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Point Stabilization Control of a Car-Like Mobil Robot in Hierarchical Skew Symmetry Chained Form

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Abstract - *Nonholonomic properties most commonly arise in mechanical systems where the non-integrable constraints are imposed on the motions, i.e. the constraints cannot be written as the time derivatives of some function of the generalized coordinates. Classical examples of nonholonomic control systems include sledges or knife-edge systems that slide on the plane, simple wheels rolling without slipping on a plane and spheres rolling without slipping on a plane. In this paper, we consider a hierarchical controller to point stabilization problem for the so-called skew symmetry chained form nonholonomic car-like mobile robot. Based on the diffeomorphic input-state transformations, we introduce a set of sufficient conditions for determining if a nonlinear kinematic model can be converted to a skew symmetry chained form. Next, a hierarchal controller is developed to incorporate the kinematic skew symmetric form into the dynamic simplified model to achieve global asymptotically stable. In hierarchical controller, we use the adaptive control features to overcome the uncertain dynamic parameters and use the sliding mode techniques to attenuate the effect of external disturbances. Finally, the efficacy of the proposed point stabilization algorithm is illustrated with car-like mobile robots. Simulation results are utilized to illustrate the effectiveness of the proposed control algorithm.*

Keywords: Nonholonomic Systems, Skew Symmetry Chained Form, Kinematic Model, Dynamic Model, Diffeomorphism, Car-Like Mobile Robot.

1 Introduction

Since 1990s, nonholonomic mechanical systems attracted a number of researches in this area. As we know, much research effort has been oriented to solving the problem of motion under nonholonomic constraints using the kinematic model of the nonholonomic mechanical systems. Murray *et al.* [1] introduced the Frobenius theory by using the technique of state-input linearization based on differential geometry. The coordinate transformation of diffeomorphism was constructed to transform a nonlinear equation of motion into a quasi-linear chained form, which significantly reduced the complexity of designing the controller, such as chained form [1], Caplygin form [2], and skew-symmetry chained form [3]. In 1995, Samson presented an interesting model called skew symmetry chained form. Besides the simplicity, its structure showed excellent characteristics in the Lyapunov stability analysis and simplify the control structure further. By Brockett's necessary stability conditions [4], a nonholonomic system with underactuated property can not be asymptotically stabilized to a desired configuration using any smooth time invariant state feedback. There are two strategies to stabilize such system based on the kinematic model. One is the time varying continuous feedback control law [5, 6] and the other one is the discontinuous control law [7, 8]. In this paper, the nonlinear kinematic model is transformed into a skew symmetric chained form which is particularly suitable to the Lyapunov synthesis design. Based on the advantages of the skew symmetric form, the controller has simple structure and is easy to realize. However, the kinematic model usually ignores the mass and inertia of the physical systems. The control inputs of the kinematics have no concern with actual applied force or torque, even more without any physical significance. In practical viewpoint, it will be more realistic to formulate the nonholonomic system by dynamic model, where the control inputs are actual torque or force. The development of the dynamic model is in the field of analytic dynamics. Fierro and Lewis [9] developed a combined kinematic /torque control law using backstepping, and induced a neural network computed-torque controller for stabilizing a nonholonomic mobile robot where exists the system uncertainty. In [10] and [11], the controller was designed based on the skew symmetric model and the system was extended to the dynamic equations with uncertain parameters. This article presented the architecture of hierarchical models and categorized the controller design in two levels. On the dynamic level, a sliding mode controller is developed, which has both adaptive and robust features, to cope with the uncertainties, disturbances and modeling errors by the dynamic model. On the kinematic level, a time varying feedback controller under the structure of skew symmetric chained form is developed to overcome the problem that control inputs is less than the degrees of freedom of the systems. This paper is organized as follows. We start with some preliminary results of nonholonomic systems and give a set of sufficient conditions for determining whether or not a nonlinear kinematic model can be converted to a skew symmetry chained form in Section 2. In Section 3, we establish the kinematics and dynamics of the car-like mobile robot. In Section 4, we use the skew symmetric chained form to design the kinematic level controller. The adaptive sliding mode controller is considered in the dynamic level. Simulation examples are presented in Section 5. Finally, we give some concluding remarks in Section 6.

2 Preliminary

2.1 Dynamics of Nonholonomic System

Consider the dynamics of a nonholonomic mechanical system in the following form:

$$M(\mathbf{q})\ddot{\mathbf{q}}+C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}+G(\mathbf{q})+\boldsymbol{\tau}_d=B(\mathbf{q})\boldsymbol{\tau}+A^T(\mathbf{q})\boldsymbol{\lambda}, \quad (1)$$

where $\mathbf{q}=[q_1,\ldots,q_n]^T$ is a n-dimensional generalized coordinate, $M(\mathbf{q})$ is a $n\times n$ positive definite symmetric inertia matrix, $C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}$ is a n-vector of centripetal and coriolis torques, $G(\mathbf{q})$ denotes the n-vector of gravitational torques, $B(\mathbf{q})$ presents the $n\times r$ input transformation matrix, $A(\mathbf{q})$ is a $(n-m)\times n$ full rank matrix, which is associated with the constraints. $\boldsymbol{\lambda}$ is $(n-m)$ vector Lagrange multiplier, $\boldsymbol{\tau}$ denotes the r-vector control input. For a nonholonomic mechanical system, we have the skew symmetric matrix $\dot{M}(\mathbf{q})-2C(\mathbf{q},\dot{\mathbf{q}})$.

