Normalised quadratic controls for a class of bilinear systems

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Abstract: Normalised quadratic control, a modification of the conventional quadratic control method, has recently been proposed to exponentially stabilise a homogeneous bilinear system whose open-loop eigenvalues all fall on the imaginary axis. The normalised quadratic control method is now extended to a broader class of bilinear systems whose open-loop eigenvalues may fall to the left or right of the imaginary axis. It is shown that for bilinear systems that are already open-loop stable, the normalised quadratic control can add to the system an extra exponential decay rate. For open-loop unstable bilinear systems, the normalised quadratic control can stabilise the system if the extra exponential decay rate provided by the control is larger than the open-loop system's growth rate.

1 Introduction

Consider a homogeneous bilinear system:

$$\dot{x}(t) = Ax(t) + u(t)Nx(t), \quad x(0) = x_0 \tag{1}$$

where $x(t) \in R^n$ is the system state vector, $u(t)$ is a scalar control input, and $A \in R^{n \times n}$ and $N \in R^{n \times n}$ are constant square matrices. It is assumed that the open-loop system matrix A has only pure imaginary eigenvalues. In this case [1], there exists a positive definite matrix P such that:

$$A^T P + PA = 0 \tag{2}$$

Under this neutrally stable assumption, the pair (A, N) satisfies the following controllability rank condition [2, 3]: for some positive integer m:

$$\text{rank}[ad^0(A, N)x_0, ad^1(A, N)x_0, \ldots, ad^m(A, N)x_0] = n \tag{3}$$

for any nonzero x_0 in R^n, where:

$$\begin{aligned} ad^0(A, N) &= N \\ ad^{k+1}(A, N) &= A \times ad^k(A, N) \\ &\quad - ad^k(A, N) \times A, \quad k = 0, 1, 2 \ldots \end{aligned}$$

In the literature, two different control methods have been proposed for an open-loop neutrally stable system (1). The first control method is quadratic feedback control [4–6]:

$$u(t) = -\gamma x^T(t)PNx(t) \quad \gamma > 0$$

which ensures global asymptotic stability of the closed-loop system. The second control method is a modification of the first one, called normalised quadratic feedback control:

$$u(t) = -\gamma e_x^T(t)PNe_x(t) \quad e_x(t) \triangleq \frac{x(t)}{\|x(t)\|}$$

which ensures global exponential stability, and hence better performance of the closed-loop system [7].

The objective of this paper is to study whether or not normalised quadratic feedback control can be extended to a broader class of systems which have left-half-plane or right-half-plane open-loop eigenvalues, and to examine under what conditions, the normalised control can provide an exponential stabilising effect for these systems.

2 Continuous Normalised Quadratic Control

When the open-loop system matrix A in (1) has eigenvalues that are not pure imaginary, the normalised quadratic feedback control is still of the form:

$$u(t) = -\gamma e_x^T(t)PNe_x(t) \quad e_x(t) = \frac{x(t)}{\|x(t)\|} \tag{4}$$

however, the matrix P is now obtained from the following modified Lyapunov equation instead of (2):

$$(A \pm \rho I)^T P + P(A \pm \rho I) = -Q \tag{5}$$

where the plus sign is for a stable A, the minus sign is for neutrally stable and unstable A, P and Q are two positive definite matrices, and $\rho > 0$ is defined by:

$$\rho = \begin{cases} -\max\{\text{Re}(\lambda_i(A))\} - \varepsilon & \text{for stable } A \\ \varepsilon & \text{for neutrally stable } A \\ \max\{\text{Re}(\lambda_i(A))\} + \varepsilon & \text{for unstable } A \end{cases} \tag{6}$$

in which $\varepsilon > 0$ is an arbitrarily small number. When ε is sufficiently small and for the case with stable A, ρ characterises the decay rate of the open-loop system. For the case with unstable A, ρ characterises the growth rate of the open-loop system.

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Before analysing the normalised control (4), one needs the following assumption: for some positive integer m:

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{R}_0\boldsymbol{x}_0 & \boldsymbol{R}_1\boldsymbol{x}_0 & \cdots & \boldsymbol{R}_m\boldsymbol{x}_0 \end{bmatrix} = n \qquad (7)$$

for any nonzero x_0 in $\boldsymbol{R}^n$, where R_k is defined recursively as

$$\boldsymbol{R}_0 = \boldsymbol{PN}, \boldsymbol{R}_k = \boldsymbol{A}^T\boldsymbol{R}_{k-1} + \boldsymbol{R}_{k-1}\boldsymbol{A}, \quad k = 1, 2, 3 \ldots$$

It is speculated that the rank condition (7) may be deduced from the rank condition (3) for any stable or unstable $\boldsymbol{A}$ matrix. Further research will be required to verify this speculation.

