VII. Conclusion

We have derived a density function for an NF-based system. The function is derived for a transformed, smooth vector field that enjoys the navigation properties of the original NF vector field. Under some assumptions, the convergence results derived on the transformed vector field are propagated to the original. This result will enable exploitation of several features of dual Lyapunov techniques to robotic navigation. Initial results from applying this approach to robotic navigation are reported in [13]. Further research includes finding density functions that are directly applicable to the primary navigation system.

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Output Feedback Control of Bilinear Systems via a Bilinear LTR Observer

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***Abstract*—In the literature, most observer-based output feedback controls for bilinear systems are only applicable to open-loop (neutrally) stable systems. This paper proposes a new observer-based output feedback control that can be applied to open-loop unstable systems. The key component of the new control is an exponentially stable *bilinear* loop transfer recovery (LTR) observer that derives from the linear LTR observer.**

***Index Terms*—Bilinear observer, bilinear system, dyadic bilinear system, loop transfer recovery (LTR) observer, output feedback control.**

I. Introduction

Bilinear systems exist in many physical phenomena that are of considerable interest to human activities [1], [2]. Recent applications of bilinear system control include heating, air conditioning control [3], power converter control [4], electromagnetic actuator control [5], and quantum system control using finite-dimensional bilinear models [6] or infinite-dimensional bilinear models [7], [8]. Even though a variety of control designs have been developed for bilinear systems, most of them are based on state feedback [9]–[15]. If only part of the state variables are accessible for measurement, one has to resort to output feedback control. Unfortunately, most output feedback controls in the literature require that the open-loop bilinear system be stable [16], neutrally stable [17] or dissipative [18]. The reason for requiring this open-loop stable condition is that they all assume the stabilizing control signal be of small magnitude so that their bilinear observer designs can be successful. There are a few bilinear observer designs proposed in the literature for the open-loop unstable bilinear system without imposing the small control condition. For example, an open-loop dead-beat observer for state estimation of open-loop unstable bilinear systems is suggested in [19], but the system must satisfy the existence condition of a control Lyapunov function [20]. In [21] and [22], bilinear observers can be constructed with the state estimation error converging independent of the control input under a set of system matrix equalities.

This paper proposes a new output feedback control for unstable bilinear systems. The key element is a bilinear loop transfer recovery (LTR) observer that derives from the linear LTR observer [23]. The new bilinear LTR observer is exponentially stable without imposing the small control condition, the existence of a control Lyapunov function, or extra matrix equalities on the system matrices. Hence, it relaxes the stringent conditions imposed by previous bilinear observer designs. Then, by combining this new bilinear LTR observer with the state feedback division control in [15], one obtains a stabilizing output feedback control for bilinear systems that may be open-loop unstable.

The remainder of this paper is arranged as follows. Section II introduces the new bilinear LTR observer. Section III presents the observer-based output feedback control, and its stability analysis. Section IV concludes the paper.

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II. BILINEAR LTR OBSERVER

Consider a dyadic bilinear system with a multiplicative control input

$$\dot{x} = Ax + bzu, \quad z = hx$$
$$y = cx \tag{1}$$

where the system state $x \in R^n$ and $z \in R^1$ are not accessible for measurement, and the only accessible signals are the control input $u \in R^1$ and the system output $y \in R^1$. All system matrices $A \in R^{n\times n}$, $b \in R^n$, $h^T \in R^n$, $c^T \in R^n$ are known, and the open-loop system matrix A is allowed to be unstable. Assume that the bilinear system (1) satisfies the following conditions:

A1: the bilinear system is observable in the sense that (A, c) is an observable pair;

A2: the bilinear system is controllable in the sense [24] that (A, b) is controllable and (A, h) observable;

A3: the system $(A + I, b, c)$ is minimum-phase (has only stable zeros).

