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線性 Boltzmann 方程解的逐點估計

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一、中文摘要

本計畫中，我們得到近似的 Green 函數，由這個 Green 函數造出來的解，滿足 Burgers 方程，並保有 ξ 方向更多的資訊。

ABSTRACT. We derive an approximate Green function for the linearized Boltzmann equation. For the full nonlinear equation, the solution constructed from the approximate Green function satisfies the Burgers equations in the variables x and t if the initial value is close to a constant Maxwellian.

1. INTRODUCTION

In this paper, we consider the Boltzmann equation

$$\frac{\partial}{\partial t}f + \xi \cdot \frac{\partial}{\partial x}f = Q(f, f) \tag{1.1}$$

in the rarefied gas dynamicd with a cut-off hard potential in the sense of Grad, where $f = f(x, \xi, t)$ with $x \in \mathbb{R}^3$, $\xi \in \mathbb{R}^3$ and $t \geq 0$, and

$$Q(f, f) = \int_{\mathbb{R}^3} \int_{S_+} (f' f'_* - f f_*) |V \cdot n| \, d\xi_* \, dn$$

is the collision term. To understand microscopic dynamics nowadays, the Boltzmann equation is more and more important. However, many fundamental topics such as, for example,

(1) rigorus validity of the Boltzmann equation,
(2) existence and uniqueness of a global solution with a general initial value,
(3) existence for more general initial-boundary value problems,
(4) hydrodynamical limits,
(5) interaction of waves of the Boltzmann equation

are still not well understood. In this paper, we mainly concern with a very first step to understand the last topic (5), that is, the estimates of the Green function.

If we consider the full equation with a quadratic collision term, then we can not avoid nonlinear wave phenomenon. Since in the hydrodynamic regime, the Euler equations and the Navier-Stokes equations have shock and travelling wave solutions, it is nature to consider similar problem for the Boltzmann quation. The existence of a weak shock wave (travelling wave) for the Boltzmann quation was obtain by Caflisch and Nicolaenko [1]. They used an exact travelling wave of the Navier-Stokes equations as an approximate solution. Then the solution was found by a Lyapunov-Schmidt method as a bifurcation from a constant Maxwellian state. Unfortunately, they can not show this solution is nonnegative and it could be of no physical meaning. Caflisch and Nicolaenko in the same paper also proved a uniqueness result for the shock profile solution near a Maxwellian. Hence if we believe there is a weak shock profile solution with physical meaning, it must be the one constructed in [1].

Inspired by the works on shock profile solutions of conservation laws in Liu and Zeng [9], Liu [6], Liu and Wang [7] and Liu and Yu [8], it seems we can understand more about shock profile solutions and wave interaction from a better estimate of the Green function. The ideas are: (1) obtain pointwise estimates for the Green function of the linearized equation near a constant state; (2) obtain pointwise estimates for the Green function of the linearized equation near a approxmate shock profile solution; (3) use these estimates to trace the interaction of waves and show the convergence to a shock profile solution. The main difficulty to apply these ideas to the Boltzmann equation is that there is one more variable ξ in the equation. When linearized around a constant Maxwellian, the known results by Ukai [12] and Nishida and Imai [11] are L^2 type estimates for the semigroup. These estimates are not sharp enough to trace movement of waves.

In this paper, we linearize the equation around a constant Maxwellian. We use the semigroup to represent a solution and drop terms which decay very fast in time. Then we transfer the dominant terms into a convolution of the initial value and the sorce term with the Green function. From this, an approximate Green function is obtained. Moreover, the main terms of a solution satisfy the Burgers equations in x and t when t is large. This is similar to a Chapman-Enskog expansion which also keep much information in the direction of ξ.