Lemma 1: Consider the closed-loop system (1) and (4). If the rank condition (7) is satisfied, and the system state $\boldsymbol{x}(t)$ satisfies:

$$\boldsymbol{x}^T(t)\boldsymbol{PNx}(t) \equiv 0 \quad \forall t \in [kT, kT + T) \qquad (8)$$

for some $T > 0$, then $\boldsymbol{x}(kT) = 0$.

Proof: If $\boldsymbol{x}^T(t)\boldsymbol{PNx}(t)$ is identically zero over the time span $[kT, kT+T)$, so is the control input (4). The system (1) then becomes open-loop; therefore:

$$\boldsymbol{x}(t) = e^{A(t-kT)}\boldsymbol{x}(kT) \quad \forall \tau \in [kT, kT + T)$$

Substituting the above equation into (8), and taking consecutively the time derivatives of the equation, one obtains $\boldsymbol{x}^T(kT)\boldsymbol{R}_i\boldsymbol{x}(kT) = 0$, $i = 1, 2, 3, \ldots, m$, or in matrix form,

$$\boldsymbol{x}^T(kT)\begin{bmatrix} \boldsymbol{R}_0\boldsymbol{x}(kT) & \boldsymbol{R}_1\boldsymbol{x}(kT) & \cdots & \boldsymbol{R}_m\boldsymbol{x}(kT) \end{bmatrix} = 0$$

The above equation shows that $\boldsymbol{x}(kT)$ is orthogonal to all vectors $\boldsymbol{R}_i\boldsymbol{x}(kT)$. Since $\boldsymbol{R}_i\boldsymbol{x}(kT)$, $i = 1, 2, \ldots, m$, span the whole space according to assumption (7), $\boldsymbol{x}(kT)$ must be the null vector. □

Further, to quantify the stabilising effect of the normalised control (4), one needs the following definition. Given a time interval length $T > 0$, define a function $\beta(\cdot): S \to R^+ \cup \{0\}$ for the controlled system (1) and (4), where S is the unit sphere in $\boldsymbol{R}^n$:

$$\beta(\boldsymbol{e}_x(kT)) \triangleq \frac{\gamma}{\bar{\sigma}_P}\frac{1}{T}\int_{kT}^{kT+T}[\boldsymbol{e}_x^T(t)\boldsymbol{PNe}_x(t)]^2 dt \quad \boldsymbol{e}_x(kT) \in S \quad (9)$$

in which γ is the control gain in (4), and $\bar{\sigma}_p$ the maximum singular value of $\boldsymbol{P}$ in (5). The following lemma shows that the rank condition (7) guarantees that $\beta\ (\boldsymbol{e}_x(kT))$, which is a measure of the exponential stabilising effect of the normalised control (4) over a time span T (see remark 3 after theorem 2), is always nonzero for all $\boldsymbol{e}_x(kT)$ on the unit sphere S.

Lemma 2: If the system (1) satisfies the rank condition (7), there exists some positive constant β^* such that for all integer k:

$$\inf_{\boldsymbol{e}_x(kT)\in S}[\beta(\boldsymbol{e}_x(kT))] = \beta^* > 0 \qquad (10)$$

Proof: Since the integrand in (9) is non-negative, $\beta(\cdot)$ must also be non-negative. To show that $\beta(\cdot)$ is nonzero (hence, positive) for all $\boldsymbol{e}_x(kT) \in S$, one can use the following contradiction argument. Assume that $\beta\ (\boldsymbol{e}_x(kT))$ is zero for some $\boldsymbol{e}_x(kT) \in S$. Based on the definition of $\beta(\cdot)$ in (9), one has $\boldsymbol{e}_x^T(t)\boldsymbol{PNe}_x(t) \equiv 0$, for all $t \in [kT, kT+T)$; in other words:

$$\boldsymbol{x}^T(t)\boldsymbol{PNx}(t) \equiv 0 \quad \forall t \in [kT, kT + T)$$