2.2 Kinematics of Nonholonomic System

Consider the catastatic pfaffian constraints in the following form: $A(\mathbf{q})\dot{\mathbf{q}}=0$, (2)

Assume a set of smooth linearly independent vector fields spanning the null space of $A(\mathbf{q})$ such that $A(\mathbf{q})\Omega(\mathbf{q})=0$, where $\Omega(\mathbf{q})=[\mathbf{w}_1(\mathbf{q}),\mathbf{w}_2(\mathbf{q}),\ldots,\mathbf{w}_m(\mathbf{q})]$. Then, there exists an m-dimensional velocity vector $\mathbf{v}=[v_1,v_2,\ldots,v_m]^T$, so that the kinematic equation can be expressed as

$$\dot{\mathbf{q}}=\Omega(\mathbf{q})\mathbf{v}=\mathbf{w}_1(\mathbf{q})v_1+\mathbf{w}_2(\mathbf{q})v_2+\cdots+\mathbf{w}_m(\mathbf{q})v_m, \quad (3)$$

where $\mathbf{v}\in R^m$ is the privileged velocity vector.

Differentiating equation (3), yields

$$\ddot{\mathbf{q}}=\dot{\Omega}(\mathbf{q})\mathbf{v}+\Omega(\mathbf{q})\dot{\mathbf{v}}, \quad (4)$$

Substituting (4) into equation (1), and multiplied by $\Omega^T(\mathbf{q})$ on both sides, we have a simplified result:

$$M_1(\mathbf{q})\dot{\mathbf{v}}+C_1(\mathbf{q},\dot{\mathbf{q}})\mathbf{v}+G_1(\mathbf{q})+\boldsymbol{\tau}_d=B_1(\mathbf{q})\boldsymbol{\tau} \quad (5)$$

where $M_1(\mathbf{q})=\Omega^T(\mathbf{q})M(\mathbf{q})\Omega(\mathbf{q})$ is a $m\times m$ positive definite symmetric inertia matrix and

$$\begin{aligned}&C_1(\mathbf{q},\dot{\mathbf{q}})=\Omega^T(\mathbf{q})M(\mathbf{q})\dot{\Omega}(\mathbf{q})+\Omega^T(\mathbf{q})C(\mathbf{q},\dot{\mathbf{q}})\Omega(\mathbf{q})\\&G_1(\mathbf{q})=\Omega^T(\mathbf{q})G(\mathbf{q})\\&B_1(\mathbf{q})=\Omega^T(\mathbf{q})B(\mathbf{q})\end{aligned} \quad (6)$$

We do not only eliminate constraint force but also take state transformation to a simplified dynamics (5), which is more appropriate for backstepping controller design. Accordingly, we integrate the kinematic model (3) with the dynamic model (5) to form one hierarchical model, which takes several intrinsic features such as the mass and the inertia of the mobile robot into account. Besides, the control input $\boldsymbol{\tau}$ of dynamics (5) signifies the actual applied force or torque.

2.3 Converting Kinematics to Chained Form

Based on the Frobenius theorem, using input - state linearization techniques can construct a diffeomorphic coordinate transformation under the condition that assures a nonlinear kinematics can be converted to chained forms of the quasi-linear system. Consider a single chain model with two inputs; a constructive procedure was given in [1]. It is given here for completeness.

[Theorem 1] Consider a two inputs nonlinear kinematics with driftless as follows.

$$\dot{\mathbf{q}}=\mathbf{w}_1(\mathbf{q})v_1+\mathbf{w}_2(\mathbf{q})v_2, \quad (7)$$

Define the distribution as

$$\Lambda_0=span\{\mathbf{w}_1,\mathbf{w}_2,ad_{\mathbf{w}_1}\mathbf{w}_2,ad_{\mathbf{w}_1}^2\mathbf{w}_2,\cdots,ad_{\mathbf{w}_1}^{n-2}\mathbf{w}_2\} \quad (8)$$

$$\Lambda_1=span\{\mathbf{w}_2,ad_{\mathbf{w}_1}\mathbf{w}_2,ad_{\mathbf{w}_1}^2\mathbf{w}_2,\cdots,ad_{\mathbf{w}_1}^{n-3}\mathbf{w}_2\} \quad (9)$$

$\mathbf{w}_1(\mathbf{q})$, $\mathbf{w}_2(\mathbf{q})$ being smooth, if and only if there exists a open set $\Omega\in\mathcal{R}^n$ such that the following conditions hold:

(N1) $\Lambda_0(\mathbf{q})=\mathcal{R}^n$ for all $\mathbf{q}\in\Omega\subset\mathcal{R}^n$.

(N2) $\Lambda_1(\mathbf{q})$ is involutive on Ω.

(N3) There exists a smooth function $\eta_1=\phi_n$ satisfying $L_{\mathbf{w}2}\phi_n=0$ and $L_{\mathbf{w}1}\phi_n=1$ conditions.

(N4) There exist another smooth function $\eta_2=\phi_1$, which can be find by $L_{\mathbf{w}2}\phi_i=0$ $(i=1,2,\ldots,n-2)$ and $L_{\mathbf{w}2}\phi_{n-1}\neq 0$ conditions from Frobrnius theorem.

(N5) To construct a diffeomorphism $\mathbf{z}=\Phi(\mathbf{q})$, using the relationship $\phi_2=L_{\mathbf{w}1}\phi_1$ and $\phi_{i+2}=L_{\mathbf{w}1}\phi_{i+1}+\phi_i$ $(i=1,..,n-3)$.

where $L_{\mathbf{f}}h(\mathbf{x})=\frac{\partial h}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x})$ is called the Lie Derivative of h with respect to $\mathbf{f}$. A diffeomorphism can be established as equation (10) and show that the Φ's Jacobian matrix is invertible. Then, the nonlinear kinematics (7) can be transformed into the skew symmetry chained form (11) under the diffeomorphism $\mathbf{z}=\Phi(\mathbf{q})$ and the state feedback $\mathbf{v}=\Psi(\mathbf{q})\mathbf{u}$.

