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Brief Paper

Exponential Stabilization of a Constrained Bilinear System*

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Key Words—Bilinear system; quadratic control; nonlinear control; exponential stability; global stability; saturation.

Abstract—For a bilinear system that is open-loop neutrally stable, a quadratic state feedback control has been proposed to ensure global asymptotical stability of the closed-loop system. In this paper, a new nonlinear control is proposed so that the closed-loop system is not only asymptotically stable but also *exponentially* stable. The new control results in a much faster state convergence rate than the quadratic control; furthermore, it can be applied to systems with tight saturation limits on the control input. © 1998 Elsevier Science Ltd. All rights reserved.

1. *Introduction*

This paper considers the control of a bilinear system

$$\dot{x}(t) = Ax(t) + u(t)Nx(t), \quad x(0) = x_0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state vector, $u(t)$ is a scalar control input, and $A \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{n\times n}$ are constant square matrices. It is assumed that there exists is a positive-definite matrix Q such that

$$A^{\mathrm{T}}Q + QA = 0, \tag{2}$$

in other words, the open-loop system is neutrally stable (Slemrod, 1978). Furthermore, the pair (A, N) satisfies the following controllability assumption (Vidyasagar, 1993): there exists an integer $m(\geq n-1)$ such that

$$span\{ad^k(A,N)x_0, k = 0, 1, 2, \ldots, m\} = \mathbb{R}^n \tag{3}$$

for any *nonzero* x_0 in $\mathbb{R}^n$, where $ad^k(A, N)$'s are defined recursively by

$$ad^0(A, N) = N,$$

$$ad^{k+1}(A, N) = A \cdot ad^k(A, N) - ad^k(A, N)\cdot A, \quad k = 0, 1, 2 \ldots .$$

Conventionally, quadratic feedback control (e.g. Jurdjevic and Quinn, 1978; Singh, 1982; Ryan and Buckingham, 1983) has been proposed for the stabilization of the system (1):

$$u(t) = -x^{\mathrm{T}}(t)QNx(t), \tag{4}$$

which ensures global asymptotic stability of the closed-loop system. However, Quinn (1980) has shown that the controlled system is *not* exponentially stable, and the state converges as

$$\|x(t)\| \sim \frac{1}{\sqrt{t}}. \tag{5}$$

The objective of this work is to introduce a new nonlinear control which stabilizes the closed-loop system *globally* and most importantly *exponentially*. The exponential stability results in a much faster time response of the system state than in equation (5); furthermore, it enhances the robustness of the controlled system (Callier and Desoer, 1991).

2. *Nonlinear control*

The proposed nonlinear control is as follows:

$$u(t) = \begin{cases} -\rho \dfrac{x^{\mathrm{T}}(t)}{\|x(t)\|} QN \dfrac{x(t)}{\|x(t)\|}, & x(t) \neq 0, \\ 0, & x(t) = 0, \end{cases} \tag{6}$$

where ρ is a positive control gain, and Q is as in equation (2). Notice that the control (6) is uniformly bounded for whatever values of the state $x(t)$:

$$|u(t)| \leq \rho n q, \quad \forall t > 0, \tag{7}$$

where q and n are, respectively, the matrix norms of Q and N. If the bilinear system (1) is subject to the control constraint

$$|u(t)| \leq u_{\max},$$

the control gain ρ in equation (6) will have to be chosen within the range:

$$\rho \in \left(0, \frac{u_{\max}}{nq}\right). \tag{8}$$

3. *Stability analysis*

Lemma 1. If the system (1) satisfies the controllability assumption (3), and there exists a constant vector x_0 such that

$$x_0^{\mathrm{T}} e^{A^{\mathrm{T}}(t-kT)} QN e^{A(t-kT)} x_0 \equiv 0, \quad \forall t \in [kT, kT+T) \tag{9}$$

for some $T > 0$, then x_0 must be the null vector.