One now proposes an observer design for the bilinear system (1) to estimate its state x from the input u and output y. If the control u of the system is uniformly bounded

$$|u| \le U \tag{2}$$

where U may be arbitrarily large, the following bilinear LTR observer is suggested

$$\dot{\hat{x}} = A\hat{x} + bh\hat{x}u + L(y - c\hat{x}) \tag{3}$$

in which the output injection gain L is designed as in the linear LTR observer [23]

$$L = \frac{1}{\mu}Qc^T, \qquad \mu = 1$$
$$Q(A+I)^T + (A+I)Q - Q\frac{c^Tc}{\mu}Q + \pi bb^T = 0$$
$$\pi(>0) \text{ sufficiently large.} \tag{4}$$

From Assumptions A1 and A2, (A, c) is observable, and (A, b) controllable. Hence, the solution $Q \in R^{n\times n}$ of the aforesaid Riccati equation is positive definite [25]. Furthermore, the relationship between the solution $Q(\pi)$ and the design parameter π satisfies the following relationship, which is a well-known result in the study of the linear LTR observer.

Theorem 1 [25]: Under the Assumption A3, the solution $Q(\pi)$ of the observer Riccati equation (4) satisfies

$$\lim_{\pi\to\infty} \frac{Q(\pi)}{\pi} = 0.$$

With Theorem 1, one can now prove the stability of the proposed bilinear LTR observer.

Theorem 2: Consider the bilinear system (1), which satisfies a control upper bound in (2). Given whatever large control bound U in (2), there always exists a sufficiently large design parameter $\pi > 0$ in the Riccati equation (4) so that the state estimate $\hat{x}$ of the bilinear LTR observer (3) approaches the true state x exponentially.

Proof: The state estimation error $\tilde{x} = x - \hat{x}$ resulting from the bilinear LTR observer satisfies

$$\dot{\tilde{x}} = (A - Lc)\tilde{x} + bh\tilde{x}u. \tag{5}$$

The goal is to prove that even for large U in (2), the error dynamics (5) is globally exponentially stable if the Riccati design parameter π in (4) is sufficiently large.

Define a Lyapunov function $W = \tilde{x}^TQ^{-1}\tilde{x}$, where $Q > 0$ is from the observer Riccati equation (4), and $\tilde{x}$ from (5). The change rate of W along the trajectory (5) satisfies

$$\begin{aligned}\dot{W} &\le -2W - \frac{1}{\mu}\|\tilde{y}\|^2 - \pi\|b^TQ^{-1}\tilde{x}\|^2 \\ &\quad + 2\|\tilde{x}\|\cdot\|h\|\cdot|u|\cdot\|b^TQ^{-1}\tilde{x}\| \\ &\le -2W - \pi\|b^TQ^{-1}\tilde{x}\|^2 \\ &\quad + 2U\|h\|\cdot\|\tilde{x}\|\cdot\|b^TQ^{-1}\tilde{x}\|\end{aligned}$$

where $\tilde{y} = c\tilde{x}$. Note that the maximum of the last two terms in the aforemnentioned equation occurs when $\|b^TQ^{-1}\tilde{x}\| = U\|h\|\cdot\|\tilde{x}\|/\pi$, with the maximum value being $(U\|h\|\cdot\|\tilde{x}\|)^2/\pi$. Hence

$$\begin{aligned}\dot{W} &\le -2W + \frac{(U\|h\|\cdot\|\tilde{x}\|)^2}{\pi} \\ &\le -\left[2 - (U\|h\|)^2\frac{\bar{\sigma}(Q)}{\pi}\right]W\end{aligned} \tag{6}$$

where the second inequality is derived using $W \ge \underline{\sigma}(Q^{-1})\|\tilde{x}\|^2 = 1/\bar{\sigma}(Q)\|\tilde{x}\|^2$. According to Theorem 1, $\bar{\sigma}(Q)/\pi$ approaches zero as the observer design parameter π approaches infinity. Hence, given whatever large constant $U\|h\|$, there always exists a sufficiently large design parameter π such that the number in the square bracket in (6) is positive. This implies that $W(t)$, and hence, $\tilde{x}(t)$ decay to zero exponentially. ■