$$\Phi=\begin{bmatrix}\phi_1\\\phi_2\\\phi_3\\\vdots\\\phi_{n-1}\\\phi_n\end{bmatrix}=\begin{bmatrix}z_1\\z_2\\z_3\\\vdots\\z_{n-1}\\z_n\end{bmatrix}=\begin{bmatrix}\eta_2\\L_{\mathbf{w}_1}\eta_2\\L_{\mathbf{w}_1}^2\eta_2+\eta_2\\\vdots\\L_{\mathbf{w}_1}z_{n-2}+z_{n-3}\\\eta_1\end{bmatrix},\ \begin{cases}u_1=v_1\\u_2=L_{\mathbf{w}_2}\phi_{n-1}v_2\end{cases} \quad (10)$$

$$\dot{\mathbf{z}}=\begin{bmatrix}\dot{\bar{\mathbf{z}}}\\\dot{z}_n\end{bmatrix}=\begin{bmatrix}\dot{z}_1\\\dot{z}_2\\\vdots\\\dot{z}_{n-2}\\\dot{z}_{n-1}\\\dot{z}_n\end{bmatrix}=\begin{bmatrix}u_1z_2\\-u_1z_1+u_1z_3\\\vdots\\-u_1z_{n-3}+u_1z_{n-1}\\-u_1z_{n-2}+\Delta\\u_1\end{bmatrix},\ \bar{\mathbf{z}}=\begin{bmatrix}z_1\\z_2\\\vdots\\z_{n-1}\end{bmatrix} \quad (11)$$

$\Delta=(z_{n-2}+\Delta_1)u_1+u_2$, $\Delta_1=L_{\mathbf{w}_1}z_{n-1}$.

Suppose the existence of the diffeomorphism $\mathbf{z}=\Phi(\mathbf{q})$ and the state feedback $\mathbf{v}=\Psi(\mathbf{q})\mathbf{u}$, the simplified dynamic model (5) can be converted into as

$$M_2(\mathbf{z})\dot{\mathbf{u}}+C_2(\mathbf{z},\dot{\mathbf{z}})\mathbf{u}+G_2(\mathbf{z})+\boldsymbol{\tau}_{d2}=B_2(\mathbf{z})\boldsymbol{\tau} \quad (12)$$

$$M_2(\mathbf{z})=M_1(\mathbf{q})\big|_{\mathbf{q}=\Phi^{-1}(\mathbf{z})},C_2(\mathbf{z},\dot{\mathbf{z}})=C_1(\mathbf{q},\dot{\mathbf{q}})\big|_{\mathbf{q}=\Phi^{-1}(\mathbf{z})}$$

$G_2(\mathbf{z}) = G_1(\mathbf{q})\big|_{\mathbf{q}=\Phi^{-1}(\mathbf{z})}$, $B_2(\mathbf{z}) = B_1(\mathbf{q})\big|_{\mathbf{q}=\Phi^{-1}(\mathbf{z})}$

Several essential properties of the dynamic equation (12) are summarized as follows.

[P1] $M_2(\mathbf{z})$ is a symmetric positive definite matrix.

[P2] A proper choice for $C_2(\mathbf{z},\dot{\mathbf{z}})$, make the matrix $\dot{M}_2 - 2C_2$ skew-symmetry.

[P3] The dynamics (12) can be expressed in a linear parameter form.

$$M_2(\mathbf{z})\dot{\xi} + C_2(\mathbf{z},\dot{\mathbf{z}})\xi + G_2(\mathbf{z}) = \Sigma(\mathbf{z},\mathbf{z},\xi,\dot{\xi})\mathbf{a} \tag{13}$$

3 Description of the Car-Like Mobile Robot

Consider the car-like robot moving on a horizontal plane, as shown in Fig.1. To simplify the kinematic model of the vehicle, each pair of wheels is modeled as a single wheel centered at the midpoint of the axle. From Figure 1, the r is the radius of the wheels, d_r and d_f are the displacement from the vertical axis at point Q_r and Q_f to the mass of center P, the wheel base is $d = d_r + d_f$, x_r and y_r are the Cartesian coordinates of the midpoint of the rear wheel axle, ϕ is the steering angle, and θ is the orientation angle. The front wheel can be steered and the rear wheel is driven by an actuated motor at fixed orientation. Based on the center of rear wheel axle Q_r, there are four constraints resulting from the mobile robot for satisfy the conditions of pure rolling and non-slipping.

[Non-slipping constraints]

$$\dot{x}_r \sin\theta - \dot{y}_r \cos\theta = 0 \tag{14}$$

$$-\dot{x}_r \sin(\theta+\phi) + \dot{y}_r \cos(\theta+\phi) + d\dot{\theta}\cos\phi = 0 \tag{15}$$

[Pure rolling constraints]

$$\dot{x}_r \cos\theta + \dot{y}_r \sin\theta = r\dot{\varphi}_r \tag{16}$$

$$\dot{x}_r \cos(\theta+\phi) + \dot{y}_r \sin(\theta+\phi) + d\dot{\theta}\sin\phi = r\dot{\varphi}_f \tag{17}$$

The robot configuration can be completely characterized by the generalized coordinate vector $\mathbf{q}$.

$\xi = [x_r \quad y_r \quad \theta \quad \phi]^T$, $\mathbf{q} = [\xi^T \quad \varphi_r \quad \varphi_f]^T$.

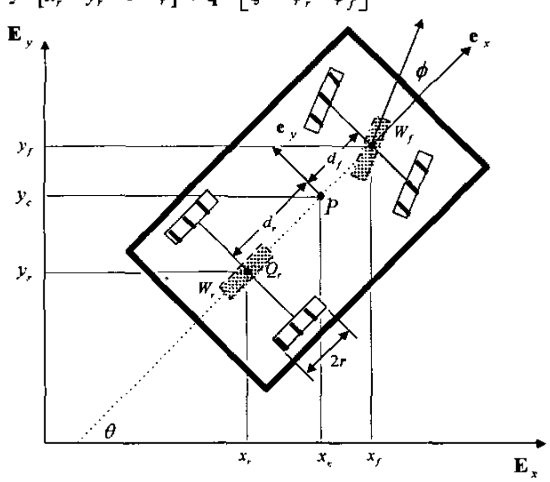

Figure 1: The Configuration of the Car-Like Robot

3.1 Kinematic model of the mobile robot

We rewrite constraint equations (14) ~ (17) to a matrix form $A(\mathbf{q})\dot{\mathbf{q}} = 0$.