Invoking lemma 1, the above equation implies $\boldsymbol{x}(kT) = 0$, which contradicts the fact that $\boldsymbol{e}_x(kT) \in S$. Therefore, $\beta(\boldsymbol{e}_x(kT))$ must be positive for all $\boldsymbol{e}_x(kT) \in S$. Furthermore, since the unit sphere S is compact, the infimum of $\beta(\boldsymbol{e}_x(kT))$ over S, denoted by β^*, must also be positive according to theorem 4.4.1 in [8]. Finally, since the dynamics of the closed-loop system (1) and (4) are the same on all different time intervals $[kT, kT+T)$, the lower bound β^* must be uniform with respect to k. This finished the proof of (10). □

One can now present the stability results for the normalised control (4), first for the open-loop stable system (1), and then for the open-loop unstable system.

Theorem 1: Consider an open-loop stable bilinear system (1). With the continuous normalised quadratic control (4), the closed-loop system is exponentially stable with the following decay rate:

$$\|\boldsymbol{x}(kT + T)\| \leq \frac{\sigma_P}{\bar{\sigma}_P} \times \exp\left(-\beta^* + \frac{\sigma_Q}{2\bar{\sigma}_P} + \rho\right)T \times \|\boldsymbol{x}(kT)\| \qquad (11)$$

Proof: Define a Lyapunov function candidate $V(t) = \boldsymbol{x}^T(t)\boldsymbol{Px}(t)$, where $\boldsymbol{P}$ is from the Lyapunov equation (5) with stable $\boldsymbol{A}$. Take the time derivative of the Lyapunov function along the closed-loop trajectory (1), (4) and (5):

$$\begin{aligned}\dot{V}(t) &= -\boldsymbol{x}^T(t)\boldsymbol{Qx}(t) - 2\rho\boldsymbol{x}^T(t)\boldsymbol{Px}(t) \\ &\quad - 2\gamma[\boldsymbol{e}_x^T(t)\boldsymbol{PNe}_x(t)]^2\|\boldsymbol{x}(t)\|^2 \\ &\leq -\left\{\frac{\sigma_Q}{\bar{\sigma}_P} + 2\rho + \frac{2\gamma[\boldsymbol{e}_x^T(t)\boldsymbol{PNe}_x(t)]^2}{\bar{\sigma}_P}\right\}V(t)\end{aligned}$$

Integrating the inequality over the time span $[kT, kT+T]$ leads to:

$$\begin{aligned}\ln\left(\frac{V(kT + T)}{V(kT)}\right) &\leq -\frac{\sigma_Q}{\bar{\sigma}_P} \times T - 2\rho \times T \\ &\quad - \frac{\gamma}{\bar{\sigma}_P}\int_{kT}^{kT+T}[\boldsymbol{e}_x^T(\tau)\boldsymbol{PNe}_x(\tau)]^2 d\tau \\ &\leq -\left(\frac{\sigma_Q}{\bar{\sigma}_P} + 2\rho + 2\beta^*\right)T\end{aligned}$$

where the last inequality results from (10) of lemma 2. Equation (11) can then be concluded from the last inequality and from the definition of $V(kT)$. □

Remark 1: Before interpreting the result (11) of theorem 1, note that $\boldsymbol{A} \pm \rho\boldsymbol{I}$ in the Lyapunov equation (4) is stable but close to neutrally stable for small ϵ (it is neutrally stable if $\epsilon = 0$). According to lemma 5 in the Appendix, one has:

$$\lim_{\epsilon\to 0}\frac{\sigma_Q}{\bar{\sigma}_P} = 0$$

and hence for small ϵ in (6), one can neglect the term $\sigma_Q/2\bar{\sigma}_P$ in the exponent of (11). Therefore, for an open-loop stable bilinear system (1), the decay rate resulting from the normalised quadratic control is given by:

$$\|\boldsymbol{x}(kT + T)\| \leq \frac{\sigma_P}{\bar{\sigma}_P} \times e^{-[\beta^*+\rho]T} \times \|\boldsymbol{x}(kT)\| \qquad (12)$$

Recall from the statement below (6) that ρ characterises the decay rate of the open-loop stable bilinear system. Equation (12) then shows that the normalised control has the effect of providing an extra exponential decay rate by an exponent of β^* for an open-loop stable bilinear system.

The second stability theorem is for open-loop unstable systems, whose proof parallels that of theorem 1, and hence is omitted.