Remark 1: Most previous bilinear observers restrict that the open-loop system be stable or the control upper bound U be small, while the proposed bilinear LTR observer has no such constraints. However, if the observer design parameter π is chosen too large, the proposed observer becomes high-gain. A disadvantage of high-gain observers is the peaking phenomenon [26], in which the estimated state $\hat{x}(t)$ peaks to extremely large values during the very initial period of the observation process. One way to relieve the peaking phenomenon is to schedule the design parameter according to $\pi(t) = \pi_i, t \in [t_k, t_{k+1})$, where π_i stepwisely jumps from 1 to the designed large value.

III. OBSERVER-BASED STATE FEEDBACK CONTROL

In the literature, a state feedback "division control" [15] has been proposed to exponentially stabilize the bilinear system (1) when the system state x is accessible for measurement. Their division control is stabilizing even for an open-loop unstable bilinear system, and is given by

$$u_x = -\frac{z}{z^2 + \epsilon^2\|x\|^2}kx \tag{7}$$

where the state feedback gain k is chosen to stabilize the system matrix $A - bk$, and ϵ is a sufficiently small parameter. The resultant exponentially stable closed-loop system is as

$$\begin{aligned}\dot{x} = f(x), \quad f(x) &= Ax - bhxu_x \\ &= Ax - bkx\frac{(hx)^2}{(hx)^2 + \epsilon^2\|x\|^2}.\end{aligned} \tag{8}$$

Since the aforementioned state feedback control system (8) is exponentially stable, one can quote the converse Lyapunov stability theorem to claim the following.

Theorem 3 [27]: Regarding the exponentially stable system $\dot{x} = f(x)$ in (8), there exists a Lypunov function $V(x)$ and positive constants

α_i such that for all $x \in R^n$

$$\alpha_1 \|x\|^2 \le V(x) \le \alpha_1 \|x\|^2 \quad (9)$$

$$\frac{d}{dt}V(x) = \frac{\partial V(x)}{\partial x} f(x) \le -\alpha_3 \|x\|^2 \quad (10)$$

$$\left| \frac{\partial V(x)}{\partial x} \right| \le \alpha_4 \|x\|. \quad (11)$$

When the system state x and $z = hx$ are not accessible for measurement, one can combine the bilinear LTR observer (3) in the previous section with the aforementioned division control (7). The resultant observer-based state feedback control is given by

$$u_{\hat{x}} = -\frac{\hat{z}}{\hat{z}^2 + \epsilon^2 \|\hat{x}\|^2} k\hat{x}, \qquad \hat{z} = h\hat{x} \quad (12)$$

where $\hat{x}$ is the estimate of x from the bilinear LTR observer (3).

Lemma 4: Both the state feedback control u_x in (7) and the observer-based state feedback control $u_{\hat{x}}$ in (12) are uniformly bounded independently of the boundedness of x and $\hat{x}$.

Proof : First, one will show that the observer-based control $u_{\hat{x}}$ is uniform bounded independently of the boundedness of $\hat{x}$. Using the inequality $|k\hat{x}| \le \|k\| \cdot \|\hat{x}\|$ and dividing (12) by $\|\hat{x}\|^2$, one can obtain the following inequality

$$|u_{\hat{x}}| \le \frac{s}{s^2 + \epsilon^2} \|k\| \overset{\text{def}}{=} g(s), \quad s = \frac{\|\hat{z}\|}{\|\hat{x}\|} \in [0, \infty).$$

A simple calculation shows that the maximum value of $g(s)$ for s ranging from zero to infinity is $\|k\|/2\epsilon$. Hence, the observer-based control $u_{\hat{x}}$ is uniformly bounded

$$|u_{\hat{x}}| \le \frac{\|k\|}{2\epsilon} \quad \forall \hat{x} \in R^n. \quad (13)$$