$$A(\mathbf{q}) = \begin{bmatrix} \sin\theta & -\cos\theta & 0 & 0 & 0 & 0 \\ \sin(\theta+\phi) & -\cos(\theta+\phi) & -d\cos\phi & 0 & 0 & 0 \\ \cos\theta & \sin\theta & 0 & 0 & -r & 0 \\ \cos(\theta+\phi) & \sin(\theta+\phi) & d\sin\phi & 0 & 0 & -r \end{bmatrix} \tag{18}$$

The following propositio

To satisfy the condition of $A(\mathbf{q})\Omega(\mathbf{q}) = 0$, we give a particular choice of $\Omega(\mathbf{q})$,

$$\Omega(\mathbf{q}) = \begin{bmatrix} r\cos\theta & r\sin\theta & \frac{r}{d}\tan\phi & 0 & 1 & \sec\phi \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T \tag{19}$$

Arbitrarily choosing privileged velocities, a general form of kinematic model based on the posture vector ξ can be obtained in (20).

$$\begin{cases} \dot{x}_r = \cos\theta v_1 \\ \dot{y}_r = \sin\theta v_1 \\ \dot{\theta} = \frac{1}{d}\tan\phi v_1 \\ \dot{\phi} = v_2 \end{cases} \Rightarrow \begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} 1 \\ \tan\theta \\ \frac{1}{d}\sec\theta\tan\phi \\ 0 \end{bmatrix} \upsilon_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \upsilon_2 = \mathbf{w}_1\upsilon_1 + \mathbf{w}_2\upsilon_2 \tag{20}$$

Where $\upsilon_1 = v_1\cos\theta$ is the driving velocity of the rear wheel, and $\upsilon_2 = v_2$ is the steering velocity input of the front wheel. Next, we construct a set of diffeomorphism $\mathbf{z}=\Phi(\xi)$ and the feedback control law $\mathbf{v}=\Psi(\xi)\mathbf{u}$ in (21) for satisfying the condition of (N1) ~ (N5), we then transform the nonlinear kinematic model (20) into the skew symmetry chained form (11) with $n = 4$.

$$\Phi(\xi) = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} y_r \\ \tan\theta \\ \frac{1}{d}\tan\phi\sec^3\theta + y_r \\ x_r \end{bmatrix}, \mathbf{v} = \begin{bmatrix} \frac{1}{\cos\theta} & 0 \\ 0 & \frac{\sec^2\phi\sec^3\theta}{d} \end{bmatrix}\mathbf{u} \tag{21}$$

In this case, the transformation is only locally defined which exclude the vehicle orientations $\theta = \pi/2 \pm k\pi$, $k \in \mathbf{N}$.

3.2 Dynamic model of the mobile robot

The Lagrange formalism is used to establish the dynamic equations of motion for the mobile robot. The kinetic energy is given by

$$K = \frac{1}{2}m\dot{x}_r^2 + \frac{1}{2}m\dot{y}_r^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}I_m(\dot{\theta}+\dot{\phi})^2 + \frac{1}{2}I_w\dot{\varphi}_r^2 + \frac{1}{2}I_w\dot{\varphi}_f^2 + md\dot{\theta}(\dot{y}_r\cos\theta - \dot{x}_r\sin\theta) \tag{22}$$

where $m = m_c + 2m_w$, m_c and m_w are the mass of body and wheel; I_c , I_w and I_m are the moment of inertia of the body about the vertical axis, the wheel about the wheel axis, and wheel about the wheel diameter, respectively, and let $I = I_c + m_c d_r^2 + m_w d^2 + I_m$, $L = m_c d_r + m_w d$. The dynamics of the mobile robot are determined by

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_i}\right) - \frac{\partial K}{\partial q_i} = B\tau + A^T\lambda \tag{23}$$

where λ is the Lagrange multiplier, τ_1 is the engine torque acting on the rear wheel, τ_2 is the steering torque. Substituting the total kinematic energy into equation (21) to derive the dynamic model, we have

$$M(\mathbf{q})\ddot{\mathbf{q}} + C(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} + \boldsymbol{\tau}_d = B(\mathbf{q})\boldsymbol{\tau} + A^T(\mathbf{q})\boldsymbol{\lambda} \tag{24}$$

Equation (24) is an alternative form of the dynamic model.

$$\begin{bmatrix} m & 0 & -L\sin\theta & 0 & 0 & 0 \\ 0 & m & L\cos\theta & 0 & 0 & 0 \\ -L\sin\theta & L\cos\theta & (I+I_m) & I_m & 0 & 0 \\ 0 & 0 & I_m & I_m & 0 & 0 \\ 0 & 0 & 0 & 0 & I_w & 0 \\ 0 & 0 & 0 & 0 & 0 & I_w \end{bmatrix} \begin{bmatrix} \ddot{x}_r \\ \ddot{y}_r \\ \ddot{\theta} \\ \ddot{\phi} \\ \ddot{\varphi}_r \\ \ddot{\varphi}_f \end{bmatrix} + \begin{bmatrix} 0 & 0 & -L\dot{\theta}\cos\theta & 0 & 0 & 0 \\ 0 & 0 & -L\dot{\theta}\sin\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_r \\ \dot{y}_r \\ \dot{\theta} \\ \dot{\phi} \\ \dot{\varphi}_r \\ \dot{\varphi}_f \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \boldsymbol{\tau} + A^T(\mathbf{q})\boldsymbol{\lambda} \tag{25}$$

