 且 $n > \alpha_j$, 對所有 $j = 1, 2, \ldots, l$, 令

$$n - \alpha_j - 1 = a_j k_j + b_j, 0 \leq b_j < k_i.$$

令 $a = \min\{a_1, a_2, \ldots, a_l\}$. 則

$$((x_1^n, x_2^n, \ldots, x_l^n) : F) = (x_1^{n-\alpha_1}, x_2^{n-\alpha_2}, \ldots, x_l^{n-\alpha_l}, p_2, p_3, \ldots, p_l, q_2, q_3, \ldots, q_l)$$

其中

$$p_j = x_1^{\theta(n-\alpha_1-(a+1)k_1)} x_j^{\theta(n-\alpha_j-(a+1)k_j)} P_a,$$
$$q_j = x_1^{\theta(n-\alpha_1-ak_1)} x_j^{\theta(n-\alpha_j-ak_j)} P_{a-1}.$$

我們也在其中作了一些相關問題的討論.

關鍵詞: Gorenstein 理想, 理想之商運算, 生成元.

ABSTRACT

Let $R = k[x_1, x_2, \cdots, x_l]$. Any zero dimensional Gorenstein ideal is of the form

$$((x_1^n, x_2^n, \cdots, x_l^n) : f(x_1, \cdots, x_l))$$

for some homogeneous polynomial f. There is a greater range in the possible number of generators of such ideals.

In this note, we prove the following result. Let

$$F = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_l^{\alpha_l} (x_1^{k_1} - x_2^{k_2} x_3^{k_3} \cdots x_l^{k_l}), \quad k_1 = k_2 + \cdots + k_l, \alpha_j \geq 0,$$

and

$$P_i = x_1^{ik_1} + x_1^{(i-1)k_1} x_2^{k_2} \cdots x_l^{k_l} + x_1^{(i-2)k_1} x_2^{2k_2} \cdots x_l^{2k_l} + \cdots + x_2^{ik_2} \cdots x_l^{ik_l}.$$

Let $\theta(j) = j$, if $j > 0$ $;= 0$, if $j \leq 0$. For any $n \in \mathbb{N}$ and $n > \alpha_j$, for all $j = 1, 2, \cdots, l$, we write

$$n - \alpha_j - 1 = a_j k_j + b_j, 0 \leq b_j < k_i.$$

Let $a = \min\{a_1, a_2, \cdots, a_l\}$. Then

$$\begin{aligned} &((x_1^n, x_2^n, \cdots, x_l^n) : F) \\ =&(x_1^{n-\alpha_1}, x_2^{n-\alpha_2}, \cdots, x_l^{n-\alpha_l}, p_2, p_3, \cdots, p_l, q_2, q_3, \cdots, q_l) \end{aligned}$$

where

$$\begin{aligned} p_j &= x_1^{\theta(n-\alpha_1-(a+1)k_1)} x_j^{\theta(n-\alpha_j-(a+1)k_j)} P_a, \\ q_j &= x_1^{\theta(n-\alpha_1-ak_1)} x_j^{\theta(n-\alpha_j-ak_j)} P_{a-1}. \end{aligned}$$

We also give some related discussions in this note.

Keywords. Gorenstein ideal, ideal quotient, generators.

ZERO-DIMENSIONAL GORENSTEIN IDEALS

§ 1. Introduction

The name *Gorenstein* comes from a duality property of Gorenstein rings that has a special case a phenomenon concerning the singularities of plane curves that studied by Daniel Gorenstein in his thesis[4] under Oscar Zariski. The theory was cast in its current form through work of Serre and Bass. Gorenstein rings are geometrically common and significant, as the title of Bass' paper " On the Ubiquity of Gorenstein Rings" [1] attests.

The most basic geometric objects associated with a smooth variety are its tangent and cotangent bundles, and various tensors derived from them. The most important is the highest exterior power of the cotangent bundle, the canonical bundle. If the variety is affine, then the sections of the canonical bundle form a module over the coordinate ring. It is called the canonical module. It is locally free of rank 1. If the variety has singularities, then there is still a canonical module, but it may not be locally free. The local Gorenstein ring is the local ring for which the canonical module is free.

Suppose that k is a field and that A is a local zero-dimension ring that is a finite-dimensional k-algebra. Define the canonical module ω_A of A to be the injective hull of the residue class field of A. The functor $M \mapsto D(M) := \mathrm{Hom}_A(M, \omega_A)$ is a dualizing functor on the category of finitely generated A-modules.

A zero-dimensional local ring A is defined to be Gorenstein if $A \cong \omega_A$. For example, fields are Gorenstein; in fact, complete intersections are also Gorenstein. More generally, let A be a local Cohen-Macaulay ring. A finitely generated A-module ω_A is a canonical module for A if there is a nonzerodivisor $x \in A$ such that $\omega_A/x\omega_A$ is a canonical module for $A/(x)$. A is Gorenstein if there a nonzerodivisor $x \in A$ such that $A/(x)$ is Gorenstein.

A zero-dimensional local ring A is Gorenstein if and only if A satisfies one of the following conditions: (a) A is injective as an A-module; (b) The socle of A is simple; (c) ω_A can be generated by one element.

One technique of constructing Gorenstein rings, Macaulay's method of inverse systems, is principally of interest in the zero-dimensional case.