Theorem 2: Consider an open-loop-unstable bilinear system (1). The continuous normalised quadratic control (4) can exponentially stabilise the closed-loop system if:

$$\beta^* + \underline{\sigma}_Q/(2\bar{\sigma}_P) > \rho$$

Furthermore, the closed-loop decay rate is given by:

$$\|x(kT+T)\| \le \frac{\underline{\sigma}_P}{\bar{\sigma}_P} \times e^{-\left(\beta^* + \frac{\underline{\sigma}_Q}{2\bar{\sigma}_P} - \rho\right)T} \times \|x(kT)\| \tag{13}$$

Remark 2: As stated in remark 1, for small ϵ in (6), one can neglect the term $\underline{\sigma}_Q/2\bar{\sigma}_P$ in the exponent of (13), which then becomes:

$$\|x(kT+T)\| \le \frac{\underline{\sigma}_P}{\bar{\sigma}_P} \times e^{-[\beta^*-\rho]T} \times \|x(kT)\| \tag{14}$$

Recall again that ρ in (4) characterises the exponential growth rate of the open-loop unstable bilinear system. Hence, (14) states that an open-loop unstable bilinear system can be stabilised if the exponential decay rate β^* provided by the normalised control is larger than the growth rate ρ of the open-loop system.

Remark 3: Based on the observations in remarks 1 and 2, one concludes that β^* is actually a measure of the strength of the normalised control (4). Metaphorically speaking, the normalised control has an equivalent effect of, as in linear system control, shifting the open-loop eigenvalues to the left by a distance of β^*. Note that β^* is a function of the control gain γ. In practical use of the normalised control, one needs to simulate control with various control gains in order to find out the optimal gain γ to maximise β^*.

3 Discontinuous Normalised Quadratic Control

In the preceding Section, it was shown that the normalised control has a stabilising effect if the bilinear system satisfies the rank condition (7). However, this rank condition depends on the matrix $\boldsymbol{P}$ in (5) and hence the matrix $\boldsymbol{Q}$, where the latter is chosen by the control designer. This suggests that the designer's choice of controller parameters may affect whether or not the specific control design has a stabilising effect. To avoid such complexity, a different normalised control design will be developed in this Section, and the new normalised control is guaranteed to have an exponential stabilising effect under the original rank condition (3) in [7], which is independent of controller parameters.

For the second normalised control design, apply the following coordinate transformation to the bilinear system (1):

$$x(t) = e^{A(t-kT)} z_k(t) \qquad t \in [kT, kT+T) \tag{15}$$

where $z_k(t)$ is the transformed state, and the transformation matrix is the fundamental matrix [9] of the open-loop system (1). Note that the transformed state $z_k(t)$ is defined only on a finite time interval $[kT, (k+1)T)$; nevertheless, this will not affect the asymptotic stability analysis presented below.

With the transformation (15), the governing equation of the transformed state becomes:

$$\dot{z}_k(t) = u(t)\boldsymbol{G}_k(t)z_k(t) \quad \boldsymbol{G}_k(t) = e^{-A(t-kT)}Ne^{A(t-kT)} \\ t \in [kT, kT+T) \tag{16}$$

The above transformed system becomes open-loop neutrally stable, similar to the case treated in [7] except that now the neutrally stable system (16) contains time-varying parameters. However, one can still follow the control design principle in [7] to construct the following normalised quadratic feedback control:

$$u(t) = -\gamma e_z^T(t)\boldsymbol{G}_k(t)e_z(t) \quad e_z(t) \triangleq \frac{z_k(t)}{\|z_k(t)\|} \quad t \in [kT, kT+T) \tag{17}$$

where γ is a positive control gain. Note that this second normalised control is discontinuous at the time instants $t = kT$ because of the reset of the transformed state at those instants.

The stability analysis of the second normalised control (17) parallels that of the first normalised control (4); therefore, proofs of the following lemmas are omitted for brevity.

Lemma 3: If the system (1) satisfies the rank condition (3), and there exists a constant vector z_0 such that $z_0^T\boldsymbol{G}_k(t)z_0 \equiv 0$ for all $t \in [kT, kT+T)$, then z_0 must be the null vector.