2. Semigroup of the Linearized Equation

Let $M = (2\pi)^{-\frac{3}{2}} exp(-\frac{|\xi|^2}{2})$. We linearize the equation around M and write $f = M + M^{\frac{1}{2}}h$ and the collision term

$$Q(f, f) = Lh + \nu\Gamma(h, h),$$

where $L = 2M^{-\frac{1}{2}}Q(M^{\frac{1}{2}}h, M)$ is the linear part. The operator L is nonpositive, i.e.,

$$(Lh, h) \leq 0 \text{ for } h \in D(L)$$

and satisfies

$$Lh = 0 \text{ iff } h \in span\{M^{\frac{1}{2}}, \xi_j M^{\frac{1}{2}}, |\xi|^2 M^{\frac{1}{2}}\}.$$

Moreover, it can be decomposed as

$$Lh = -\nu(|\xi|)h + Kh,$$

where $\nu(|\xi|)$ satisfies

$$0 < \nu_o \leq \nu(|\xi|) \leq \nu_1(1 + |\xi|)$$

and K is a compact operator in L^2. Now we consider the linearized equation

$$\frac{\partial}{\partial t}h + \xi \cdot \frac{\partial}{\partial x}h = Lh$$

with

$$h(x, \xi, 0) = h_o(x, \xi).$$

Let $\hat{h}$ denote the Fourier transform of h in x. Then $\hat{h}(k, \xi, t)$ satisfies

$$\frac{\partial}{\partial t}\hat{h} + i\xi \cdot k\hat{h} = L\hat{h}.$$

Let

$$B(k) = L - i\xi \cdot kI.$$

We can represent $\hat{h}$ in the form of semigroup. See [12] and [11].

Theorem 1. *There exist $\delta > 0$, $b_1 > 0$ and $b_2 > 0$ such that for $\hat{h}_o = \hat{h}(k,\xi,0) \in D(B(k))$,*
(a) For any k with $|k| \leq \delta$,

$$\hat{h} = e^{tB(k)}\hat{h}_o = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{-b_1-ir}^{-b_1+ir} e^{t\lambda}(\lambda - B(k))^{-1}\hat{h}_o \, d\lambda + \sum_{i=1}^{5} e^{td_j(k)}(\psi_j(-k), \hat{h}_o)\psi_j(k),$$

where $d_j(k)$ and $\psi_j(k)$ are the eigenvalues and the corresponding eigenfunctions of $B(k)$.
(b) For $|k| \geq \delta$,

$$\hat{h} = e^{tB(k)}\hat{h}_o = \lim_{r\to\infty} \frac{1}{2\pi i} \int_{-b_2-ir}^{-b_2+ir} e^{t\lambda}(\lambda - B(k))^{-1}\hat{h}_o \, d\lambda$$

3. Approximate Green Function

Taking inverse Fourier transform in k, we have

$$h(x,\xi,t) = \int \hat{h}\, dk = \int_{|k|\geq\delta} \hat{h}\, dk + \int_{|k|\leq\delta} \hat{h}\, dk.$$

By the spectrum property of $B(k)$, we have

$$d_j(k) = i\alpha_j\kappa - \beta_j\kappa^2 - O(|k|^3),$$

where $\kappa = \pm|k|$, $\alpha_j \in \mathbb{R}$ and $\beta_j > 0$ for $j = 1, 2, .., 5$ and the limit terms in Theorem 1 satisfy with some $b > 0$

$$\lim_{r\to\infty} \frac{1}{2\pi i} \int_{-b_1-ir}^{-b_1+ir} e^{t\lambda}(\lambda - B(k))^{-1}\hat{h}_o \, d\lambda = O(e^{-bt})$$

for $|k| \leq \delta$ and

$$\lim_{r\to\infty} \frac{1}{2\pi i} \int_{-b_2-ir}^{-b_2+ir} e^{t\lambda}(\lambda - B(k))^{-1}\hat{h}_o \, d\lambda = O(e^{-bt})$$

for $|k| \geq \delta$. Hence

$$\int_{|k|\geq\delta} \hat{h}\, dk = O(e^{-bt})$$

and

$$\begin{aligned}
\int_{|k|\leq\delta} \hat{h}\, dk =& O(e^{-bt}) + \sum_{j=1}^{5} \int_{|k|\leq\delta} e^{-t[i\alpha_j\kappa - \beta_j\kappa^2 + O(|k|^3)]}(\psi_j(-k), \hat{h}_o)\psi_j(k)\, dk \\
=& O(e^{-bt}) + \sum_{j=1}^{5} \int_{|k|\leq\delta} e^{-t[i\alpha_j\kappa - \beta_j\kappa^2]}(\psi_j(-k), \hat{h}_o)\psi_j(k)\, dk \\
&+ \sum_{j=1}^{5} \int_{|k|\leq\delta} e^{-t[i\alpha_j\kappa - \beta_j\kappa^2]}[e^{O(|k|^3)} - 1](\psi_j(-k), \hat{h}_o)\psi_j(k)\, dk \\
=& \sum_{j=1}^{5} \int_{\mathbb{R}^3} e^{-t[i\alpha_j\kappa - \beta_j\kappa^2]}(\psi_j(-k), \hat{h}_o)\psi_j(k)\, dk \\
&+ O(t^{-2}\log^3 t) + O(e^{-b_3 t})
\end{aligned}$$

for some $b_3 > 0$. The final form we obtained is as follows.