In a similar way as one derives (13), one can also show that the state feedback control u_x in (7) is uniformly bounded independently of the boundedness of x

$$|u_x| \le \frac{\|k\|}{2\epsilon} \quad \forall x \in R^n. \quad (14)$$

■

According to Lemma 4, the proposed observer-based control (12) satisfies the uniform upper bound condition (2) required in Theorem 2. One can, therefore, legally quote Theorem 2 to conclude that $\hat{x}$ approaches x exponentially fast. In other words, exponential stability of the proposed bilinear LTR observer can be concluded before the stability analysis of the controlled closed-loop system.

Corollary 5: If the observer design parameter π in (4) is sufficiently large, the bilinear LTR observer (3) for the bilinear system (1) under the proposed observer-based control (12) is exponentially stable; hence, there exist positive constants K and γ such that

$$\|\tilde{x}(t)\| \le K e^{-\gamma t} \|\tilde{x}(0)\| \to 0, \quad \text{as} \quad t \to \infty. \quad (15)$$

The stability analysis of the proposed controlled system will proceed as follows.

Theorem 6: Consider the bilinear system (1) under the observer-based state feedback control (12). The controlled system state will not explode to infinity in finite time, nor will it decay to zero in finite time.

Proof : One can check, using (1) and (13), that the system state under the observer-based state feedback control (12) satisfies $\|\dot{x}\| \le M\|x\|$ for some bounded constant $M > 0$. In other words, the closed-loop system equation is Lipschitz. By quoting [27, Proposition 1.4.1], one concludes that

$$0 < \|x(0)\| e^{-Mt} \le \|x(t)\| \le \|x(0)\| e^{Mt} < \infty.$$

The inequality on the left of $\|x(t)\|$ shows that $x(t)$ cannot decay to zero in finite time, and the inequality on the right shows that $x(t)$ cannot explode to infinity in finite time. ■

Theorem 7: Consider the bilinear system (1) under the observer-based state feedback control (12). If the observer design parameter π in (4) is sufficiently large, the proposed control stabilizes the system (1) in the sense that the system state x will converge asymptotically to the origin.

Proof: Define an arbitrarily small neighborhood around the origin

$$V_\eta = \{x | V(x) \le \eta\}$$

where $V(x)$ is the Lyapunov function in Theorem 3, and $\eta > 0$ an arbitrarily small number. Also, define a critical time constant T^*

$$T^* = \frac{1}{\gamma} \ln \left(\frac{2\alpha_4 \beta m K \|\tilde{x}(0)\|}{\alpha_3 \sqrt{\eta/\alpha_1}} \right) \quad (16)$$

where α_i are as in Theorem 3, K, γ, and $\tilde{x}(0)$ are as in Corollary 5, $\beta = \|b\| \cdot \|h\|$, and m is the upper bound in (18), as shown next. According to Theorem 6, the controlled system state x can grow at most exponentially fast; therefore, given any bounded initial condition $x(0)$, $x(T^*)$ is also bounded. It will now be shown that starting from the bounded $x(T^*)$, $x(t)$ will enter V_η within a finite time, say at $t = T^* + T_1$, and $x(t)$ will never exit V_η again. Since η is arbitrarily small, this result is equivalent to asymptotic stability of the controlled system.