To quantify the stabilising effect of the new normalised control (17), define a function $\alpha(\cdot)$: $S \to R^+ \cup \{0\}$, where S is the unit sphere in $\boldsymbol{R}^n$, for the transformed closed-loop system (16) and (17):

$$\alpha(e_z(kT)) \triangleq \frac{\gamma}{T}\int_{kT}^{(k+1)T} [e_z^T(t)\boldsymbol{G}_k(t)e_z(t)]^2 dt \quad e_z(kT) \in S \tag{18}$$

Lemma 4: If the bilinear system (16) satisfies the rank condition (3), there exists some positive constant α^* such that for all positive integer k:

$$\inf_{e_z(kT)\in S} [\alpha(e_z(kT))] = \alpha^* > 0 \tag{19}$$

One can now prove the global exponential stability of the bilinear system (1) under the second normalised control (17).

Theorem 3: Consider the bilinear system (1) and the normalised control (17). If:

$$e^{\alpha^* T} > \|e^{AT}\| \tag{20}$$

the closed-loop system is exponentially stable with the following decay rate

$$\|x(kT+T)\| \le \pi\|x(kT)\| \quad \text{where} \quad \pi = \frac{\|e^{AT}\|}{e^{\alpha^* T}} < 1 \tag{21}$$

Proof: In this proof, one will first estimate the variation of the transformed state $\|z_k(t)\|$ over the time interval $[kT, kT+T)$, and then utilises the obtained result to estimate the variation of the original state $\|x(t)\|$. To do so, first define:

$$V_k(t) = \frac{1}{2}\|z_k(t)\|^2 \quad t \in [kT, (k+1)T)$$

and take the time derivative of $V_k(t)$ along (16) and (17):

$$\dot{V}_k(t) = z_k^T(t)\boldsymbol{G}_k(t)z_k(t)u(t) = -2\gamma[e_z^T(t)G_k(t)e_z(t)]^2 V_k(t).$$

Integrating the above equation from kT to $(k+1)T$ gives:

$$\ln\frac{V_k(kT+T)}{V_k(kT)} = -2\gamma\int_{kT}^{(k+1)T} |e_z^T(t)\boldsymbol{G}_k(t)e_z(t)|^2 dt \le -2\alpha^* T$$

where the last inequality is obtained via lemma 4. Re-arranging the equation gives $V_k(kT+T) \le e^{-2\alpha^* T}V_k(kT)$, or equivalently:

$$\|z_k(kT+T)\| \le e^{-\alpha^* T}\|z_k(kT)\| \tag{22}$$

Next, to find out how the original system state $\|x(kT)\|$ varies, recall from (15) that:

$$x(kT) = z_k(kT) \quad \text{and} \quad x(kT+T) = e^{AT}z_k(kT+T) \tag{23}$$

Taking the norm of $x(kT+T)$ in (23), and using (22), one can then prove (21):

$$\|x(kT+T)\| \le \frac{\|e^{AT}\|}{e^{\alpha^* T}} \times \|z_k(kT)\| = \frac{\|e^{AT}\|}{e^{\alpha^* T}} \times \|x(kT)\|$$

where the last equality results from the first identity in (23). Finally, the assumption (20) guarantees that the closed-loop system is exponentially stable. □

Remark 4: Given any system matrix A, there exists a positive constant m such that:

$$\|e^{AT}\| \le m\, e^{\pm \rho T} \tag{24}$$

where the minus sign is for stable A, and the plus sign is for unstable A, and ρ as defined in (6). With (24), (21) becomes, for stable A:

$$\|x(kT+T)\| \le m\, e^{-(\alpha^*+\rho)T} \times \|x(kT)\| \tag{25}$$

and for unstable A:

$$\|x(kT+T)\| \le m\, e^{-(\alpha^*-\rho)T} \times \|x(kT)\| \tag{26}$$

Equation (25), which is similar to (12) in the preceding Section, suggests that when the system (1) is open-loop stable, the second normalised control (17) can add to the open-loop decay rate ρ an extra decay rate α^*. Equation (26), which is similar to (14) in the preceding Section, suggests that when the system (1) is open-loop unstable, the second normalised control (17) can still stabilise the closed-loop system if the decay rate α^* provided by the control is larger than the open-loop growth rate ρ. In conclusion, α^* defined in (19) plays the same role as β^* in the preceding Section; that is, α^* is a measure of the exponentially stabilising effect for the second normalised control (17).