Let $R = k[x_1, \cdots, x_r]$ and $S = k[x_1^{-1}, \cdots, x_r^{-1}] \subset K(R) = k(x_1, \cdots, x_r)$ be the polynomial ring on the inverses of the x_i. We make S into an R-module as follows: Let $L \subset K(R)$ be the vector space generated by the monomials in the x_i that are not in S. We identify the R-module $K(R)/L$ with S by means of the maps $S \subset k(R) \to K(R)/L$. More directly put: If $m \in R$ and $n \in S$ are monomials, then $m \cdot n$ is the monomial $mn \in K(R)$ if this happens to lie in S, and 0 otherwise.

There is a one-to-one inclusion reversing correspondence between finitely generated R-modules $M \subset S$ and the ideals $I \subset R$ such that $I \subset (x_1, \cdots, x_r)$ and R/I is a local zero-dimensional ring, given by

$$\begin{aligned} M &\longmapsto (0 :_R M), \text{ the annihilator of } M \text{ in } R; \\ I &\longmapsto (0 :_S I), \quad \text{the submodule of } S \text{ annihilated by } I. \end{aligned}$$

If M and I correspond then $M \cong \omega_{R/I}$, so the ideals $I \subset (x_1, \cdots, x_r)$ such that R/I is a local zero-dimensional Gorenstein ring are precisely the ideals of the form $I = (0 :_R f)$ for some nonzero element $f \in S$.

For the sake of convenience for computations, we give another description of the ideal $I = (0 :_R f)$.

Lemma 1. Let $f \in S$ be a homogeneous polynomial, we write $f = \frac{\tilde{f}}{x_1^{l_1} x_2^{l_2} \cdots x_r^{l_r}}$ where $\tilde{f} \in R$ and all terms of $\tilde{f}$ do not have a common divisor. Set $l = \max_{1 \le i \le r}\{l_i\}$. Then

$$(0 :_R f) = ((x_1^{l+1}, x_2^{l+1}, \cdots, x_r^{l+1}) :_R x_1^{l-l_1} x_2^{l-l_2} \cdots x_r^{l-l_r} \tilde{f}).$$

Proof. Suppose $g \in R$ and

$$g\tilde{f} x_1^{l-l_1} x_2^{l-l_2} \cdots x_r^{l-l_r} = h_1 x_1^{l+1} + \cdots + h_r x_r^{l+1} \in (x_1^{l+1}, x_2^{l+1}, \cdots, x_r^{l+1}).$$

Dividing it by $x_1^l x_2^l \cdots x_r^l$, we get

$$\begin{aligned} gf &= g \frac{\tilde{f}}{x_1^{l_1} x_2^{l_2} \cdots x_r^{l_r}} \\ &= h_1 \frac{x_1}{x_2^l \cdots x_r^l} + h_2 \frac{x_2}{x_1^l x_3^l \cdots x_r^l} + \cdots + h_n \frac{x_r}{x_1^l \cdots x_{r-1}^l}. \end{aligned}$$

Since all terms in $h_i \frac{x_i}{x_1^l \cdots \hat{x}_i^l \cdots x_r^l}$ are zero in S. Thus $gf = 0$ and we have proved that

$$(0 :_R f) \supset ((x_1^{l+1}, x_2^{l+1}, \cdots, x_r^{l+1}) :_R x_1^{l-l_1} x_2^{l-l_2} \cdots x_r^{l-l_r} \tilde{f}).$$

Conversely, let $g \in (0 :_R f)$. Then

$$gf = \sum_I a_I x_1^{i_1} x_2^{i_2} \cdots x_r^{i_r} = 0 \quad \text{in } S,$$

where, for each $I = (i_1, i_2, \cdots, i_r)$, there exists j such that $i_j > 0$. So

$$\begin{aligned} g x_1^{l-l_1} \cdots x_r^{l-l_r} \tilde{f} &= g \frac{\tilde{f}}{x_1^{l_1} x_2^{l_2} \cdots x_r^{l_r}} x_1^l \cdots x_r^l \\ &= g f x_1^l \cdots x_r^l \\ &= \sum_I a_I x_1^{l-i_1} \cdots x_r^{l+i_r} \in R. \end{aligned}$$

For any I, there exists j such that $l + i_j \geq l + 1$, so $x_1^{l+i_1} \cdots x_r^{l+i_r} \in (x_1^{l+1}, \cdots, x_r^{l-1})$ and we have $g x_1^{l-l_1} \cdots x_r^{l-l_r} \tilde{f} \in (x_1^{l+1}, \cdots, x_r^{l-1})$.

Example 2. In $R = k[x, y, z]$, let $f = x^{-1}y^{-1} + z^{-2} \in S$. We have

$$I := (0 :_R f) = ((x^3, y^3, z^3) : xy(xy + z^2)) = (x^2, y^2, xz, yz, xy - z^2).$$

David Eisenbud has proposed a project in [3]. Here we quote:

"Compute some ideals of the form $I = ((x_1^s, \cdots, x_r^s) : p)$,where p is a homogeneous polynomial. It's easy to do by hand the case $r = 1$ and the case p a monomial, r =anything. Try the case $r = 2$ with more complicated p on the machine. How many generators does I require? Next try $r = 3$, various p. Here there is a greater range in the possible number of generators. Is there any restrictions? Make a conjecture! How about with $r = 4$? ... ... This is actually the first project that involved me personally with computer algebra. David Buchsbaum and I were interest in Gorenstein ideals around 1971-72. ... You can find the results inspired by our computations in Buchsbaum and Eisenbud[2]."

Example 3 We first give some examples which are easy to do by hand.

(1) Let $R = k[x]$ and $f, g \in R$. Then $(f : g) = (\frac{f}{\mathrm{GCD}(f,g)})$.