Therorem 2. *Define the approximate Green function*

$$G(x,y,t,\xi) = \sum_{j=1}^{5} G_j(x,y,t,\xi)$$

with G_j satisfying

$$\begin{aligned}
&G_j *_y h_o \\
&= \int_{\mathbb{R}^3} (4\beta_j \pi t)^{-\frac{3}{2}} e^{-\frac{(x-y-\alpha_j t)^2}{4\beta_j t}} \Big\{ (\psi_j(0), h_o(\bar{\xi}, y))\psi_j(0) \\
&+ i(\frac{x-\alpha_j t}{\beta_j t})[(\psi_j(0), h_o(\bar{\xi}, y))\frac{\partial}{\partial k}\psi_j(0) + (\frac{\partial}{\partial k}\psi_j(0), h_o(y,\bar{\xi}))\psi_j(0)] \\
&+ \frac{1}{2}[\frac{1}{2\beta_j t} - (\frac{x-\alpha_j t}{\beta_j t})^2] \\
&\times [2(\frac{\partial}{\partial k}\psi_j(0), h_o(y,\bar{\xi}))\frac{\partial}{\partial k}\psi_j(0) + (\psi_j(0), h_o(y,\bar{\xi}))\frac{\partial^2}{\partial k^2}\psi_j(0) \\
&+ (\frac{\partial^2}{\partial k^2}\psi_j(0), h_o(y,\bar{\xi}))\psi_j(0)] \Big\}\, dy.
\end{aligned}$$

Then

$$h(x,\xi,t) = \sum_{j=1}^{5} G_j * h_o + O(t^{-2}\log^3 t).$$

Moreover, we can write h in the form

$$\begin{aligned}
h(x,\xi,t) = &\sum_{j=1}^{5} \{\eta_{0,j}\psi_j(0) - \frac{\partial}{\partial x}(i\eta_{0,j}\frac{\partial}{\partial k}\psi_j(0) + i\eta_{1,j}\psi_j(0)) \\
&- \frac{1}{2}\frac{\partial^2}{\partial x^2}[2\eta_{1,j}\frac{\partial}{\partial k}\psi_j(0) + \eta_{2,j}\psi_j(0) + \eta_{0,j}\frac{\partial^2}{\partial k^2}\psi_j(0)]\} + O(t^{-2}\log^3 t),
\end{aligned}$$

where

$$\begin{aligned}
\eta_{0,j} &= \int_{\mathbb{R}^3} (4\beta_j \pi t)^{-\frac{3}{2}} e^{-\frac{(x-y-\alpha_j t)^2}{4\beta_j t}} (\psi_j(0), h_o(y,\bar{\xi}))\, dy, \\
\eta_{1,j} &= \int_{\mathbb{R}^3} (4\beta_j \pi t)^{-\frac{3}{2}} e^{-\frac{(x-y-\alpha_j t)^2}{4\beta_j t}} (\frac{\partial}{\partial k}\psi_j(0), h_o(y,\bar{\xi}))\, dy, \\
\eta_{2,j} &= \int_{\mathbb{R}^3} (4\beta_j \pi t)^{-\frac{3}{2}} e^{-\frac{(x-y-\alpha_j t)^2}{4\beta_j t}} (\frac{\partial^2}{\partial k^2}\psi_j(0), h_o(y,\bar{\xi}))\, dy.
\end{aligned}$$

One interesting consequence of Theorem 2 is: if we omit the terms decaying fast in t and consider the full nonlinear equation, then the integrals in ξ of the leading terms in the expansion of h satisfy the Burgers equations. One advantage to use the approximate Green function is that we can get more information in ξ direction when passing from the Boltzmann equation to the Burgers equations.

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