行政院國家科學委員會專題研究計畫　成果報告

高階布氏方程式之非線性特性分析

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高階布氏方程式之非線性特性分析
The Study of Nonlinear Properties of high-order Boussinesq equations

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一、 中文摘要

關鍵詞：*布氏方程式，二階史托克交互作用，非線性參數*

欲瞭解高階布氏方程式的非線性效應，吾人首先將推導四階之布氏方程式。並由二階史托克交互作用分析其超諧和及次諧和轉換係數，選擇適當的水深參數，吾人可獲得最佳化之布氏方程式。

二、英文摘要

Keywords： *Boussinesq equations, second-order Stokes-type Interaction, nonlinear parameters*

The nonlinear characteristics of the $O(\mu^4)$ Boussinesq equations are investigated herein. After deriving the set of high-order Boussinesq equations, the super- and subharmonic transfer coefficients of the wave interaction are analyzed. These coefficients represent the behaviors of the second-order Stokes-type interaction. By comparing the present coefficients to the exact solution, the optimal model can be determined by choosing a suitable water-depth parameter.

三、研究方法及內容

3.1 控制方程式與邊界條件

首先將無因次控制方程式與邊界條件表示如下

$$\mu^2\nabla^2\Phi+\frac{\partial^2\Phi}{\partial z^2}=0 \quad at \quad -h\le z\le\varepsilon\eta \qquad (1)$$

$$\frac{\partial\Phi}{\partial z}=\mu^2\left(\frac{\partial\eta}{\partial t}+\varepsilon\nabla\Phi\cdot\nabla\eta\right) \quad at \quad z=\varepsilon\eta \qquad (2)$$

$$\frac{\partial\Phi}{\partial z}=-\mu^2\left(\nabla\Phi\cdot\nabla h\right) \quad at \quad z=-h \qquad (3)$$

$$\frac{\partial\Phi}{\partial t}+\eta+\frac{\varepsilon}{2}\left[\left(\nabla\Phi\right)^2+\frac{1}{\mu^2}\left(\frac{\partial\Phi}{\partial z}\right)^2\right]=0 \quad at \quad z=\varepsilon\eta \qquad (4)$$

此處$\varepsilon=h_0a_0^{-1}$ 和 $\mu=k_0h_0$為代表非線性及分散性之兩個參數。對於弱非線性條件而言，必須滿足$\varepsilon=\mu^2<1$此關係。將式(1)由$z=-h$積分至$z=\varepsilon\eta$，並考慮式(2)及式(3)之邊界條件，我們可得

$$\nabla\cdot\int_{-h}^{\varepsilon\eta}\nabla\Phi dz+\frac{\partial\eta}{\partial t}=0 \qquad (5)$$

以上即為控制方程式與邊界條件。

3.2 理論解析

令速度勢Φ為

$$\Phi(x,y,z,t)=\sum_{0}^{\infty}\mu^{2n}\Phi_n(x,y,z,t) \qquad (6)$$

將式(6)代入式(1)及式(3)，則可得

$$\Phi_0=\Phi_{00}(x,y,t)$$

$$\Phi_1=\Phi_{10}(x,y,t)-z\nabla\cdot\left(h\nabla\Phi_{00}\right)-\frac{z^2}{2}\nabla^2\Phi_{00}$$

$$\Phi_2 = \Phi_{20}(x,y,t) + z$$
$$\left[-\nabla\cdot(h\nabla\Phi_{10}) - \frac{h^2}{2}\nabla^2\nabla\cdot(h\nabla\Phi_{00}) \right.$$
$$\frac{h^3}{6}\nabla^2\nabla^2\Phi_{00} - h\nabla\nabla\cdot(h\nabla\Phi_{00})\cdot\nabla h$$
$$\left. + \frac{h^2}{2}\nabla\nabla^2\Phi_{00}\cdot\nabla h \right]$$
$$-\frac{z^2}{2}\nabla^2\Phi_{10} + \frac{z^3}{6}\nabla^2\nabla\cdot(h\nabla\Phi_{00})$$
$$+\frac{z^4}{24}\nabla^2\nabla^2\Phi_{00}$$