(2) Let $R = k[x_1, x_2, \cdots, x_r]$ and $f \in R$. Then $((x^n) : f) = (\frac{x^n}{\mathrm{GCD}(x^n, f)})$.

(3) Let $R = k[x_1, x_2, \cdots, x_r]$ and $m_1, \cdots, m_t, n \in R$ be monomials . Then $((m_1, \cdots, m_t) : n) = (\frac{m_1}{\mathrm{GCD}(m_1, n)}, \cdots, \frac{m_t}{\mathrm{GCD}(m_t, n)})$.

In this note, we restrict ourself to the case of f being a binomial.

§ 2 Two variable case

In this section let $R = k[x, y]$ and $f \in R$ be any binomial. In the following Theorem 4, we first compute the ideal $((x^n, y^n) : f)$ for the special type of $f = x^l - y^l$.

Theorem 4. Let

$$P_{sl} = x^{sl} + x^{(s-1)l}y^l + x^{(s-2)l}y^{2l} + \cdots + y^{sl}.$$

For any $n \in \mathbb{N}$, write $n = ls + t$, $t = 1, 2, \cdots, l$. We have

$$((x^n, y^n) : x^l - y^l) = (P_{sl}, x^t y^t P_{s-1,l}).$$

First Proof. Let $g_m \in ((x^n, y^n) : x^l - y^l)$ be a homogeneous polynomial of degree m. It is clear that $g_m = 0$ for $m = 0, 1, 2, \cdots, n-1-l$.

Now we consider the case $n - l \leq m \leq 2n - l - 2$. Let $g_m = a_0 x^m + a_1 x^{m-1} y + a_2 x^{m-2} y^2 + \cdots + a_m y^m$, we have

$$\begin{aligned} g_m(x^l - y^l) =& a_0 x^{m+l} + \cdots + a_l x^m y^l + \cdots + a_m x^l y^m + \cdots + a_{m+l} y^{m+l} \\ &- (a_{-l} x^{m+l} + \cdots + a_0 x^m y^l + \cdots + a_{m-l} x^l y^m + \cdots + a_m y^{m+l}) \\ =& \sum_{k=0}^{m+l} (a_k - a_{k-l}) x^{m+l-k} y^k \in (x^n, y^n) \end{aligned}$$

where $a_t = 0$ if $t \neq 0, 1, \cdots, m$, so

$$a_{k-l} = a_k \qquad m - n + l + 1 \leq k \leq n - 1.$$

Thus we can rewrite g_m as the following

$$g_m = x^n f + \sum_{k=m-n+1}^{m-n+l} a_k \Big(\sum_{i=0}^{[\frac{n-1-k}{l}]} x^{m-k-il} y^{k+li} \Big) + y^n h, \qquad f, h \in K[x, y].$$

We introduce another ideal : Let $h_{mk} = \sum_{i=0}^{[\frac{n-1-k}{l}]} x^{m-k-il} y^{k+li}$ and define

$$\begin{aligned} I := (x^n, y^n, h_{mk} | n - l \leq m \leq 2n - l - 2, m - n + 1 \leq k \leq m - n + l, k \geq 0, \\ k + l[\frac{n-1-k}{l}] \leq m). \end{aligned}$$

Thus we have proved that , if $m < 2n - l - 1$, then $g_m \in I$. If $m \geq 2n - l - 1$, for any g_m, we have $g_m(x^l - y^l) \in (x^n, y^n)$. But $g_m \in (x^n, y^n, x^{n-l}y^{n-l}) \subset I$ since $h_{2n-2l,m-n+l} = x^{n-l}y^{n-l}$. Thus $((x^n, y^n) : x^l - y^l) \subset I$.

Let $J := (P_{sl}, x^t y^t P_{s-1,l})$ for simplicity. Now we want to prove $I \subset J$. Since

$$\begin{aligned} x^t P_{sl} - y^{l-t} x^t y^t P_{s-1,l} &= x^n + \sum_{i=1}^{s} x^{ls-li-t} y^{li} - \sum_{i=0}^{s-1} x^{l(s-1)-li+t} y^{l(i+1)} \\ &= x^n, \end{aligned}$$

$$\begin{aligned} y^t P_{sl} - x^{l-t} x^t y^t P_{s-1,l} &= y^n + \sum_{i=0}^{s-1} x^{ls-li} y^{li-t} - \sum_{i=0}^{s-1} x^{ls-li} y^{li+t} \\ &= y^n, \end{aligned}$$

we have $x^n, y^n \in J$. To show that $h_{mk} \in J$, we separate into 3 cases.
Case 1. $m - k \geq ls$.

$$\begin{aligned} x^{m-k-ls} y^k P_{sl} &= \sum_{i=0}^{s} x^{m-k-li} y^{k+li} \\ &= \sum_{i=0}^{[\frac{n-1-k}{l}]} x^{m-k-li} y^{k-li} + \sum_{[\frac{n-1-k}{l}]+1}^{s} x^{m-k-li} y^{k+li} \\ &= h_{mk} + y^n p(x, y). \end{aligned}$$

$p(x, y) \in k[x, y]$, since $[\frac{n-1-k}{l}] = [\frac{ls+t-1-k}{l}] \leq s$ and the power of y in the second term $= k + li \geq k + l([\frac{n-1-k}{l}] + 1) \geq k + l(\frac{n-k-l}{l} + 1) = n$.
Case 2. $m - k < ls$ and $k \geq t$.

$$\begin{aligned} x^{m-k-(n-l)} y^{k-t} x^t y^t P_{s-1,l} &= \sum_{i=0}^{s} x^{m-k-n+l(s-i)+t} y^{k+li} \\ &= \sum_{i=0}^{s-1} x^{m-k-li} y^{k+li} \\ &= h_{mk} + y^n p'(x, y) \end{aligned}$$

$p'(x, y) \in k[x, y]$, since $\frac{n-1-k}{l} < s$, so $[\frac{n-1-k}{l}] \leq s - 1$, and the power of $y \geq n$ by the similar arguments as case 1.
Case 3. $m - k < ls$ and $k < t$. In this case.

$$k + l[\frac{n-1-k}{l}] = k + l[\frac{ls+t-1-k}{l}] \geq k + ls > m,$$

contradicts to the last condition in the definition of I. Thus $((x^n, y^n : x^l - y^l) \subset I \subset J$ and the proof is complete.