Define the normalized state $e_x = x/\|x\|$, and normalized estimated state $e_{\hat{x}} = \hat{x}/\|\hat{x}\|$. It is shown in the Appendix that they satisfy

$$\|e_x - e_{\hat{x}}\| \le \frac{2}{\|x\|} \|x - \hat{x}\|. \quad (17)$$

Also, notice that the observer-based control (12) and the state feedback control (7) can be expressed as the same function of the (normalized) state

$$u_{\hat{x}} = q(e_{\hat{x}}), \quad \text{and} \quad u_x = q(e_x), \quad \text{where} \quad q(e) = \frac{he \cdot ke}{|he|^2 + \epsilon^2}.$$

It is easy to see that $q(e)$ has a bounded derivative for all bounded e

$$\left\| \frac{\partial q(e)}{\partial e} \right\| \le m < \infty \quad \forall \|e\| = 1. \quad (18)$$

Using the mean value theorem [28] and the inequalities (17), (18) yields

$$|\Delta(t)| \overset{\text{def}}{=} |u_{\hat{x}} - u_x| = |q(e_{\hat{x}}) - q(e_x)| = \left\| \frac{\partial q(e)}{\partial e} \right\| \cdot \|e_{\hat{x}} - e_x\| \le \frac{2m}{\|x\|} \|\hat{x} - x\|. \quad (19)$$

One will now use a contradiction argument to show that starting from the bounded $x(T^*)$, $x(t)$ will enter the small neighborhood V_η within a finite time at $T^* + T_1$. Assume the contrary; that is, x is always outside of V_η for all $t \ge T^*$. As a result of this assumption and (9) in Theorem 3, one has $\|x(t)\| > \sqrt{\eta/\alpha_1}$ for all $t \ge T^*$. Then, using (15) in Corollary 5 and (19), one concludes that

$$|\Delta(t)| \le \frac{2m}{\sqrt{\eta/\alpha_1}} \|\hat{x} - x\| \to 0, \quad as \quad t \to \infty. \quad (20)$$

Now check the time derivative of $V(x)$ along the trajectory under the proposed observer-based control $u_{\hat{x}}$

$$\begin{aligned}\frac{d}{dt}V(x) &= \frac{\partial V(x)}{\partial x}(Ax - bhxu_{\hat{x}} \pm bhxu_x)\\ &= \frac{\partial V(x)}{\partial x}f(x) + \frac{\partial V(x)}{\partial x}bhx\Delta(t)\\ &\leq (-\alpha_3 + \alpha_4\beta|\Delta(t)|)\cdot\|x\|^2\\ &\leq \left(-\frac{\alpha_3}{\alpha_1} + \frac{\alpha_4\beta}{\alpha_1}|\Delta(t)|\right)V(x) \qquad (21)\end{aligned}$$

where $f(x)$ is as defined in (8), $\beta = \|b\|\cdot\|h\|$, and the inequalities in Theorem 3 have been used to derive the last two inequalities. The last inequality (21) and equation (20) imply that V will eventually decay exponentially to values smaller than η, contradicting the earlier assumption that x will always stay outside of V_η ($V(x) > \eta$). Hence, one concludes that if initially $x(T^*)$ is outside of V_η, $x(t)$ must enter V_η within a finite time at $T^* + T_1$ for some finite T_1.

Now, it remains to show that once $x(t)$ enters V_η at $T^* + T_1$, it will never exit V_η again. To show this, notice that whenever x intends to exit V_η, it must go across the boundary of V_η, where $V(x) = \eta$. One will now check the change rate of $V(x)$ at the boundary of V_η. At the boundary, one has $V(x) = \eta$ and hence, $\|x\| \geq \sqrt{\eta/\alpha_1}$, according to (9) in Theorem 3. Substituting $\|x\| \geq \sqrt{\eta/\alpha_1}$, (15), and (19) into (21) leads to

$$\frac{d}{dt}V(x) \leq \left(-\frac{\alpha_3}{\alpha_1} + \frac{\alpha_4\beta}{\alpha_1}\frac{2mK}{\sqrt{\eta/\alpha_1}}\|\tilde{x}(0)\|e^{-\gamma t}\right)V(x). \qquad (22)$$

