COMPUTING PSEUDO WIGNER DISTRIBUTION BY THE FAST HARTLEY TRANSFORM

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ABSTRACT

The Wigner distribution (WD) is useful in time-frequency signal analysis. In this paper the fast Hartley transform (FHT) is proposed to evaluate the WD entirely in the real domain, the computation complexity is reduced greatly from 3 complex FFT's to 3 real FHT's.

1. Introduction

The Wigner distribution (WD) is a potentially useful tool for analyzing the time-varying signals, it has some important properties over the conventional time-frequency transform [1]. The existing fast Fourier transform (FFT) method is often used to calculate the Wigner distribution, recently a new method is proposed to evaluate the WD with a real signal using the fast Hartley transform (FHT) by Berriel-Valdós, Gonzalo and Bescós [2]. However, this WD with a real signal has the aliasing problem and exhibits low-frequency artifacts [3], these low-frequency artifacts are caused by interaction between positive and negative frequencies. Boashash [3] has shown that all these problems do not occur when one uses the Wigner-Ville distribution with the analytic signal, no aliasing happens because the spectrum of the analytic signal is zero for negative frequencies. The analytic signal of a real signal is complex, and consists of the real part with the signal itself and the imaginary part with its Hilbert transform, Pei and Jaw [4] develop a fast algorithm to compute the discrete Hilbert transform through fast Hartley transform (FHT); Here the FHT is used to evaluate the Wigner-Ville distribution with the analytic signal, the operation is performed entirely in the real domain, the computation complexity can be greatly reduced from 3 complex FFT's into 3 real FHT's.

2. Pseudo Wigner Distribution with Windowing and FHT

Suppose f(n) is a discrete-time signal, then the Wigner distribution for f(n) is defined by [1]:

$$WD_f(n,\theta)=2\sum_{k=-\infty}^{\infty} e^{-j2k\theta} f(n+k)f^*(n-k) \quad (1)$$

where f^* is the complex conjugate of f . This formulation is only useful for finite duration signal; for analyzing infinite duration signal, it's general to weight the signal f by a function W before evaluating the Wigner distribution, this weighting function is often called the window and will slide along the time axis with the instant n where the Wigner distribution has to be evaluated, this windowed Wigner distribution known as the Pseudo Wigner distribution (PWD) [1] may be expressed as

$$PWD_f(n,m\pi/M)=\sum_{k=-(L-1)}^{L-1} e^{-j2\pi km/M} w(k)w^*(-k)\cdot f(n+k)f^*(n-k) \quad (2)$$

The window w(k) has a length 2L-1 and w(k)=0 for $|k|\geq L$, the pseudo Wigner distribution is often calculated by the FFT procedure. However, such an FFT requires an even number of points. This can be solved easily by adding a zero, so that M=2L.

Let G(n,-L)=0

$$G(n,k)=w(k)f(n+k) \quad k=-(L-1),\dots,0,\dots,(L-1)$$

$$\text{then } PWD(n,m\pi/M)=2\sum_{k=-L}^{L-1} e^{-j2\pi km/M} G(n,k)G^*(n,-k) \quad (3)$$

Since an FFT is usually evaluated with the boundaries 0 and M-1. This can also be overcome by renumbering the sequence [5] or introducing a 180° phase change shown as below (see also Fig.1):

A) Renumbering method:

Let

$$x(n,k)=\begin{cases} G(n,k)G^*(n,-k) & k=0,\dots,L-1 \\ G(n,k-2L)G^*(n,-k+2L) & k=L,\dots,2L-1 \end{cases} \quad (4)$$

$$\text{Then } PWD(n,m\pi/M)=2\sum_{k=0}^{M-1} e^{-j2\pi km/M} x(n,k) \quad (5)$$

B) Phase change method:

We find out that the PWD can also be calculated without renumbering the sequence by introducing a 180° phase change

Let $x'(n,k)=G(n,k-L)G^*(n,-(k-L)) \quad k=0,\dots,2L-1$

$$\text{Then } PWD(n,m\pi/M)=2\sum_{k=0}^{M-1} (-1)^m e^{-j2\pi km/M} x'(n,k) \quad (6)$$

Due to the low-frequency artifacts of the PWD with a real signal, it's desired to convert the original signal into its analytic singal before

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calculating its PWD.

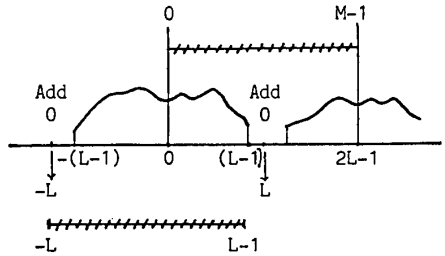

Fig. 1 Renumbering the Sequence for PWD Calculation.