Second Proof. Let $p_1 = P_{sl}$ and $p_2 = x^t y^t P_{s-1,l}$ and

$$Q_1 = p_1(x^l - y^l) = x^{(s-1)l} - y^{(s-1)l} \in (x^n, y^n),$$
$$Q_2 = p_2(x^l - y^l) = x^n y^t - x^t y^n \in (x^n, y^n).$$

Now let $f \in ((x^n, y^n) : x^l - y^l)$ be a homogeneous polynomial of degree k and let

$$g = f(x^l - y^l) \in (x^n, y^n).$$

By long division with respect to x.

$$g = q_1 Q_1 + g_1, \quad \deg_x g_1 < (s+1)l.$$

Case 1. $\deg g = k + l \geq n + l - 1$.
Then $n + l - 1 - (s+1)l = t - 1$,so $\deg_y g_1 > t - 1$ and we divide g_1 by Q_2:

$$g_1 = q_2 Q_2 + g_2, \quad \deg_x g_2 \leq n - 1.$$

Note that $g_2 \in (x^n, y^n)$. Thus $y^n | g^2$. Moreover. $x^l - y^l | g_2$. So we write

$$g_2 = y^n (x^l - y^l) g_3$$

and

$$g = q_1 Q_1 + q_2 Q_2 + y^n (x^l - y^l) g_3,$$
$$f = q_1 p_1 + q_2 p_2 + y^n g_3.$$

On the other hand,

$$y^n = y^{sl+t} = y^t p_1 - x^{l-t} p_2.$$

So $f \in (p_1, p_2)$ and our theorem is proved in this case.
Case 2. $\deg g = k + l < n + l - 1$.
Write

$$g = x^n (a_0 x^j + a_1 x^{j-1} y + \cdots + a_{j-1} x y^{j-1} + a_j y^j)$$
$$+ y^n (b_0 x^j + b_1 x^{j-1} y + \cdots + b_{j-1} x y^{j-1} + b_j y^j).$$

Since $\deg g = n + j < n + l - 1$, so $j \leq l - 2$.

$$\begin{aligned} g =& x^{ls}(a_0 x^{j+t} + a_1 x^{j+t-1} y + \cdots + a_{j-1} x^{t+1} y^{j-1} + a_j x^t y^j) \\ &+ y^n (b_0 x^j + b_1 x^{j-1} y + \cdots + b_{j-1} x y^{j-1} + b_j y^j). \end{aligned}$$

Case 2.1. If $j + t < l$, then

$$\begin{aligned} g \equiv& y^{ls}(a_0 x^{j+t} + a_1 x^{j+t-1} y + \cdots + a_t x^j y^t + \cdots + a_{j-1} x^{t+1} y^{j-1} + a_j x^t y^j) \\ &+ y^n (b_0 x^j + b_1 x^{j-1} y + \cdots + b_{j-1} x y^{j-1} + b_j y^j) \equiv 0 (\mathrm{mod}(x^l - y^l)). \end{aligned}$$

Since all exponents of x are less than l, so when $j \geq t$, we have

$$b_0 + a_t = 0, b_1 + a_{t+1} = 0, \cdots, b_{j-t} + a_j = 0$$

and all other coefficients $a_0, \cdots, a_{t-1}, b_{j-t+1}, \cdots, b_j$ are 0. Therefore

$$g = a_t(x^{n+j-t} y^t - y^n x^j) + a_{t+1}(x^{n+j-t-1} y^{t+1} - x^{j-1} y^{n-1}) + \cdots + a_j (x^n y^j - x^t y^{n+j-t}).$$

Since

$$\begin{aligned} & x^{n+j-t-i} y^{t+i} - x^{j-i} y^{n-i} \\ =& x^{j-t-i} y^i (x^n y^t - x^t y^n), \quad i \leq j - t. \end{aligned}$$

Hence $Q_2 | g$ and $p_2 | f$.

If $j < t$ then all coefficients a_i, b_j are zero and $g = 0$.