From this inequality, it is not difficult to see that the change rate of $V(x)$ at the boundary of V_η is always negative *if* $t \geq T^*$; meaning that x can never exit V_η at $t \geq T^*$. Since we have already shown that x enters V_η at $T^* + T_1$, we conclude that x will stay within V_η thereafter. ■

Remark 2: In the analysis of Theorem 7, there is no telling where the control signal will converge asymptotically. However, one has proved in (13) that the proposed control signal $u_{\hat{x}} = q(e_{\hat{x}})$ is uniformly bounded. One can further show that the time derivative of $u_{\hat{x}} = q(e_{\hat{x}})$

$$\frac{d}{dt}u_{\hat{x}} = \frac{\partial q(e_{\hat{x}})}{\partial e_{\hat{x}}}\left(I - e_{\hat{x}}e_{\hat{x}}^T\right)(A\hat{x} + bh\hat{x}u_{\hat{x}} + L(y - c\hat{x}))$$

is also uniformly bounded due to (18) and boundedness of $x, \hat{x}, u_{\hat{x}}$. In other words, the control signal is *smooth* to some extent so that it can be implemented without difficulty on physical actuators.

Remark 3: In proving Theorem 7, one implicitly assumes in the definition of $e_x = x/\|x\|$ that $x \neq 0$ for any finite time. This assumption is supported by Theorem 6 which states that x will not decay to zero in finite time. Also, in Corollary 5, it has been shown that $\hat{x}$ converges to x with a specific exponential rate independently of the stability of x. In case when the system state x converges to zero faster than $\hat{x}$ converges to x, then the stability proof of x is trivially done without referring to the arguments in Theorem 7. In the other case when x has not converged to zero before $\hat{x}$ converges to x, one then applies the argument in Theorem 7 to conclude the stability of x. In this case, after $\hat{x}$ has converged to $x (\neq 0)$, one can say that $\hat{x} \neq 0$ in the definition of $e_{\hat{x}} = \hat{x}/\|\hat{x}\|$.

APPENDIX

Define state estimation error $\tilde{x} = x - \hat{x}$. It is easy to show that

$$e_x - e_{\hat{x}} = \frac{\hat{x}(\|\hat{x}\| - \|\hat{x} + \tilde{x}\|) + \tilde{x}\|\hat{x}\|}{\|\hat{x}\|\cdot\|\hat{x} + \tilde{x}\|}.$$

Using the inequality $\|\vec{a}\| - \|\vec{b}\| \leq \|\vec{a} - \vec{b}\| = \|\vec{b} - \vec{a}\|$ for any two vectors $\vec{a}$ and $\vec{b}$, one can show that

$$|\|\hat{x}\| - \|\hat{x} + \tilde{x}\|| \leq \|\tilde{x}\|.$$

Combining the aforementioned two equations, one obtains

$$\|e_x - e_{\hat{x}}\| \leq \frac{2\|\hat{x}\|\cdot\|\tilde{x}\|}{\|\hat{x}\|\cdot\|\hat{x} + \tilde{x}\|} = \frac{2\|\tilde{x}\|}{\|\hat{x} + \tilde{x}\|} = \frac{2}{\|x\|}\|x - \hat{x}\|.$$

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On Linear-Quadratic Stackelberg Games With Time Preference Rates

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***Abstract*—This note deals with linear-quadratic Stackelberg differential games including time preference rates with an open-loop information structure. The properties of the characteristic matrix associated with the necessary conditions for a Stackelberg strategy are pointed out. It is shown that such a matrix exhibits a special symmetry property of its eigenvalues. Sufficient conditions to guarantee a predefined degree of stability are given based on the distribution of the eigenvalues in the complex plane.**

***Index Terms*—Game theory, Hamiltonian matrix, Riccati equation, α-stability, Stackelberg strategy, time preference rate.**

I. Introduction

The Stackelberg strategies are an elegant concept for dealing with hierarchical differential games [1], [2]. In the framework of an open-loop information structure [3], the necessary conditions are well known and could be obtained explicitly within the context of linear-quadratic problems [1], [2]. Nevertheless, it seems that an explicit solution, coping with differential games with criteria including time preference rates, does not exist.