Using $\ddot{\mathbf{q}} = \dot{\Omega}(\mathbf{q})\mathbf{u} + \Omega(\mathbf{q})\dot{\mathbf{u}}$ and $\mathbf{z} = \Phi(\mathbf{q})$, after some simple manipulations, equation (25) can be expressed as

$$M_2(\mathbf{z})\dot{\mathbf{u}} + C_2(\mathbf{z},\dot{\mathbf{z}})\mathbf{u} + \boldsymbol{\tau}_d = B(\mathbf{q})\boldsymbol{\tau} \tag{26}$$

$$M_2(\mathbf{z}) = \Omega^T(\mathbf{q})M(\mathbf{q})\Omega(\mathbf{q})\Big|_{q=\Phi^{-1}(z)} = \begin{bmatrix} (I + I_m + d^2\frac{I_w}{r^2})T_\phi^2 + (m + \frac{2I_w}{r^2}) & I_m T_\phi \\ I_m T_\phi & I_m \end{bmatrix}$$

$$C_2(\mathbf{z},\dot{\mathbf{z}}) = \Omega^T(\mathbf{q})M(\mathbf{q})\dot{\Omega}(\mathbf{q}) + \Omega^T(\mathbf{q})C(\mathbf{q})\Omega(\mathbf{q})\Big|_{q=\Phi^{-1}(z)} = \begin{bmatrix} (I + I_m + d^2\frac{I_w}{r^2})T_\phi \dot{T}_\phi & 0 \\ I_m \dot{T}_\phi & 0 \end{bmatrix}$$

Taking diffeomorphism inverse transformation, some parameters of equation (26) can be substituted as

$$\tan^2\phi = d^2T_\phi^2 = d^2 z_3^2(1+z_2^2)^{-3},$$

$$\sec^2\phi = 1 + d^2T_\phi^2 = [(1+z_2^2)^3 + d^2z_3^2](1+z_2^2)^{-3},$$

$$\sec^2\phi \cdot \dot{\phi} = d\dot{T}_\phi = d\dot{z}_3(1+z_2^2)^{-\frac{3}{2}} - 3dz_2z_3\dot{z}_2(1+z_2^2)^{-\frac{5}{2}},$$

$$\sec^2\phi\tan\phi \cdot \dot{\phi} = d^2T_\phi\dot{T}_\phi = d^2z_3\dot{z}_3(1+z_2^2)^{-3} - 3d^2z_2z_3^2\dot{z}_2(1+z_2^2)^{-4}.$$

Observably, equation (26) satisfy the condition of [P2], i.e., $\dot{M}_2 - 2C_2$ is a skew-symmetric matrix satisfying $\boldsymbol{\chi}^T(\dot{M}_2 - 2C_2)\boldsymbol{\chi} = 0$. According to equations (13), the linear parameter form of the car-like mobile robot can be converted as

$$M_2(\mathbf{z})\dot{\mathbf{u}}_s + C_2(\mathbf{z},\dot{\mathbf{z}})\mathbf{u}_s = \Sigma(\mathbf{z},\mathbf{z},\mathbf{u}_s,\dot{\mathbf{u}}_s)\mathbf{a}$$

$$\begin{bmatrix} (I + I_m + \frac{d^2I_w}{r^2})T_\phi^2\dot{u}_{s1} + (m + \frac{2I_w}{r^2})\dot{u}_{s1} + I_mT_\phi\dot{u}_{s2} + (I + I_m + d^2\frac{I_w}{r^2})T_\phi\dot{T}_\phi u_{s1} \\ I_mT_\phi\dot{u}_{s1} + I_m\dot{u}_{s2} + I_m\dot{T}_\phi u_{s1} \end{bmatrix}$$

$$= \begin{bmatrix} T_\phi^2\dot{u}_{s1} & \dot{u}_{s1} & T_\phi\dot{u}_{s2} & T_\phi\dot{T}_\phi u_{s1} \\ 0 & 0 & T_\phi\dot{u}_{s1} + \dot{u}_{s2} + \dot{T}_\phi u_{s1} & 0 \end{bmatrix} \begin{bmatrix} \frac{r^2I + r^2I_m + d^2I_w}{r^2} \\ \frac{mr^2 + 2I_w}{r^2} \\ I_m \\ \frac{r^2I + r^2I_m + d^2I_w}{r^2} \end{bmatrix} \tag{27}$$

4 Design of the Hierarchical Controller

Consider the hierarchical model constructed by a skew symmetric chained form (11) and dynamic equations (26). This paper uses the advantage of skew symmetric chained form to design the time-varying controller whose structure not only is simple but is easy to realize. The higher level is based on the kinematic model under nonholonomic constraints, $z_1 \sim z_n$ is desired to approach zero in order to achieve the aim of posture stabilization. The lower level controller is to find the control torque $\boldsymbol{\tau}$, the privileged velocity $\mathbf{u}$ of the dynamics (26) can be converged to the kinematic level of control input signal $\mathbf{u}_c$. Define the desired velocity $\mathbf{u}_c$, the compensated signal $\boldsymbol{\Pi}$ and relative signals as follows.

$$\mathbf{u}_c = \begin{bmatrix} u_{c1} \\ u_{c2} \end{bmatrix} = \begin{bmatrix} -k_1z_n + \kappa(\bar{\mathbf{z}},t) \\ -z_{n-2}u_1 - L_{\kappa_1}z_{n-1}u_1 - k_2z_{n-1} \end{bmatrix} + \Lambda\mathbf{u}_i \tag{28}$$

$$\boldsymbol{\Pi} = [z_{n-1}z_{n-2} + z_{n-1}L_{\kappa_1}z_{n-1} \quad z_{n-1}]^T \tag{29}$$

Consider the z_n-subsystem of $\dot{z}_n = u_1$. In order to guarantee z_1 asymptotically stable, we can choose the control input u_{c1} as shown in (28), where $\kappa(\bar{\mathbf{z}},t)$ is called exciting function. The exciting function must satisfy the following conditions: [Samson 1995]

[C1] $\kappa(\bar{\mathbf{z}},t)$ is a uniformly continuous function. Suppose $\kappa(\bar{\mathbf{z}},t) \in C^{p+1}(p \geq 1)$ and all partial derivatives are uniformly bounded with respect to time.