Case 2.2. If $j + t \geq l$, then

$$\begin{aligned} g \equiv& y^{ls}(a_0 x^{j+t-l} y^l + \cdots + a_{j+t-l} y^{j+t} + a_{j+t-l+1} x^{l-1} y^{j+t-l+1} + \cdots + a_t x^j y^t + \\ & a_{t+1} x^{j-1} y^{t+1} + \cdots + a_j x^t y^j) + y^n (b_0 x^j + \cdots + b_j y_j) \\ =& y^{ls}(a_{j+t-l+1} x^{l-1} y^{j+t-l+1} + \cdots + a_t x^j y^t + a_{t+1} x^{j-1} y^{t+1} + \cdots + a_j x^t y^j + \\ & a_0 x^{j+t-l} y^l + \cdots + a_{j+t-l} y^{j+t}) + \\ & (b_0 x^j y^n + \cdots + b_{l-t-1} x^{j+t-l+1} y^{n-t+l-1} + b_{l-t} x^{j+t-l} y^{n-t+l} + \cdots + b_j y^{n+j}). \end{aligned}$$

Hence we have

$$\begin{cases} a_0 + b_{l-t} = 0, a_1 + b_{l-t+1} = 0, \cdots, a_{j+t-l} + b_j = 0, \\ \qquad b_0 + a_t = 0, \cdots, b_{j-t} + a_j = 0, \qquad \text{if } t \le j. \end{cases}$$

and all other coefficients $a_{j-t-l+1}, \cdots, a_{t-1}, b_{j-t-1}, b_{l-t-1}$ are zero. Thus

$$\begin{aligned} & x^{n-j-i}y^{n-i} - x^{j-l-t-i}y^{n+l-t-i} \\ =& x^{j-l+t-i}y^{i}(x^{n+l-t} - y^{n+l-t}) \\ =& x^{j-l+t-i}y^{i}Q_1. \end{aligned}$$

So $g \in (Q_1, Q_2)$, and $f \in (p_1, p_2)$.

Remark. In the above Theorem, if $f = ax^l - by^l$, then we replace P_{sl} by $a^s x^{sl} + a^{s-1}bx^{(s-1)l}y^l + a^{s-2}b^2x^{(s-2)l}y^{2l} + \cdots + b^s y^{sl}$. The results are the same as before and the proof is similar.

By the above remark, for any binomial in $k[x,y]$, we may assume that it has the form $x^k y^m(x^l - y^l)$. In the following we let θ to be the Heaviside function : $\theta(i) = 0$, if $i \le 0$; $= i$, if $i > 0$.

Theorem 5. Let $F = x^k y^m(x^l - y^l) \in k[x,y]$. Suppose $k \ge m$. Define $N = \max\{k, m+l\}$, $d = k-m$ and define s, t as the following: $n = N+ls+t, t = 1, 2, \cdots, l$. Then we have

$$((x^n, y^n) : x^k y^m(x^l - y^l)) = \begin{cases} (1), & n \le N \\ (y^{\theta(d-l)+t}x^{\theta(t-d)}P_{sl}, h(x,y)), & n > N. \end{cases}$$

and

$$h(x,y) = \begin{cases} x^{n-k}, & \text{if } \theta(t-d) = 0 \\ P_{s-1,l}, & \text{if } \theta(t-d) > 0. \end{cases}$$

Proof. Case 1. $m = 0$. In this case, $N = \max\{k,l\}, d = k$. We will prove by induction on k. When $k = 0$, it holds by Theorem 4. Let $f \in ((x^n, y^n) : x^k(x^l - y^l))$, then $fx \in ((x^n, y^n) : x^{k-1}(x^l - y^l))$.
Case 1.1. $1 \le k \le l$. In this case, $N = l, n = l + ls + t$. If $t \le k-1$, then

$$fx = gy^t P_{ls} + hx^{n-k+1},$$

which implies $x|g$. We write $g = xg'$, so $f = g'y^tP_{ls} + hx^{n-k} \in (y^tP_{ls}, x^{n-k})$.

If $t > k$,then

$$\begin{aligned} fx &= gy^t x^{t-k+1}P_{ls} + hP_{l,s-1}, \\ f &= gyx^{t-k}P_{ls} + h'P_{l,s-1} \in (y^t x^{t-d}P_{ls}, P_{l,s+1}). \end{aligned}$$

If $t = k$,. then

$$fx = gy^t xP_{ls} + hP_{l,s+1}.$$

We write $h = xh'$, so

$$\begin{aligned} f =& gy^t P_{ls} + hP_{l,s-1} \\ =& gy^t P_{ls} + h'(x^{l(s-1)} - y^t P_{ls}) \in (y^t P_{ls}, x^{n-k}). \end{aligned}$$

Case 1.2. $l < k$. Now $N = k, n = k + ls + t$. If $t < l - 1$, then

$$\begin{aligned} fx =& gy^{k-l+t} P_{ls} + hx^{n-k-1} \\ f =& g'y^{k-l+t} P_{ls} + hx^{n-k} \in (y^{d-l+t} P_{ls}, x^{n-k}). \end{aligned}$$

If $t = l$. then

$$\begin{aligned} fx =& gy^{k-1-l+1} P_{l(s+1)} + hx^{n-k+1}. \\ f =& g'y^{k-l}(x^{l(s+1)} - y^t P_{ls}) + hx^{n-k} \\ =& g'y^{k-l}x^{n-k} - g'y^k P_{ls} + hx^{n-k} \in (y^{d-l+t} P_{ls}, x^{n-k}). \end{aligned}$$

Case 2. General Cases. If $m \geq n$,then $I = (1)$ and $N \geq n$, so the result holds. Now we assume that $n > m$. The following two formula are clear.

$$((x^n, y^n) : x^m y^m) = (x^{n-m}, y^{n-m});$$

$$((x^n, y^n) : x^k y^m (x^l - y^l)) = ((x^{n-m}, y^{n-m}) : x^{k-m}(x^l - y^l)).$$

Now we apply the results in Case 1. to the ideal

$$(x^{n-m}, y^{n-m}) : x^{k-m}(x^l - y^l)).$$

In this case, set $N' = \max\{k - m, l\} = N - m$ and from $n - m = N' + ls' + t'$, it is clearly that $s' = s, t' = t$. Moreover,$n - m > N'$ if and only in $n > N$. Thus we have

$$(x^{n-m}, y^{n-m}) : x^{k-m}(x^l - y^l)) = \begin{cases} (1), & n - m \leq N' \\ (y^{\theta(d-l)+t} x^{\theta(t-d)} P_{sl}, h'(x, y)), & n - m > N'. \end{cases}$$

. where $h'(x, y) = h(x, y)$. Hence the proof is finished.