令 Φ_m 為任意水深處 $z = z_m(x,y)$ 之勢函數，並令 $z = z_m$，可得

$$\Phi_m = \Phi_{00} + \mu^2\left[\Phi_{10} - z_m\nabla\cdot(h\nabla\Phi_{00}) - \frac{1}{2}z_m^2\nabla^2\Phi_{00}\right]$$
$$+\mu^4\left\{\Phi_{20} + z_m\left[-\nabla\cdot(h\nabla\Phi_{10})\right.\right.$$
$$-\frac{h^2}{2}\nabla^2\nabla\cdot(h\nabla\Phi_{00}) + \frac{h^3}{6}\nabla^2\nabla^2\Phi_{00}$$
$$\left. - h\nabla\nabla\cdot(h\nabla\Phi_{00})\cdot\nabla h + \frac{h^2}{2}\nabla\nabla^2\Phi_{00}\cdot\nabla h\right]$$
$$-\frac{z_m^2}{2}\nabla^2\Phi_{10} + \frac{z_m^3}{6}\nabla^2\nabla\cdot(h\nabla\Phi_{00})$$
$$\left. + \frac{z_m^4}{24}\nabla^2\nabla^2\Phi_{00}\right\} + O(\mu^6) \qquad (7)$$

故

$$\left.\begin{aligned} \Phi_{00} &= \Phi_m + \mu^2\left[z_m\nabla\cdot(h\nabla\Phi_m) + \frac{z_m^2}{2}\nabla^2\Phi_m\right] \\ &\quad + O(\mu^4) \\ \Phi_{10} &= O(\mu^2) \\ \Phi_{20} &= O(\mu^4) \end{aligned}\right\} \qquad (8)$$

因此長波方程式可表為

$$\frac{\partial\eta}{\partial t} + \nabla\cdot\left[(h+\varepsilon\eta)\nabla\Phi_m\right]$$
$$+\mu^2\nabla\cdot\left\{h\nabla\left[z_m\nabla\cdot(h\nabla\Phi_m) + \frac{z_m^2}{2}\nabla^2\Phi_m\right]\right\}$$
$$+\mu^2\nabla\cdot\left\{+\frac{h^2}{2}\nabla\nabla\cdot(h\nabla\Phi_m) - \frac{h^3}{6}\nabla\nabla^2\Phi_m\right\}$$
$$+\varepsilon\mu^2\nabla\cdot\left\{\eta\nabla\left[z_m\nabla\cdot(h\nabla\Phi_m) + \frac{z_m^2}{2}\nabla^2\Phi_m\right]\right\}$$
$$+\mu^4\nabla\cdot\left\{-\frac{h^3}{6}\nabla\nabla^2\left[z_m\nabla\cdot(h\nabla\Phi_m) + \frac{z_m^2}{2}\nabla^2\Phi_m\right]\right.$$
$$\left. h\nabla G_1 + \frac{h^2}{2}\nabla G_2 - \frac{h^4}{24}\nabla\nabla^2\nabla\cdot(h\nabla\Phi_m) + \frac{h^5}{120}\nabla\nabla^2\nabla^2\Phi_m\right\} = 0$$
$$(9)$$

及

$$\eta + \frac{\partial\Phi_m}{\partial t}$$
$$+\mu^2\left[z_m\nabla\cdot\left(h\nabla\frac{\partial\Phi_m}{\partial t}\right) + \frac{z_m^2}{2}\nabla^2\left(\frac{\partial\Phi_m}{\partial t}\right)\right] + \frac{\varepsilon}{2}(\nabla\Phi_m)^2$$
$$+\mu^4\left\{z_m G_3 + \frac{z_m^2}{2}\nabla^2\left[z_m\nabla\cdot\left(h\nabla\frac{\partial\Phi_m}{\partial t}\right) + \frac{z_m^2}{2}\nabla^2\left(\frac{\partial\Phi_m}{\partial t}\right)\right]\right.$$
$$\left. -\frac{z_m^3}{6}\nabla^2\nabla\cdot\left(h\nabla\frac{\partial\Phi_m}{\partial t}\right) - \frac{z_m^4}{24}\nabla^2\nabla^2\left(\frac{\partial\Phi_m}{\partial t}\right)\right\}$$
$$+\varepsilon\mu^2\left\{\frac{1}{2}\left[\nabla\cdot(h\Phi_m)\right]^2 - \eta\nabla\cdot\left(h\nabla\frac{\partial\Phi_m}{\partial t}\right)\right\} = 0$$
$$(10)$$