[C2] $\kappa(0,t) = 0, \forall t$.

[C3] As we know, z_i and $\dot{z}_i$ being bounded, assume $\dot{z}_\kappa(\bar{\mathbf{z}})$ tending to zero imply $\lim_{t\to\infty} z_i = 0, (2 \leq i \leq n)$.

The choice of exciting function $\kappa(\bar{\mathbf{z}},t)$ is critical, the type of function not only forces motion as long as the system has not reached the desired configuration but also determine the motion trajectory of the mobile robot. In this paper, we adopt the following function which satisfy [C1] ~ [C3] conditions.

$$\kappa(\bar{\mathbf{z}},t) = \sum_{i=1}^{n-1}\alpha_i\sin(\beta_i t)z_i \tag{30}$$

For point stabilization problem, such as auto-parking case, the approaching part is quite natural and the final part of the motion resembles a parallel parking maneuver. The whole response is very similar to the real parking of vehicle. Next, $\mathbf{u}_e = \mathbf{u} - \mathbf{u}_c = [u_{e1}\ u_{e2}]^T$ is the velocity error signal, $\mathbf{u}_i = \int \mathbf{u}_e dt = [u_{i1}\ u_{i2}]^T$ is the integral signal of $\mathbf{u}_e$. Define the integral type of sliding surface as [11]

$$\mathbf{S} = [S_1 \quad S_2]^T = \mathbf{u}_e + \Lambda\int\mathbf{u}_e dt = \mathbf{u}_e + \Lambda\mathbf{u}_i \tag{31}$$

A new variable $\mathbf{u}_s$ is introduced such that $\mathbf{S} = \mathbf{u} - \mathbf{u}_s$, i.e.,

$$\mathbf{u}_s = \mathbf{u} - \mathbf{u}_e - \Lambda\mathbf{u}_i = \mathbf{u}_c - \Lambda\mathbf{u}_i \tag{32}$$

We denote $\hat{\mathbf{a}}$ be the estimation vector of $\mathbf{a}$, $\hat{M}_2$ and $\hat{C}_2$ are the corresponding estimations of M_2 and C_2. From the property of [P3], we have the linear parameter form

$$M_2(\mathbf{z})\dot{\mathbf{u}}_s + C_2(\mathbf{z},\dot{\mathbf{z}})\mathbf{u}_s + G_2(\mathbf{z}) = \Sigma(\mathbf{z},\dot{\mathbf{z}},\mathbf{u}_s,\dot{\mathbf{u}}_s)\mathbf{a} \tag{33}$$

[Theorem 2] Given the car-like mobile robot described by a hierarchical model of (11) and (26) containing unknown parameters and disturbances. A smooth auxiliary velocity signal $\mathbf{u}_s$ is given by (32). Further then, the control law of point stabilization problem is obtained by the adaptive sliding mode techniques can be expressed:

$$\boldsymbol{\tau} = B_2^{\#}[\Sigma(\mathbf{z}, \dot{\mathbf{z}}, \mathbf{v}_s, \dot{\mathbf{v}}_s)\hat{\mathbf{a}} - c \cdot sgn(\mathbf{S}) - \mathbf{\Pi}] \quad (34)$$

The adaptive law of parameter estimates is updated by

$$\dot{\hat{\mathbf{a}}} = -\Gamma^{-1}\Sigma^T(\mathbf{z}, \dot{\mathbf{z}}, \mathbf{v}_s, \dot{\mathbf{v}}_s)\mathbf{S} \quad (35)$$

where Γ is a symmetric positive define constant matrix, $B_2^{\#}$ is the left inverse matrix of B_2 and $c = \max(\boldsymbol{\tau}_d) + \varepsilon$, $\varepsilon > 0$, is the design parameter for suppress the external disturbances. Assume both $|u_{c1}(t)|$ and $|u_{c1}(t)|$ are bounded, then the close loop system is globally asymptotically stable, this implies that $\mathbf{z} \to 0$ and the velocity $\mathbf{u}$ converges to the desired velocity $\mathbf{u}_s$ as $t \to \infty$, achieving the purpose of point stabilization controller.

Proof: Our goal is to make $z_1 \sim z_n$ tend to zero and $\mathbf{u} \to \mathbf{u}_s$ within limited time. Starting with $\bar{\mathbf{z}}$ subsystem and define Lyapunov candidate function as

$$V_1(\bar{\mathbf{z}}) = \frac{1}{2}(z_1^2 + z_2^2 + z_3^2 + \cdots + z_{n-1}^2) \quad (36)$$

Differentiating $V_1(\bar{\mathbf{z}})$ along the closed loop system (11),

$$\dot{V}_1 = z_1(u_1 z_2) + z_2(-u_1 z_1 + u_1 z_3) + \cdots + z_{n-2}(-u_1 z_{n-3} + u_1 z_{n-1})$$
$$+ z_{n-1}(-u_1 z_{n-2} + \Delta) = z_{n-1}\Delta$$

where $z_{n-1}\Delta = (z_{n-1}z_{n-2} + z_{n-1}L_{w_1}z_{n-1})u_1 + z_{n-1}u_2 = \Pi^T\mathbf{u}$

Define Lyapunov candidate function as

$$V_2(\bar{\mathbf{z}}, \mathbf{S}, \tilde{\mathbf{a}}) = V_1(\bar{\mathbf{z}}) + \frac{1}{2}\mathbf{S}^T M_2(\mathbf{z})\mathbf{S} + \frac{1}{2}\tilde{\mathbf{a}}^T\Gamma\tilde{\mathbf{a}} \quad (37)$$

Using the result of (28) and differentiating the Lyapunov candidate function along the system (26).