§ 3 Three variable case

Let $F = x^{\alpha}y^{\beta}z^{\gamma}(x^k - y^l z^m)$, $k = l + m$, $l, m > 0$, $\alpha, \beta, \gamma \geq 0$. Define

$$P_i = x^{ik} + x^{(i-1)k}y^l z^m + \cdots + y^{il}z^{im},$$

$P_{-1} = 0$. For any $n \in \mathbb{N}$ and $n > \alpha, \beta, \gamma$, we write

$$\begin{aligned} n - \alpha - 1 &= a_1 k + b_1, & 0 \leq b_1 < k; \\ n - \beta - 1 &= a_2 l + b_2, & 0 \leq b_2 < l; \\ n - \gamma - 1 &= a_3 m + b_3, & 0 \leq b_3 < m. \end{aligned}$$

Let $a = \min\{a_1, a_2, a_3\}$.

Theorem 6

$$\begin{aligned} &((x^n, y^n, z^n) : x^{\alpha}y^{\beta}z^{\gamma}(x^k - y^l z^m)) \\ =&(x^{n-\alpha}, y^{n-\beta}, z^{n-\gamma}, x^{\theta(n-\alpha-(a+1)k)}y^{\theta(n-\beta-(a+1)l)}P_a, x^{\theta(n-\alpha-(a+1)k)}z^{\theta(n-\gamma-(a+1)m)}P_a, \\ &x^{\theta(n-\alpha-ak)}y^{\theta(n-\beta-al)}P_{a-1}, x^{\theta(n-\alpha-ak)}z^{\theta(n-\gamma-am)}P_{a-1}). \end{aligned}$$

Proof. We write

$$p_1 = x^{\theta(n-\alpha-(a+1)k)}y^{\theta(n-\beta-(a+1)l)}P_a.$$

Note that $\theta(i) \geq i$ and $P_a F = x^{\alpha}y^{\beta}z^{\gamma}(x^{(a+1)k} - y^{(a+1)l}z^{(a+1)m})$.Thus

$$p_1 F = x^{\theta(n-\alpha-(a+1)k)+\alpha}y^{\theta(n-\beta-(a+1)l)-\beta}(x^{(a+1)k} - y^{(a+1)l}z^{(a+1)m}) \in (x^n, y^n, z^n).$$

Similarly, let $p_2 = x^{\theta(n-\alpha-(a+1)k)}z^{\theta(n-\gamma-(a+1)m)}P_a$, $p_3 = x^{\theta(n-\alpha-ak)}y^{\theta(n-\beta-al)}P_{a-1}$, $p_4 = x^{\theta(n-\alpha-ak)}z^{\theta(n-\gamma-am)}P_{a-1}$. Then $p_i F \in (x^n, y^n, z^n)$, $i = 2, 3, 4$.

Now given any $f \in ((x^n, y^n, z^n) : F)$, we have to show that $f \in (x^{n-\alpha}, y^{n-\beta}, z^{n-\gamma}, p_1, p_2, p_3, p_4)$. We prove it by induction on $\deg_x f$. Firstly, we may assume that $\deg_x f < n - \alpha, \deg_y f < n - \beta, \deg_z f < n - \gamma$. Let

$$f = q_0 x^i + q_1 x^{i-1} + \cdots + q_{i-1}x + q_i, \quad q_0 \neq 0, q_i \in k[x, y].$$

$$\begin{aligned} g =fF = &q_0 x^{k+i+\alpha}y^{\beta}z^{\gamma} + q_1 x^{k+i-1+\alpha}y^{\beta}z^{\gamma} + \cdots + q_{k-1}x^{i+\alpha+1}y^{\beta}z^{\gamma} + \\ &(q_k - q_0 y^l z^m)x^{i+\alpha}y^{\beta}z^{\gamma} + (q_{k+1} - q_1 y^l z^m)x^{i+\alpha-1}y^{\beta}z^{\gamma} + \cdots + (q_i - q_{i-k}y^l z^m) \\ &x^{k+\alpha}y^{\beta}z^{\gamma} + (-q_{i-k+1}y^l z^m)x^{k+\alpha-1}y^{\beta}z^{\gamma} + \cdots + (-q_i y^l z^m)x^{\alpha}y^{\beta}z^{\gamma} \in (x^n, y^n, z^n). \end{aligned} \qquad (*)$$

Case(I). Now we assume that $a = a_1$. Note that $n - \alpha - (a+1)k = a_1 k + b_1 + 1 - (a+1)k = b_1 + 1 - k \leq 0$. Then

$$p_1 = y^{\theta(n-\beta-(a+1)l)} P_a$$

and

$$p_2 = z^{\theta(n-\gamma-(a+1)m)} P_a$$

They have x-degree ak.

Case 1. $ak \leq \deg_x f < n - \alpha$, that is $ak \leq i \leq ak + b_1$.