其中 G_1、G_2 及 G_3 為與深度 z 有關的變數。(9)式及(10)式即為以任意水深處之速度勢為變數之長波方程式。現在考慮定水深且流場為二維流場，並令水深參數 $m = \frac{z_m}{h}$ ($-1 \le m \le 0$)，則上二式可改寫為

$$\frac{\partial\eta}{\partial t} + \frac{\partial}{\partial x}\left[(h+\varepsilon\eta)\frac{\partial\Phi_m}{\partial x}\right] + \mu^2 h^3 H_1\frac{\partial^4\Phi_m}{\partial x^4}$$
$$+\mu^4 h^5 H_2\frac{\partial^6\Phi_m}{\partial x^6} + \varepsilon\mu^2 h^2 H_3\frac{\partial}{\partial x}\left[\eta\frac{\partial^3\Phi_m}{\partial x^3}\right] = 0$$
$$(11)$$

$$\frac{\partial\Phi_m}{\partial t} + \eta + \mu^2 h^2 H_3\frac{\partial^3\Phi_m}{\partial x^2\partial t} + \frac{\varepsilon}{2}\left(\frac{\partial\Phi_m}{\partial x}\right)^2$$
$$+\mu^4 h^4 H_4\frac{\partial^5\Phi_m}{\partial x^4\partial t} + \varepsilon\mu^2\left[-h\eta\frac{\partial^3\Phi_m}{\partial x^2\partial t} + \frac{h^2}{2}\left(\frac{\partial^2\Phi_m}{\partial x^2}\right)^2\right] = 0$$

(12)

其中

$$H_1 = \frac{1}{2}m^2 + m + \frac{1}{3} \qquad (13.a)$$

$$H_2 = \frac{5}{24}m^4 + \frac{5}{6}m^3 + \frac{7}{6}m^2 + \frac{2}{3}m + \frac{2}{15} \qquad (13.b)$$

$$H_3 = \frac{1}{2}m^2 + m \qquad (13.c)$$

$$H_4 = \frac{5}{24}m^4 + \frac{5}{6}m^3 + m^2 + \frac{1}{3}m \qquad (13.d)$$

(9)、(10)式及(11)、(12)式即為以任意水深之速度勢為變數之弱非線性長波方程式，在此吾人稱其為「新布氏方程式」。

四、布氏方程式之非線性特性探討

為分析布氏方程式之非線性特性，在此吾人引入二階史托克交互作用。首先，吾人令

$$\eta = \eta_0 + \varepsilon\eta_1 + \varepsilon^2\eta_2 + \dots \qquad (12)$$

$$\Phi_m = \Phi_0 + \varepsilon\Phi_1 + \varepsilon^2\Phi_2 + \dots \qquad (13)$$

吾人將η及Φ代入原布氏方程式，並依ε的冪次排序，可得

$$\frac{\partial \eta_n}{\partial t} + \mathrm{L}_1\tilde{\Phi}_n = -\nabla\cdot F_n \qquad (14)$$

$$\eta_n + \mathrm{L}_2\frac{\partial\tilde{\Phi}_n}{\partial t} = -E_n \qquad (15)$$

其中運算子$\mathrm{L}_1, \mathrm{L}_2$為

$$\mathrm{L}_1 = \left(G_1\nabla^2 + \mu^2 G_2\nabla^2\nabla^2 + \mu^4 G_3\nabla^2\nabla^2\nabla^2\right)$$

$$\mathrm{L}_2 = 1 + \mu^2 H_1\nabla^2 + \mu^4 H_2\nabla^2\nabla^2$$

(14)式及(15)式之右邊項可解得為

$$\begin{cases} F_0 = 0 \\ F_1 = \eta_0\left(\nabla + \mu^2 H_1\nabla\nabla^2 + \mu^4 H_2\nabla\nabla^2\nabla^2\right)\tilde{\Phi}_0 \\ E_0 = 0 \\ E_1 = -\mathrm{L}_{2a}\dfrac{\partial\tilde{\Phi}_0}{\partial t} + \dfrac{1}{2}\left(\nabla \mathrm{L}_2\tilde{\Phi}_0\right)^2 + \\ \mu^2\left[\dfrac{G_1^2}{2}\left(\nabla^2\tilde{\Phi}_0\right)^2\right] + \mu^4\left[G_1G_2\nabla^2\tilde{\Phi}_0\nabla^2\nabla^2\tilde{\Phi}_0\right] \end{cases}$$