Example 1: Consider an open-loop stable bilinear system (1) with:

$$A = \begin{bmatrix} -1 & -5 \\ 4 & -1 \end{bmatrix} \qquad N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and the initial condition $x^T(0) = [5, -4]$. Fig. 1 shows the open-loop response (the dash line) and the closed-loop response (the solid line) of the system under the second normalised quadratic control (17) with $\gamma = 11$ and $T = 1$. It is seen that the normalised quadratic control makes the decay rate three times faster than the open-loop. Fig. 2 depicts the normalised control signal.

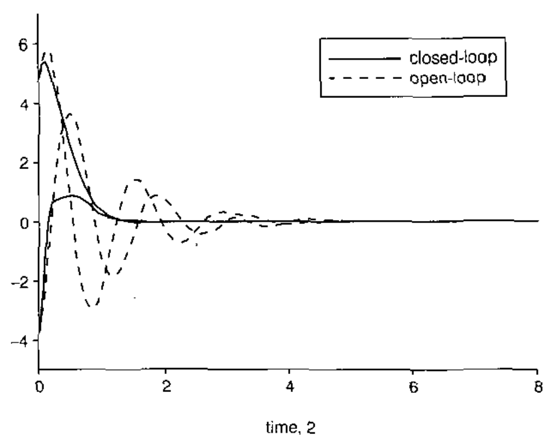

Fig. 1 *Controlled state of open-loop stable system*

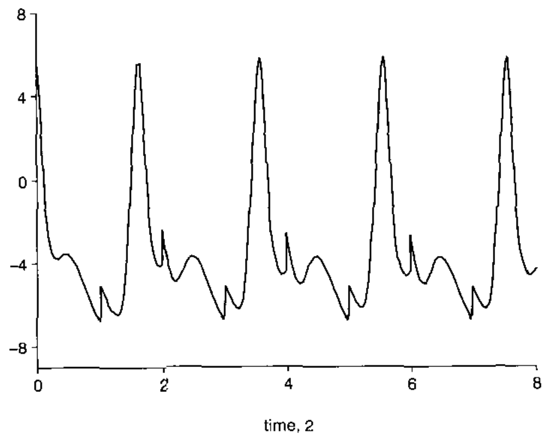

Fig. 2 *Control signal for open-loop system*

Example 2: Consider an open-loop unstable bilinear system (1) with:

$$A = \begin{bmatrix} 0.5 & 1 \\ -1 & 0.5 \end{bmatrix} \quad N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and $x^T(0) = [5, -4]$. Fig. 3 shows that the normalised quadratic control (17) with $\gamma = 2$ and $T = 1$ effectively stabilises the closed-loop system. Fig. 4 shows the control

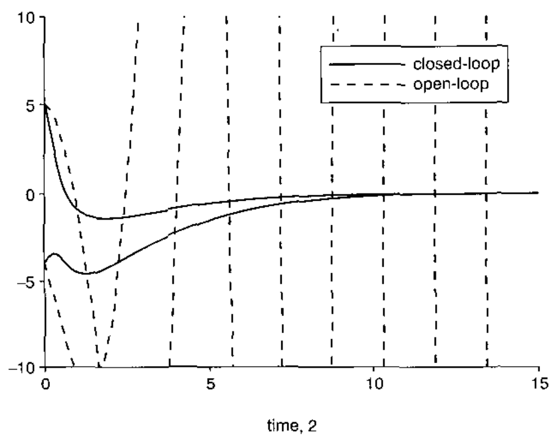

Fig. 3 *Controlled state of open-loop unstable system*

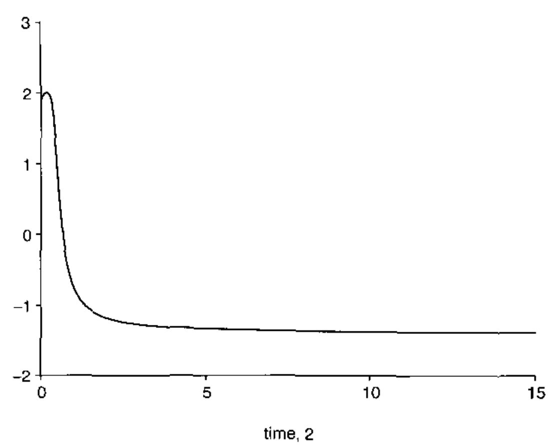

Fig. 4 *Control signal for open-loop unstable system*

signal, which seems to asymptotically approach a constant $u^* = -1.39$. However, one can verify that a constant control design $u(t) = u^*$ will not stabilise the bilinear system.