Proof. Taking consecutively the time derivatives of equation (9) at $t = kT$, and using equation (2) repeatedly, one obtains

$$x_0^{\mathrm{T}} Q ad^0(A, N) x_0 = z_0^{\mathrm{T}} ad^1(A, N)\, Q x_0 = \cdots = x_0^{\mathrm{T}} Q ad^m(A, N) x_0 = 0,$$

where $ad^k(A, N)$ is as given in equation (3). These identities can be put into a matrix form

$$x_0^{\mathrm{T}} Q [ad^0(A, N)x_0, ad^1(A, N)x_0, \ldots, ad^m(A, N)x_0] = 0. \tag{10}$$

From assumption (3), the matrix in equation (10) has full rank. Therefore, $x_0^{\mathrm{T}} Q = 0$ and hence $x_0 = 0$ since Q is positive definite. □

Given any time interval length $T > 0$, define a scalar function $B(\cdot): S \to R$ for the controlled system (1) and (6), where S is the unit sphere in $\mathbb{R}^n$,

$$B\left(\frac{x(kT)}{\|x(kT)\|}\right) \triangleq \int_{kT}^{(k+1)T} \left(\frac{x^{\mathrm{T}}(t)}{\|x(t)\|} QN \frac{x(t)}{\|x(t)\|}\right)^2 \mathrm{d}t, \quad x(t) \neq 0. \tag{11}$$

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Note that given Q and N, the function $B(\cdot)$ is determined by the normalized closed-loop trajectory $x(t)/\|x(t)\|$, $t \in [kT, kT + T)$, which is uniquely determined by its initial condition $x(kT)/\|x(kT)\|$. Therefore, $B(\cdot)$ in equation (11) is defined as a function of the initial condition $x(kT)/\|x(kT)\|$.

Lemma 2. There exists some positive constant β such that

$$\inf\left[B\left(\frac{x(kT)}{\|x(kT)\|}\right)\right] = \beta > 0, \tag{12}$$

where the *inf(imum)* is taken over all $x(kT)/\|x(kT)\| \in S$ (that is, over all $x(kT) \neq 0$).

Proof. Since the integrand in equation (11) is nonnegative, $B(\cdot)$ must be nonnegative. It will further be shown that $B(\cdot)$ is, in fact, positive for each $x(kT)/\|x(kT)\| \in S$. A contradiction argument will be used for this purpose. Assume that $B(x(kT)/\|x(kT)\|)$ is zero for some $x(kT)/\|x(kT)\| \in S$ (that is, for some *nonzero* $x(kT)$). By the definition of $B(\cdot)$, one has

$$\frac{x(t)^{\mathrm{T}}}{\|x(t)\|} QN \frac{x(t)}{\|x(t)\|} \equiv 0, \quad \forall t \in [kT, kT + T) \tag{13}$$

suggesting that

$$u(t) \equiv 0, \quad \forall t \in [kT, kT + T),$$

according to equation (6). Hence, following the open-loop dynamics (1), one has

$$x(t) = \mathrm{e}^{A(t-kT)} x(kT), \quad \forall t \in [kT, kT + T). \tag{14}$$

Substituting equation (14) into equation (13) gives

$$\frac{x(kT)^{\mathrm{T}}}{\|x(kT)\|} \mathrm{e}^{A^{\mathrm{T}}(t-kT)} QN \mathrm{e}^{A(t-kT)} \frac{x(kT)}{\|x(kT)\|} \equiv 0, \quad \forall t \in [kT, kT + T).$$

Now, by applying Lemma 1 to the above equation, one can deduce that

$$\frac{x(kT)}{\|x(kT)\|} = 0,$$

contradicting the fact that $x(kT)/\|x(kT)\| \in S$ (or $x(kT) \neq 0$). Therefore, one concludes that

$$B\left(\frac{x(kT)}{\|x(kT)\|}\right) > 0 \quad \text{for each } \frac{x(kT)}{\|x(kT)\|} \in S. \tag{15}$$

Further, note that $B(\cdot)$ in equation (11) depends continuously on $x(t)$, $t \in [kT, kT + T)$. Since the right-hand side of equation (1) with $u(t)$ given by the control law (6) has continuous first-order derivative with respect to $x(t)$ (for nonzero $x(t)$), it follows from Theorem 7.2 (Coddington and Levinson, 1955) that the closed-loop solution $x(t)$ depends *continuously* on the initial condition $x(kT)$. As a result, $B(\cdot)$ depends continuously on its argument $x(kT)/\|x(kT)\|$. Since the domain of $B(\cdot)$, S, is compact, it follows from equation (15) and Theorem 4.4.1 (Marsden and Hoffman, 1993) that there exists a positive constant β such that equation (12) holds. □

One can now prove the global exponential stability of the controlled bilinear system.

Theorem. Consider the bilinear system (1) and the nonlinear control (6) subject to the constraint (8). Given any initial condition, the controlled state $x(t)$ converges to zero *exponentially*.