Let $\hat{G}(n,k)$ be the discrete Hilbert transform of $G(n,k)$, the analytic signal is defined as

$$G_a(n,k) = G(n,k) + j\hat{G}(n,k) \qquad (7)$$

Pei and Jaw [4] have developed a fast algorithm to compute the discrete Hilbert transform of $G(n,k)$ through FHT, Fig.2 shows the flow-chart of this algorithm. Also the PWD with the analytic signal is of the form:

$$PWD(n,m\pi/M)=2\sum_{k=-L}^{L-1} e^{-j2\pi km/M} G_a(n,k)G_a^*(n,-k) \qquad (8)$$

Let $z(n,k)=G_a(n,k)G_a^*(n,-k)$ (9)

Since $z^*(n,-k)=z(n,k)$, $z(n,k)$ is hermitian symmetric with respect to k, thus $z(n,k)$ can be written as

$$z(n,k) = E(n,k) + jO(n,k) \qquad (10)$$

where $E(n,k)$ is an even function of k and $O(n,k)$ is an o d function of k. Then

$$PWD(n,m\pi/M)$$

$$=2\sum_{k=-L}^{L-1} e^{-j2\pi km/M} z(n,k)$$

$$=2\sum_{k=-L}^{L-1} \left(\cos\frac{2\pi Km}{M}-j\sin\frac{2\pi km}{M}\right)\cdot\left(E(n,k) + jO(n,k)\right)$$

$$=2\sum_{k=-L}^{L-1} \left[\cos\frac{2\pi km}{M}E(n,k)+\sin\frac{2\pi km}{M}O(n,k)\right]$$

$$=2\sum_{k=-L}^{L-1} [E(n,k)+O(n,k)]\left[\cos\frac{2\pi km}{M}+\sin\frac{2\pi km}{M}\right]$$

$$=2\,DHT\{\text{ Renumber }[E(n,k)+O(n,k)]\} \qquad (11)$$

Where DHT is the discrete Hartley transform and is defined as [6]:

$$H_f(m)=\sum_{k=0}^{M-1} f(k)cas\left(\frac{2\pi km}{M}\right) \quad m=0,1,\ldots,M-1 \qquad (12)$$

Where $cas(x)=cos(x)+sin(x)$, the inverse relation is

$$f(k)=\frac{1}{M}\sum_{m=0}^{M-1} H_f(m)cas\left(\frac{2\pi km}{M}\right) \quad k=0,1,\ldots,M-1 \qquad (13)$$

Recently, the discrete Hartley transform [6] has been considered as an interesting alternative to the FFT for spectral analysis and fast convolution of real data [7], and many efficient fast algorithms have been developed for computing the discrete Hartley transform [8][9][10].

If $f(n)$ and $w(n)$ are real, then $G(n,k)$ and $\hat{G}(n,k)$ are real, using Eqs.(9) and (10), Eq.(11) can be further simplified as:

$$z(n,k)$$

$$=G_a(n,k)\cdot G_a^*(n,-k)$$

$$=[G(n,k)+j\,\hat{G}(n,k)][G(n,-k)-j\cdot\hat{G}(n,-k)]$$

$$=[G(n,k)G(n,-k)+\hat{G}(n,k)\hat{G}(n,-k)]+ j\cdot[\hat{G}(n,k)G(n,-k)-G(n,k)\hat{G}(n,-k)]$$

$$=E(n,k)+j\cdot O(n,k) \qquad (14)$$

Thus $E(n,k)=G(n,k)G(n,-k)+\hat{G}(n,k)\hat{G}(n,-k)$ (15a)

$$O(n,k)=\hat{G}(n,k)G(n,-k)-G(n,k)\hat{G}(n,-k) \qquad (15b)$$

Let $S(n,k)=E(n,k)+O(n,k)$

$$=G(n,k)[G(n,-k)-\hat{G}(n,-k)]+ \hat{G}(n,k)[G(n,-k)+\hat{G}(n,-k)] \qquad (16)$$

Then Eq.(11) can be written as follows:

$$PWD\left(n,\frac{m\pi}{M}\right) = 2DHT\{\text{ Renumber }[s(n,k)]\}$$

$$m=0,\ldots,M-1. \quad n=n_{min},\ n_{max}. \qquad (17)$$

Fig.3 shows the flow chart of the PWD calculation by FHT. Compare with the conventional FFT approach, it's performed entirely in the real domain, also many efficient FHT algorithms exist and can be applied here, the computation complexity is greatly reduced from 3 complex FFT's into 3 real FHT's, almost half of the operations are saved with the FHT approach.