$$\dot{V}_2 = \mathbf{u}^T\Pi + \tilde{\mathbf{a}}^T\Gamma\dot{\tilde{\mathbf{a}}} + \mathbf{S}^T M_2\dot{\mathbf{S}} + \frac{1}{2}\mathbf{S}^T\dot{M}_2\mathbf{S}$$
$$= \mathbf{S}^T[B_2\boldsymbol{\tau} - \boldsymbol{\tau}_d - \Sigma(\mathbf{z}, \dot{\mathbf{z}}, \mathbf{u}_s, \dot{\mathbf{u}}_s)\mathbf{a} + \Pi] + \mathbf{S}^T[\frac{1}{2}\dot{M}_2(\mathbf{z}) - C_2(\mathbf{z}, \dot{\mathbf{z}})]\mathbf{S}$$
$$+ \tilde{\mathbf{a}}^T\Gamma\dot{\tilde{\mathbf{a}}} + \Pi^T(\mathbf{u}_c - \Upsilon\mathbf{u}_1)$$
$$= \mathbf{S}^T[B_2\boldsymbol{\tau} - \boldsymbol{\tau}_d - \Sigma(\mathbf{z}, \dot{\mathbf{z}}, \mathbf{u}_s, \dot{\mathbf{u}}_s)\boldsymbol{\alpha} + \Pi] + \tilde{\boldsymbol{\alpha}}^T\Gamma\dot{\tilde{\boldsymbol{\alpha}}} - k_2 z_{n-1}^2 \quad (38)$$

Consider the point stabilization controller (34) and [P3], equation (38) can be further written as:

$$\dot{V}_2 = \mathbf{S}^T[\Sigma(\mathbf{z}, \dot{\mathbf{z}}, \mathbf{u}_s, \dot{\mathbf{u}}_s)\tilde{\boldsymbol{\alpha}} - c \cdot \mathrm{sgn}(\mathbf{S}) - \boldsymbol{\tau}_d] + \tilde{\boldsymbol{\alpha}}^T\Gamma\dot{\tilde{\boldsymbol{\alpha}}} - k_2 z_{n-1}^2 \quad (39)$$

Substituting the adaptive law (35) into (39), we yield the finial result.

$$\dot{V}_2 = -\mathbf{S}^T(c \cdot \mathrm{sgn}(\mathbf{S}) + \boldsymbol{\tau}_d) - k_2 z_{n-1}^2 \le -\varepsilon\, \mathbf{S}^T \mathrm{sgn}(\mathbf{S}) - k_2 z_{n-1}^2 \le 0 \quad (40)$$

From equation (40), $\dot{V}_2 \le 0$ means $V_2(\bar{\mathbf{z}}, \mathbf{S}, \tilde{\mathbf{a}})$ is a non-increasing function. That is, $\tilde{\mathbf{a}}$, $\mathbf{S}$, $z_1, z_2, \ldots, z_{n-1}$ are bounded. Besides, we assume $|u_{c1}(t)|$ as a function with boundary. The skew symmetric chained form (11) reveals the information that $z_1, z_2, \ldots, \dot{z}_{n-1}$ is bounded. Differentiating (40) again, we can get equation (41). z_{n-1}, z_{n-1}, are bounded, so $\ddot{V}_2$ is bounded function.

$$\ddot{V} = \frac{d}{dt}(-\varepsilon\, \mathbf{S}^T \mathrm{sgn}(\mathbf{S}) - k_2 z_{n-1}^2) = -\varepsilon\, \dot{\mathbf{S}}^T \mathrm{sgn}(\mathbf{S}) - 2k_2 z_{n-1}\dot{z}_{n-1} \quad (41)$$

From Barbalat's lemma, we can realize $\dot{V}_2$ is a uniform continuous function and $\dot{V}_2$ tends to zero. Therefore, z_{n-1} tends to zero from equation (40). Next, we define the function $u_{c1}^2 z_{n-1}$ and take time derivative.

$$\frac{d}{dt}(u_{c1}^2 z_{n-1}) = (2u_{c1}\dot{u}_{c1}z_{n-1} - u_{c1}^2 k_2 z_{n-1}) - u_{c1}^3 z_{n-2} \quad (42)$$

The function $v_{c1}^3 z_{n-2}$ is uniformly continuous because its time derivative is bounded. Equation (42) will approach to zero due to z_{n-1} tends to zero, implying $\lim_{t\to\infty} z_{n-2} = 0$. Taking the time derivative of $u_{c1}^2 z_j$ and repeating the above procedure, we can prove that $z_1, z_2, \ldots, z_{n-1}$ all tends to zero and achieving the purpose of global asymptotically stable.

5 Computer Simulation

The simulations have been carried out for the type of car-like mobile robot on the point stabilization problem, which is shown in Figure 1. The parameter specifications of physical model are given as $m_w = 1$, $m_c = 20$, $I_w = 0.1$, $I_c = 17.75$, $I_m = 2$, $b = 0.5$, $d_r = 0.25$, $d_f = 0.75$, $r = 0.1$. The simulation is a parking maneuver and the track of the autonomous parallel parking is similar to the behavior of the real situation. The kinematic model and dynamic model are described in equations (20) and (25). Using a set of diffeomorphism (21), we then transform the nonlinear kinematic model (20) into the skew symmetry chained form (11). In the condition that the dynamic model is uncertain, the linear parametric form can be expressed as equation (27) and estimation vector is

$$\mathbf{a} = \begin{bmatrix} \frac{r^2 I + r^2 I_m + d^2 I_w}{r^2} & \frac{mr^2 + 2I_w}{r^2} & I_m & \frac{r^2 I + r^2 I_m + d^2 I_w}{r^2} \end{bmatrix}^T$$

From (28), (29), design of the desired velocity $\mathbf{u}_c$ on kinematic model and compensated signal $\mathbf{\Pi}$.