Proof. Define a Lyapunov function candidate

$$V(t) = x^{\mathrm{T}}(t) Q x(t),$$

where Q is as in equation (2). Notice that

$$x^{\mathrm{T}}(t) Q x(t) \leq \bar{\lambda} \|x\|^2, \tag{16}$$

where $\bar{\lambda}$ is the maximum eigenvalue of the positive-definite matrix Q. The time derivative of $V(t)$ along equations (1) and (6) is given by

$$\begin{aligned} \dot{V}(t) &= x^{\mathrm{T}}(t)(A^{\mathrm{T}}Q + QA)x(t) + 2x^{\mathrm{T}}(t)QNx(t)u(t) \\ &= -2\rho\left(\frac{x^{\mathrm{T}}(t)}{\|x(t)\|} QN \frac{x(t)}{\|x(t)\|}\right)^2 \|x(t)\|^2 \leq 0. \end{aligned} \tag{17}$$

Since $V(t)$ is nonincreasing, one has

$$V(kT + T) \leq V(t), \quad \forall t \in [kT, kT + T). \tag{18}$$

Integrating equation (17) from kT to $(k + 1)T$ yields

$$\begin{aligned} &V(kT + T) - V(kT) \\ &= -2\rho \int_{kT}^{(k+1)T} \left(\frac{x^{\mathrm{T}}(t)}{\|x(t)\|} QN \frac{x(t)}{\|x(t)\|}\right)^2 \frac{\|x(t)\|^2}{x^{\mathrm{T}}(t)Qx(t)} V(t)\mathrm{d}t, \\ &\leq -2\frac{\rho}{\bar{\lambda}} V(kT + T) \int_{kT}^{(k+1)T} \left(\frac{x^{\mathrm{T}}(t)}{\|x(t)\|} QN \frac{x(t)}{\|x(t)\|}\right)^2 \mathrm{d}t, \\ &\leq -2\frac{\rho\beta}{\bar{\lambda}} V(kT + T), \end{aligned}$$

where the first inequality results from equations (16) and (18), and the second from equation (12) in Lemma 2. Rearranging the last inequality gives

$$V(kT + T) \leq \frac{1}{1 + 2\rho\beta/\bar{\lambda}} V(kT) \tag{19}$$

proving that the Lyapunov function $V(kT)$ decreases exponentially to zero as k approaches infinity, and so does $x(kT)$.

Finally, it remains to show that the *continuous* state $x(t)$ remains bounded and also converges to zero exponentially. To this end, note from equations (1) and (7) that

$$\|\dot{x}(t)\| \leq (a + \rho n^2 q)\|x(t)\|, \tag{20}$$

where a is the matrix norm of the open-loop system matrix A. Taking the time derivative of the identity $\|x(t)\|^2 = x^{\mathrm{T}}(t)x(t)$, one obtains

$$2\|x(t)\| \frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\| = 2x^{\mathrm{T}}(t)\dot{x}(t) \leq 2\|x(t)\| \cdot \|\dot{x}(t)\|, \tag{21}$$

where the inequality results from the Schwartz inequality (Marsden and Hoffman, 1993). Cancelling $\|x(t)\|$ from equation (21) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\| \leq \|\dot{x}(t)\|, \tag{22}$$

One can then derive from equations (20) and (22) that

$$\|x(t)\| \leq \mathrm{e}^{(a + \rho n^2 q)(t - kT)} \|x(kT)\|, \quad \forall t \in [kT, kT + T).$$

Hence, the continuous state $x(t)$ remains bounded and converges exponentially to zero as the discrete state $x(kT)$ does. □

Remark. Notice that the theorem holds for whatever value of the control saturation limit $u_{\max} > 0$ as long as the control gain ρ satisfies equation (8). As a result, even if the control actuator can provide only a small amount of energy (i.e. a tight saturation limit $u_{\max}$), the proposed control can still stabilize the system globally and exponentially. Such a property is not shared by the conventional quadratic control (4), for which the amount of energy required is proportional to the square of $\|x(t)\|$. Hence, large control input is required by the quadratic control (4) if $x(t)$ is far from the origin.