3. Experimental Results

A numerical example is demonstrated below to calculate the PWD by FHT. The signal is the sum of two linear chirp with different frequency increasing rate and different time duration intervals.

$$x(t) = \cos[2\pi Bt^2/2+2\pi f_0 t]$$

$$S_1(t) = \begin{cases} x(t-4), & \text{for } t \text{ in } [1,7], \text{ with } B=4,\ f_0=22. \\ 0, & \text{elsewhere.} \end{cases}$$

$$S_2(t) = \begin{cases} x(t-6), & \text{for } t \text{ in } [3,9], \text{ with } B=2,\ f_0=10. \\ 0, & \text{elsewhere.} \end{cases}$$

A zero mean white gaussian noise $N(0,0.1,n)$ with standard deviation 0.1 is added to the signal, we have sampled the signal with the sampling interval T=0.01, a 31-point Hamming window is used in the experiment.

$$s(n)=s_1(nT)+s_2(nT)+N(0,0.1,n)$$

with $T=0.01$, $n_{min}=30$, $n_{max}=960$, $L=16$, $M=32$.

Fig.4(a) and (b) show the PWD's of s(n) indistinguishably with FFT and FHT respectively. It's composed of two linear chirps and the cross-term in between, and doesn't introduce any low-frequency artifacts for a better time-frequency signal analysis. The FHT procedure runs much faster at no loss in accuracy than the usual FFT approach.

4. Conclusions

In this paper, we have presented a new fast algorithm for computing the pseudo Wigner distribution through fast Hartley transform. Compare with the conventional FFT approach, it's performed entirely in the real domain, also many efficient FHT algorithms exist and can be applied here, the computation complexity is greatly reduced from 3 complex FFT's into 3 real FHT's.

References

[1] T.A.C.M. Claasen and W.F.G. Mecklenbraüker, "The Wigner distribution--A tool for time-frequency signal analysis. Part I: Continuous-time signals; Part II: Discrete-time signals; Part III: Relation with other time-frequency signal transformations", Phillips J. Res., Vol.35, pp.217-250, pp.276-300, and pp.372-389, 1980.

[2] L.R. Berriel-Valdós, C. Gonzalo and J. Bescós, "Computation of the Wigner distribution function by the Hartley transform, application to image restoration", Optics Communications, Vol.68, No.5, pp.339-344, 1 Nov. 1988.

[3] B. Boashash, "Note on the use of the Wigner distribution for time-frequency signal analysis", IEEE Trans. on Acoustics, Speech and Signal Proc., Vol.38, No.9, pp.1518-1521, Sept. 1988.

[4] S.C. Pei and S.B. Jaw, "Computation of discrete Hilbert transform through fast Hartley transform," published IEEE Trans. on Circuits and Systems, Sept. 1989, Vol.CAS-36, No.9, pp.1251-1252.

[5] C.P. Janse and A.J.M. Kaizer, "Time-frequency distributions of loudspeakers: The application of the Wigner distribution," J. Audio Eng. Soc., Vol.31, No.4, pp.198-222, 1983.

[6] R.N. Bracewell, "Discrete Hartley transform", J. Opt. Soc. Amer. Vol.73, pp.1832-1835, Dec. 1983.

[7] R.N. Bracewell, "The fast Hartley transform," Proc. IEEE, Vol.72, No.8, Aug. 1984, pp.1010-1018.

[8] S.C. Pei and J.L. Wu, "Split-radix fast Hartley transform," Electronics Letters, Vol.22, No.1, 2nd Jan. 1986, pp.26-27.

[9] H.V. Sovensen, D.L. Jones, C.S. Burrus and M.T. Heideman, "On computing the discrete Hartley transform," IEEE Trans. on Acoustics Speech and Signal Proc., Vol.ASSP-33, No.5, pp.1231-1238, Oct. 1985.

[10] H.S. Hou, "The fast Hartley transform algorithm," IEEE Trans. on Computers, Vol.C-36, No.2, pp.147-156, Feb. 1987.

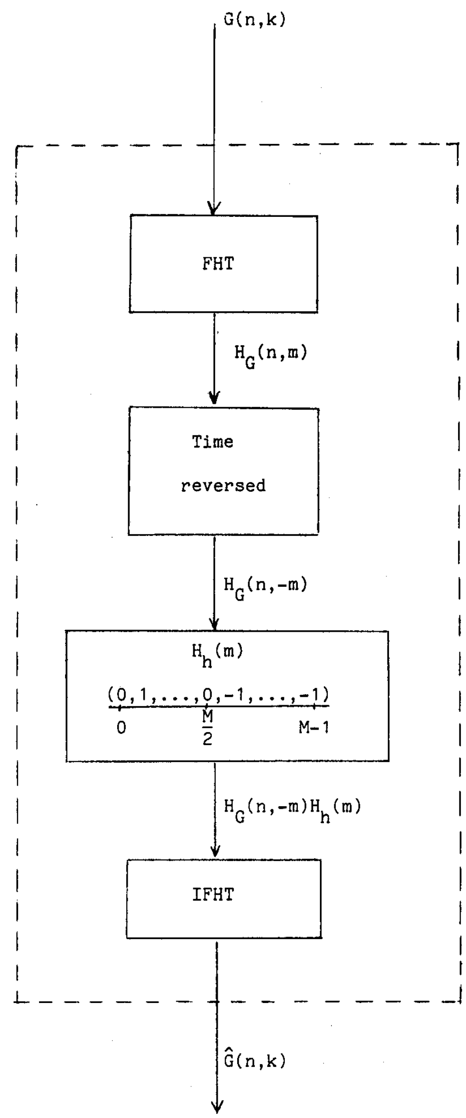

Fig. 2 Flow-chart of Discrete Hilbert Transform by FHT.