$$\begin{cases} u_{c1} = -k_1 z_4 + \alpha_1 \sin(\beta_1 t) z_1 + \alpha_2 \sin(\beta_2 t) z_2 + \alpha_3 \sin(\beta_3 t) z_3 + \lambda_1 \int u_{c1} dt \\ u_{c2} = -z_2 u_{c1} - z_3 u_{c1} - \frac{z_3(z_4 - z_2)^2}{1 + z_3^2} u_{c1} - k_2 z_3 + \lambda_2 \int u_{c2} dt \end{cases}$$

$$\mathbf{\Pi} = [z_3 z_2 + z_3^2 + \frac{z_3^2(z_4 - z_2)^2}{1 + z_3^2} \quad z_3]^T, \Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Apply the sliding mode control scheme $\boldsymbol{\tau}$ in (34) to the uncertain car-like mobile robot. The adaptive law of parameter estimates is updated by (35). $\mathbf{S} = [S_1\ S_2]^T$ is integral type of sliding surface, the design parameters in simulation are assumed to be k_1=2, k_2=6, α_1=4, α_2=20, α_3=6, $\beta_1 = \beta_2 = \beta_3 = 1$, $c = 50$, $\Gamma = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}$, $\Lambda = \begin{bmatrix} 6 & 0 \\ 0 & 6 \end{bmatrix}$ and disturbances $\boldsymbol{\tau}_d$ are random number in the range [-1,1]. The reference posture is x_r=0, y_r=0, θ=0, ϕ=0. The initial estimate $\hat{\boldsymbol{\alpha}}(0) = [1\ 1\ 1\ 1]^T$ is different from the true value. The initial posture of the mobile robot is $x_r(0)$=0, $y_r(0)$=-10, $\theta(0)$=0, $\phi(0)$=0. Simulation results are shown on Figure 2 ~ Figure 5. Figure 2 demonstrate the response of the mobile robot; we can see that the posture vector x_r, y_r, θ, ϕ and the states of the kinematics $z_1 \sim z_4$ asymptotically tend to zero. Figure 3 presents the mobile robot's Cartesian trajectory. The tracking effect of privileged velocities $\mathbf{u}_s$ is shown in Figure 4. The estimated parameters $\hat{a}$ are all bounded as shown in Figure 5.

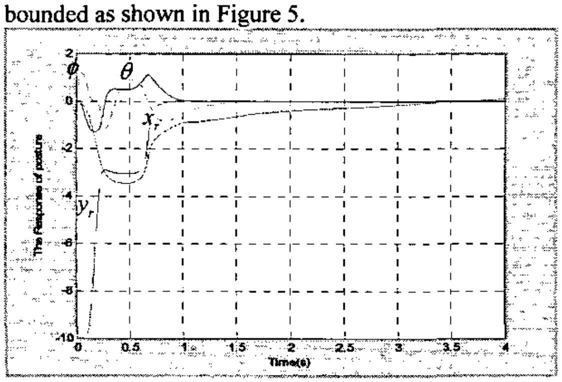

Figure 2: The response of posture x_r, y_r, θ, ϕ

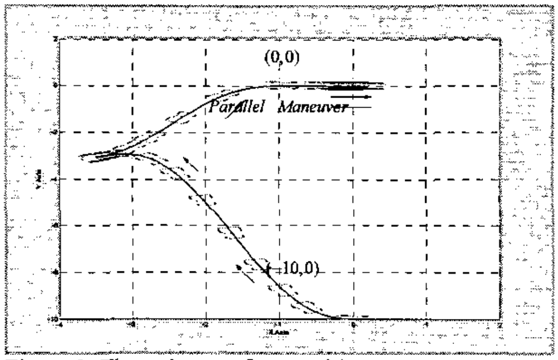

Figure 3: The trajectory for a parking maneuver

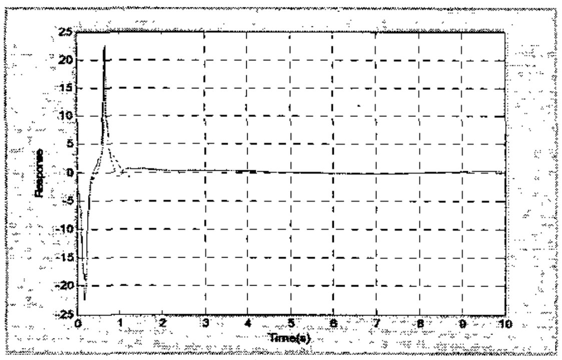

Figure 4: The tracking effect of privileged velocities $\dot{u}_1$-u_{s1}
(Solid line '-' is desired velocity u_{s1}, dashed line '- -' is actual velocity u_1)

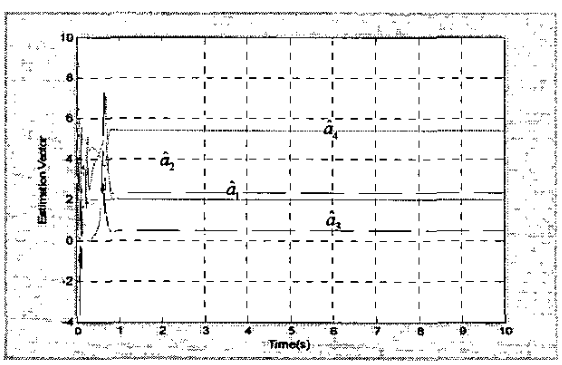

Figure 5: Estimated parameter of $\hat{a}_1$, $\hat{a}_2$, $\hat{a}_3$, $\hat{a}_4$

6 Conclusions

A two-level hierarchical point stabilization control structure has been proposed and demonstrated for car-like mobile robot under plant uncertainties and external disturbances. In order to achieve the control objective of different levels, the kinematic model uses the skew symmetric chained form. The time-varying controller is simply structured and easy to realize. Furthermore, the adaptive sliding mode controller is constructed by dynamic simplified model, which has both adaptive and robust features. The proposed algorithm, which is used to cope with the parametric uncertainties and attenuate the effect of external disturbances, guarantees the stabilization of the motion to a desired configuration. From the simulation results, it is concluded that the proposed hierarchical strategy achieves the effectiveness of desired performance.

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