令 $\eta_0 = \sum_n a_n\cos\theta_n$, $\tilde{\Phi}_0 = \sum_n b_n\sin\theta_n$ 代入(14)及(15)，可得

$$-\nabla\cdot F_1 = \frac{1}{4}\sum_n\sum_l a_n a_l\left[\mathbf{F}_{nl}^{+}\sin(\theta_n+\theta_l) + \mathbf{F}_{nl}^{-}\sin(\theta_n-\theta_l)\right] \qquad (16)$$

$$\mathbf{F}_{nl}^{\pm} = \frac{\mu_l k_n^2 \pm \mu_n k_l^2 + (\mu_n \pm \mu_l)(\bar{k}_n\cdot\bar{k}_l)}{\mu_n\mu_l} \qquad (17)$$

$$-E_1 = \frac{1}{4}\sum_n\sum_l a_n a_l\left[\mathbf{E}_{nl}^{+}\cos(\theta_n+\theta_l) + \mathbf{E}_{nl}^{-}\cos(\theta_n-\theta_l)\right] \qquad (18)$$

$$\begin{aligned}\mathbf{E}_{nl}^{\pm} = \frac{1}{\mu_n\mu_l\hat{H}_n\hat{H}_l}\Big[&-\left(\bar{k}_n\cdot\bar{k}_l\right)\cdot\left(\hat{H}_n+\hat{H}_l-1\right) \\ &\pm k_n^2k_l^2\left(\mu^2G_1 + \mu^4G_2\left(k_n^2+k_l^2\right)\right) \\ &+ k_n^3k_l^3\left(-\mu^4H_1^2\right) + \mu_n\mu_l\hat{H}_n\hat{G}_l + \mu_n\mu_l\hat{H}_l\hat{G}_n\Big]\end{aligned} \qquad (19)$$

$$\hat{G}_n = \mu^2G_1k_n^2 - \mu^4G_2k_n^4 \qquad (20)$$

故可得η_1為

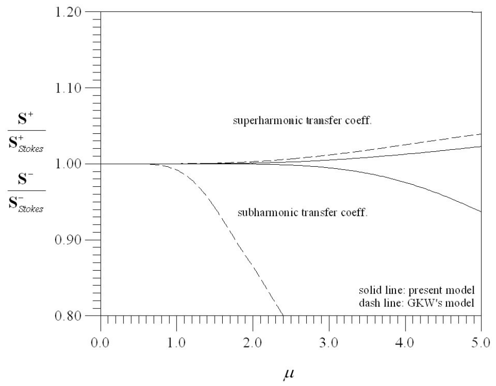
$$\eta_1 = \sum_n\sum_l a_n a_l\left\{\mathbf{S}_{nl}^{+}\cos(\theta_n+\theta_l) + \mathbf{S}_{nl}^{-}\cos(\theta_n-\theta_l)\right\} \qquad (21)$$

其中$\mathbf{S}_{nl}^{+}$及$\mathbf{S}_{nl}^{-}$分別稱為超諧和及次諧和轉換係數。

五、結論與討論

為探討上述兩係數的效應，吾人令

$\mu_n = \mu_l$，$k_n = k_l$。由於選擇不同的水深參數 m 值將會造成不同的結果，我們選定水深範圍為 0 < « < 5，此時的最佳水深參數為 −0.328。圖一表示 超諧和及次諧和轉換係數與正確解之間的關係。和 GKW 於 2000 年的研究成果比較，可發現吾人的成果優於 GKW 的成果，此原因在於 GKW 的最佳化方程式是使用帕德近似所推得，此基本假設並不足以使其非線性效應達到最精確的目標。在爾後的研究中，吾人將試圖將可應用之水深範圍擴大，以加強本模式的適用性。



圖一　超諧和及次諧和轉換係數

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