Suppose $\deg_y q_0 \geq \theta(n - \beta - (a+1)l)$ or $\deg_z q_0 \geq \theta(n - \gamma - (a+1)m)$, we can divide f by p_1 or p_2, respectively, and get

$$f = q_1 p_j + f_1, \quad j = 1, 2,$$

$\deg_x f_1 \leq \deg f$ and $f_1 \in ((x^n, y^n, z^n) : F)$. Then the statement is proved by induction. Thus we may assume that

$$\begin{aligned} 0 < n - \beta - (a+1)l \quad &\text{and} \quad \deg_y q_0 < n - \beta - (a+1)l, \\ 0 < n - \gamma - (a+1)m \quad &\text{and} \quad \deg_z q_0 < n - \gamma - (a+1)m. \end{aligned}$$

Since $i \leq ak + b_1 = n - \alpha - 1 < n$, Thus from $(*)$ we have

$$q_k = q_0 y^l z^m, q_{2k} = q_k y^l z^m = q_0 y^{2l} z^{2m}, \cdots q_{jk} = q_0 y^{jl} z^{jm}.$$

In particular, $q_{ak} = q_0 y^{al} z^{am}$.

Now $ak \leq i$, and from $i \leq ak + b_1 \leq ak + k - 1$, so $ak \geq i - k + 1$ and q_{ak} must be one of $q_i, q_{i-1}, \cdots$, or q_{i-k+1}. Hence the coefficients of $x^{\alpha+i-ak}$ in $(*)$ must be zero. That is,

$$q_{ak} y^l z^m y^\beta z^\gamma = q_0 y^{(a+1)l+\beta} z^{(a+1)m+\gamma} \in (y^n, z^n).$$

It has y-degree $< n - \beta - (a+1)l + \beta < n$ and z-degree $< n - \gamma - (a+1)m + \gamma < n$. A contradiction.

Case 2. $a > 0$ and $(a-1)k + b_1 + 1 \leq \deg_x f < ak$.

Similar to case 1, we have

$$q_{(a-1)k} = q_0 y^{(a-1)l} z^{(a-1)m}.$$

and we may assume that

$$\deg_y q_0 < n - \beta - al, \quad \deg_z q_0 < n - \gamma - am,$$

otherwise f can be divided by p_3 or p_4. Now $i \le ak-1$ so $i-k+1 \le (a-1)k$ and $i \ge (a-1)k+b_1+1 \ge (a-1)k$. Thus $q_{(a-1)k}$ must be one of $q_i, q_{i-1}, \cdots$, or q_{i-k+1}. Therefore we consider

$$h = q_{(a-1)k}y^l z^m y^\beta z^\gamma = q_0 y^{al+\beta} z^{am+\gamma}.$$

$\deg_y h < n-\beta-al+al+\beta = n$ and $\deg_z h < n$. A contradiction.

Case 3. $a_1 \ge 1$ and $i < n-k-\alpha = (a_1-1)k+b_1+1$.
In this case, $q_0 y^\beta z^\gamma \in (y^n, z^n)$, which implies $q_0 = 0$. A contradiction.

Hence the proof of the case $a = a_1$ is complete.

Case(II). Now we assume that

$$a = a_2 < a_1, a_2 \le a_3.$$

In this case, $p_1 = x^{\theta(n-\alpha-(a+1)k)}P_a$, which is monic in x. $p_2 = x^{\theta(n-\alpha-(a+1)k)} z^{\theta(n-\gamma-(a+1)m)}P_a$. Note that p_2 is a multiple of p_1. $\theta(n-\alpha-(a+1)k) = (a_1-a-1)k+b_1+1 > 0$.
Case 1. If $\deg_x f \ge (a_1-1)k+b_1+1 = n-\alpha-k$, then f can be divided by p_1 and $\deg_x f$ is reduced.
Case 2. Let $\deg_x f < n-\alpha-k$.

$$g = Ff = q_0 x^{k+i-\alpha} y^\beta z^\gamma + \cdots \in (x^n, y^n, z^n).$$

$\deg_x g = i+\alpha+k < n-\alpha-k+\alpha+k = n$ and $\deg_y g$, $\deg_z g < n$, so $q_0 = 0$. A contradiction.

In case $a = a_3$, the proof is the same as the case $a = a_2$. The proof is complete.

§ 4 Discussions

More generally, we have the following result. The proof is the same as that of Theorem 6.

Let

$$F = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_l^{\alpha_l}(x_1^{k_1} - x_2^{k_2}x_3^{k_3}\cdots x_l^{k_l}), \quad k_1 = k_2+\cdots+k_l, \alpha_j \ge 0,$$

and

$$P_i = x_1^{ik_1} + x_1^{(i-1)k_1}x_2^{k_2}\cdots x_l^{k_l} + x_1^{(i-2)k_1}x_2^{2k_2}\cdots x_l^{2k_l} + \cdots + x_2^{ik_2}\cdots x_l^{ik_l}.$$

Let $\theta(j)$ be the Heaviside function. For any $n \in \mathbb{N}$ and $n > \alpha_j$, for all $j = 1, 2, \cdots, l$, we write

$$n - \alpha_j - 1 = a_j k_j + b_j, 0 \le b_j < k_i.$$

Let $a = \min\{a_1, a_2, \cdots, a_l\}$.