It was proved in [4] that the linear-quadratic optimal control problem with a single criterion including a constant time preference rate α could be restated as a standard one with a shift of the eigenvalues of the drift matrix by α. The reformulation uses a change of variable, which is closely connected with asymptotic stability of degree α.

Besides, a criterion with a time preference rate is quite frequent especially in economic applications of game theory (see [5], [6] for more details) and are recognized as the discount rate associated with the cost functionals. In order to emphasize the fact that each player has its own objective, the time preference rates are not necessarily identical [8].

When there is no time preference rate, the necessary conditions for obtaining an open-loop Stackelberg equilibrium are characterized by a Hamiltonian matrix (see [9]–[11] for an overview). This leads to a symmetry of the eigenvalues with respect to the origin of the complex plane. However, for the general case where time preference rates are different and not null, this property does not hold. The main contribution of this note is to consider such a general case. Two points are examined. First, the eigenvalues distribution of the characteristic matrix associated with an open-loop Stackelberg strategy applied on the differential game is studied. Second, it is shown that a predefined degree of stability could be imposed to the controlled system.

The note is organized as follows. In Section II, the Stackelberg strategy with an open-loop information structure is recalled and the associated necessary conditions are derived. The cases of finite and infinite time horizon are considered. The characteristic matrix and the corresponding coupled Riccati equations are presented. A nontrivial symmetry for the eigenvalues is described in Section III. Sufficient conditions for a strict α-stability are provided in the same section, followed by an interpretation in terms of game theory. An example illustrates the main result. Some concluding remarks make up Section IV.

II. Stackelberg Strategy

A. Problem Statement

Consider a two-players linear-quadratic differential game on a finite time horizon, defined by

$$\dot{x}(t) = Ax(t) + B_1 u_1(t) + B_2 u_2(t), \qquad x(t_0) = x_0 \tag{1}$$

where $x \in \mathbb{R}^n$, $u_i \in \mathcal{U}_{ad,i} \subset \mathbb{R}^{r_i}$ ($i \in \{1,2\}$ and n, $r_i \in \mathbb{N}$, $\mathcal{U}_{ad,i}$ is the admissible set of the controls u_i) and with the cost functionals J_i ($i \in \{1,2\}$) including a time preference rate α_i

$$J_i = \frac{1}{2} x_f^T e^{2\alpha_i t_f} K_{if} x_f + \frac{1}{2}\int_{t_0}^{t_f} e^{2\alpha_i t}\left(x^T Q_i x + u_1^T R_{i1} u_1 + u_2^T R_{i2} u_2\right) \mathrm{d}\tau \tag{2}$$

where $x_f = x(t_f)$. All weighting matrices are constant and symmetric with $Q_i = C_i^T C_i \geq 0$, $K_{if} \geq 0$, $R_{ij} \geq 0$ ($i \neq j$), and $R_{ii} > 0$. The matrices C_i are of full rank $C_i \in \mathbb{R}^{m_i \times n}$.

Stackelberg strategy with an open-loop information structure is applied for the differential game (1)–(2). Player 2 is assumed to be the leader while player 1 is the follower. The hierarchy in the game comes from the fact that the leader knows the rational reaction of the follower and reveals first his/her strategy. The follower does not know the rational reaction of the leader and must optimize his/her criterion J_1 for a given control $u_2^*(t)$ of the leader. Define the rational reaction set of the follower $\mathcal{R}_1(u)$

$$\left\{\tilde{u}_1 \mid J_1(\tilde{u}_1, u) \leq J_1(u_1, u), \forall u_1 \in \mathcal{U}_{ad,1}\right\}. \tag{3}$$

For a differential game with an open-loop information structure [12], i.e., the players are committed to follow a predetermined strategy or no

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