4 Conclusions

This paper studies two normalised quadratic controls (NQC) for a homogeneous bilinear system which satisfies the controllability rank condition. It is shown that the NQC can provide an extra exponential decay rate if the open-loop system is already stable, and can stabilise an open-loop unstable system if the extra exponential decay rate provided by the control is larger than the growth rate of the open-loop system. However, the theoretical analysis in this paper is more qualitative than quantitative in the sense that the extra decay rate provided by NQC, namely β^* in (10) or α^* in (19), is difficult to calculate. In practical use of NQC, one must simulate control with different control gains γ to find out if the NQC can stabilise a particular open-loop unstable system, and, if it can, what is the optimal control gain γ for achieving the fastest decay rate.

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6 References

1 SLEMROD, M.: 'Stabilization of bilinear control systems with applications to nonconservative problems in elasticity', *SIAM J. Control Optim.*, 1978, **16**, pp. 131–141
2 JURDJEVIC, V., and QUINN, J.P.: 'Controllability and stability', *J. Differ. Equ.*, 1978, **28**, pp. 381–389
3 VIDYASAGAR, M.: 'Nonlinear systems analysis' (Prentice Hall, Englewood Cliffs, New Jersey, 1993)
4 RYAN, E.P., and BUCKINGHAM, N.J.: 'On asymptotically stabilizing feedback control of bilinear systems', *IEEE Trans. Autom. Control*, 1983, **28**, pp. 863–864
5 SINGH, S.N.: 'Stabilizing feedback controls for nonlinear hamiltonian systems and nonconservative bilinear systems in elasticity', *Trans. ASME, J. Dyn. Syst. Meas. Control*, 1982, **104**, pp. 27–32
6 QUINN, J.P.: 'Stabilization of bilinear systems by quadratic feedback controls', *J. Math. Anal. Appl.*, 1980, **75**, pp. 66–80
7 CHEN, M.S.: 'Exponential stabilization of a constrained bilinear system', *Automatica*, 1998, **34**, pp. 989–992
8 MARSDEN, J.E., and HOFFMAN, M.J.: 'Elementary classical analysis' (W. H. Freeman and Company, New York, 1993)
9 CALLIER, F., and DESOER, C.A.: 'Linear system theory' (Springer-Verlag, Hong Kong, 1992)

7 Appendix

Lemma 5: Given the Lyapunov equation (5), one has

$$\lim_{\varepsilon \to 0} \frac{\underline{\sigma}_Q}{\bar{\sigma}_P} = 0$$

where $\bar{\sigma}_P$ denotes the maximum singular value of the matrix $\boldsymbol{P}$, and $\underline{\sigma}_Q$ the minimum singular value of $\boldsymbol{Q}$.

Proof: Without loss of generality, consider the case where $\boldsymbol{Q} = \boldsymbol{I}$ in (5). Therefore:

$$\underline{\sigma}_Q = 1 \tag{27}$$

$$\begin{aligned}\bar{\sigma}_P &= \max_{\|y_0\|=1} \int_0^\infty \boldsymbol{y}_0^T e^{(A \pm \rho I)^T t} e^{(A \pm \rho I)t} \boldsymbol{y}_0 dt \\ &= \max_{\|y_0\|=1} \int_0^\infty \|y(t)\|^2 dt \end{aligned} \tag{28}$$

where $y(t) = e^{(A \pm \rho I)t} \boldsymbol{y}_0$ denotes the solution of the differential equation $\dot{y} = (\boldsymbol{A} \pm \rho I)y$ subject to the initial condition $y(0) = \boldsymbol{y}_0$. By the definition of ρ in (6), the matrix $\boldsymbol{A} \pm \rho I$ is stable. Denote its eigenvalues by $\lambda_i = -\sigma_i + j\omega_i$ with $\sigma_{i+1} \geq \sigma_i$, $i = 1, 2, \ldots, n$. Then, according to (6):

$$\sigma_1 = \varepsilon \tag{29}$$

Asymptotically, the time history of $y(t)$ is determined by the dominant mode $i = 1$; hence, $\lim_{t \to \infty} y(t) = c_1 e^{-\sigma_1 t} \sin(\omega_1 t + \phi_1)$ for some constants c_1 and ϕ_1. Substituting $y(t)$ into (28), and using (29), one can show that:

$$\bar{\sigma}_P = O\left(\frac{1}{\varepsilon}\right) \tag{30}$$

The lemma is then proved by combining (27) and (30).