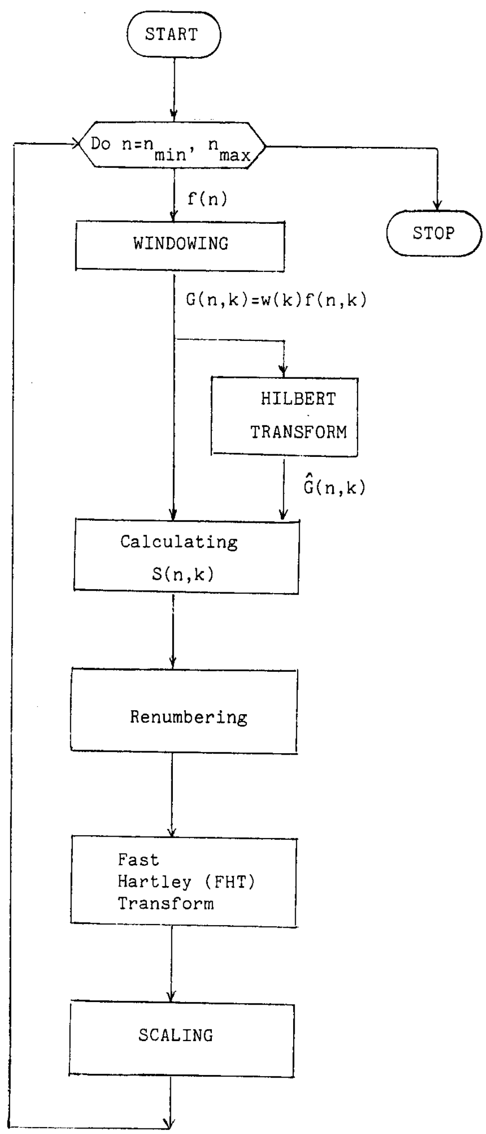

Fig. 3 Flow-Chart of Pseudo.Wigner Distribution by FHT.

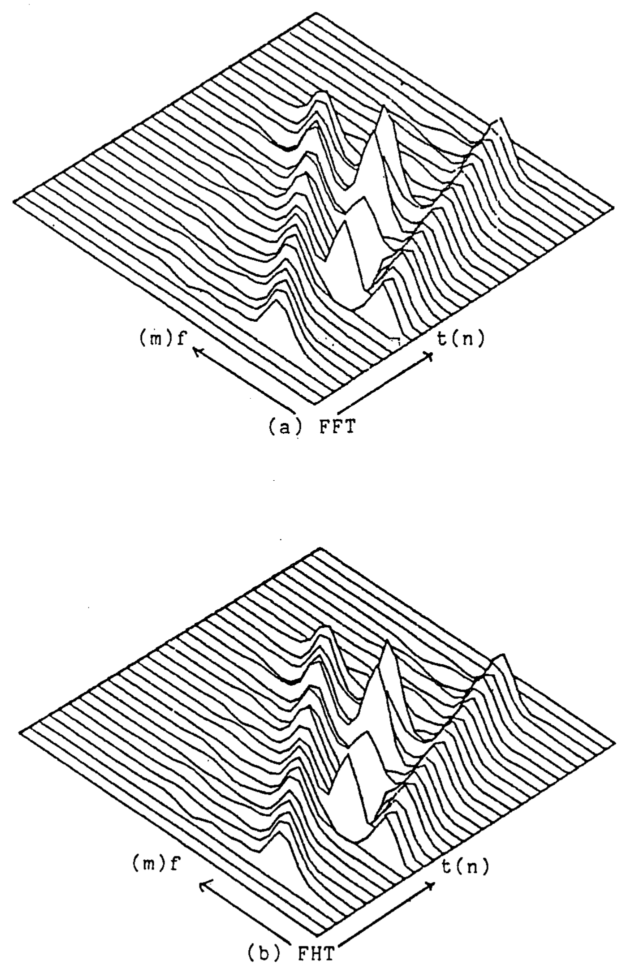

Fig. 4 Pseudo Wigner Distribution of Two Linear Chirp Signals by
(a) FFT Approach (b) FHT Approach,