Theorem 7.

$$\begin{aligned} &((x_1^n, x_2^n, \cdots, x_l^n) : F) \\ =&(x_1^{n-\alpha_1}, x_2^{n-\alpha_2}, \cdots, x_l^{n-\alpha_l}, p_2, p_3, \cdots, p_l, q_2, q_3, \cdots, q_l) \end{aligned}$$

where

$$\begin{aligned} p_j &= x_1^{\theta(n-\alpha_1-(a-1)k_1)} x_j^{\theta(n-\alpha_j-(a+1)k_j)} P_a \\ q_j &= x_1^{\theta(n-\alpha_1-ak_1)} x_j^{\theta(n-\alpha_j-ak_j)} P_{a-1}. \end{aligned}$$

Remark 8. The ideal $((x^n, y^n) : x^i y^j - x^k y^l), i + j = k + l$, in Theorem 5 is always generated by 2 elements. But the ideal

$$((x_1^n, x_2^n, \cdots, x_l^n) : x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_l^{\alpha_l}(x_1^{k_1} - x_2^{k_2} x_3^{k_3} \cdots x_l^{k_l}), k_1 = k_2 + \cdots + k_l,$$

in Theorem 7 need $3l - 2$ generators. As $l = 3, 3l - 2 = 7$. In fact, the number of generators can be reduced in some special cases.

Now we consider

$$I = ((x^n, y^n, z^n) : x^k - y^l z^m), l, m \ne 0, k = l + m.$$

In this case, $k > l$, so $n - 1 = ak + b, 0 \le b < k$. Let

$$\begin{aligned} p_2 &= y^{\theta(am+b+1-l)} P_a, \\ p_3 &= y^{\theta(al+b+1-m)} P_a, \\ q_2 &= x^{b+1} y^{am+b+1} P_{a-1}, \\ q_3 &= x^{b+1} z^{al+b+1} P_{a-1}. \end{aligned}$$

Then $I = (x^n, y^n, z^n, p_2, p_3, q_2, q_3)$.

We first give two fundamental identities:

$$\begin{aligned} P_a &= x^{ak} + y^l z^m P_{a-1} \\ &= x^k P_{a-1} + y^{al} z^{am}. \end{aligned}$$

(1)When $a \neq 0$, $n \leq (a+1)l$ or $n \leq (a+1)m$, we have

$$am + b + 1 - l = ak - al + b + 1 - l = n - (a+1)l.$$

So $p_2 = P_a$ or $p_3 = P_a$. Moreover ,

$$P_a x^{b+1} = x^n + x^{n-k} y^l z^m + \cdots + x^{b-1} y^{al} z^{am}$$

and

$$q_2 y^{(a+1)l-n} z^m = x^{n-k} y^l z^m + \cdots + x^{b-1} y^{al} z^{am}.$$

Then $x^n = P_a x^{b+1} - q_2 y^{(a+1)l-n} z^m$. Thus

$$I = (y^n, z^n, q_2, q_3, P_a).$$

(2) When $n > (a+1)l$, $n > (a+1)m$ and $b = k+1$, we have

$$p_2 = x^{ak} y^{am+k-l} + x^{(a-1)k} y^{am+k} z^m + x^{(a-2)k} y^{am+k+l} z^{2m} + \cdots + y^{(a+1)k-l} z^{am},$$

$$q_2 = x^{ak} y^{am+k} + x^{(a-1)k} y^{am+k+l} z^m + x^{(a-2)k} y^{am+k+2l} z^{2m} + \cdots + y^{(a+1)k-l} z^{(a-1)m}.$$

So

$$q_2 = p_2 y^l - y^n z^{am},$$
$$q_3 = p_3 z^m - z^n y^{al}.$$

Hence $I = (x^n, y^n, z^n, p_2, p_3)$.
(3) When $a = 0$, we have

$$\begin{aligned} I &= (x^n, y^n, z^n, y^{b-1-l}, z^{b+1-m}) \\ &= (x^n, y^{b-1-l}, z^{b-1-m}). \end{aligned}$$

Remark 9. In the cases discussed above, the upper bound of the numbers of the generators of the ideal $(x_1^n, x_2^n, \cdots, x_l^n) : F)$ is independent of n. But it is not true in general. The followings are examples computed by MACAULAY 2.

Example 9.1. Let $F = xy - zw$. Define $P_a = x^a y^a + x^{a-1} y^{a-1} zw + \cdots + z^a w^a$. Let

$$p_i = x^{n-1-i} z^{n-1-i} P_i, q_i = x^{n-1-i} w^{n-1-i} P_i,$$
$$r_i = y^{n-1-i} z^{n-1-i} P_i, s_i = y^{n-1-i} w^{n-1-i} P_i, 0 \leq i \leq n-1.$$

Then

$$((x^n, y^n, z^n, w^n) : xy - zw)$$
$$=(x^n, y^n, z^n, w^n, p_0, q_0, r_0, s_0, , p_1, q_1, r_1, s_1, \cdots, p_{n-2}, q_{n-2}, r_{n-2}, s_{n-2}, P_{n-1}).$$

It is generated by $4n+1$ elements.

Example 9.2. Let $F = x^2y - z^2w$. Define $P_a = x^{2a}y^a + x^{2a-2}y^{a-1}z^2w + \cdots + z^{2a}w^a$. Let

$$p_i = x^{n-2-i}z^{n-2-i}P_i, q_i = x^{n-2-i}w^{n-1-i}P_i,$$
$$r_i = y^{n-1-i}z^{n-2-i}P_i, s_i = y^{n-1-i}w^{n-1-i}P_i, 0 \leq i \leq n-1.$$

Then

$$((x^n, y^n, z^n, w^n) : x^2y - z^2w)$$
$$=(x^n, y^n, z^n, w^n, p_0, q_0, r_0, s_0, , p_1, q_1, r_1, s_1, \cdots, , p_{[\frac{n-2}{2}]}, q_{[\frac{n-2}{2}]}, r_{[\frac{n-2}{2}]}, s_{[\frac{n-2}{2}]}, P_{[\frac{n}{2}]}).$$

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