4. *Simulation examples*

Example 1. Consider the bilinear system (1) with

$$A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

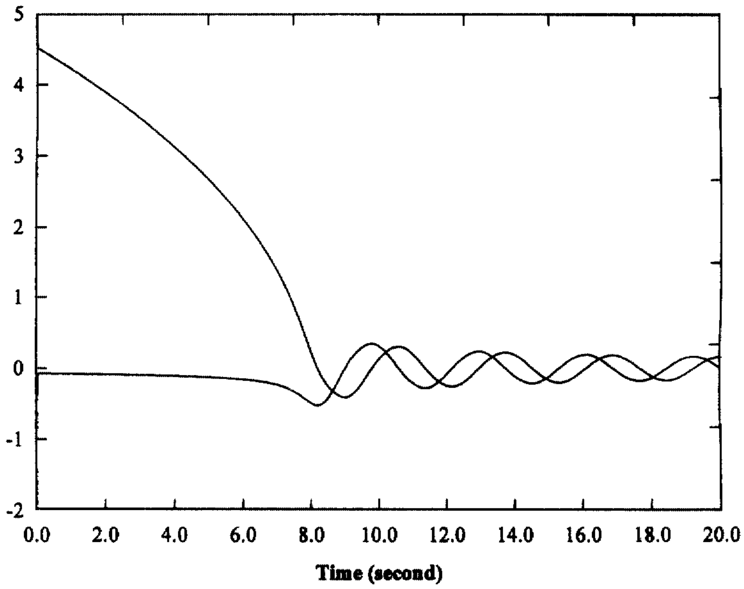

Fig. 1. State response with quadratic control.

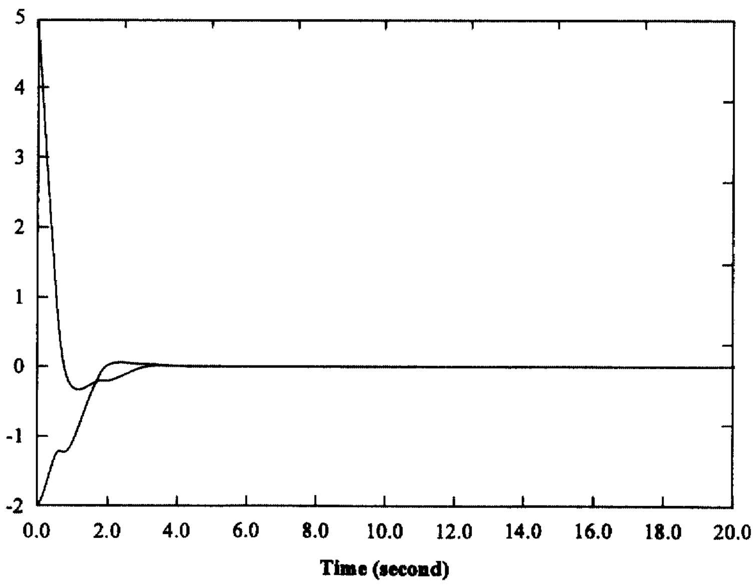

Fig. 2. State response with new nonlinear control.

and the initial condition $x^T(0) = [5, -2]$. Figure 1 shows the state response of the system with the conventional quadratic control (4) with $Q = 3I$, and Fig. 2 the state response with the new nonlinear control (5) with $\rho = 3$, $Q = I$. It is obvious that the new nonlinear control results in a much faster time response since the state now decays exponentially.

Example 2. Consider the same system as in the previous example but with a perturbation on the open-loop system matrix

$$\Delta A = \begin{bmatrix} 0.4 & 0 \\ 0 & 0 \end{bmatrix}.$$

The perturbed open-loop system becomes slightly unstable, but still controllable in the sense of equation (3). The initial condition is $x^T(0) = [5, -2]$. Figure 3 shows that the closed-loop system becomes unstable under the conventional quadratic control (4). However, the new nonlinear control ($\rho = 3$, $Q = I$) can still stabilize the perturbed system as is indicated by Fig. 4. The reason why the new control can stabilize a slightly perturbed system is as follows. Since the closed-loop system with the proposed control is *exponential* stable, one can show, following Theorem 121 in Chap. 7 in Callier and Desoer (1991), that the exponential stability is retained given any *small* perturbation in the open-loop system matrix A in equation (1). Note that such slightly perturbed system may not be stabilized by the conventional quadratic control (4) due to the lack of exponential stability for the nominal closed-loop system.

5. *Conclusions*

In this paper, a new nonlinear control different from the conventional quadratic feedback control is proposed to stabilize a homogeneous-in-the-state bilinear system. The new control results in exponential stability of the closed-loop system, and hence a much faster time response than with the quadratic control.

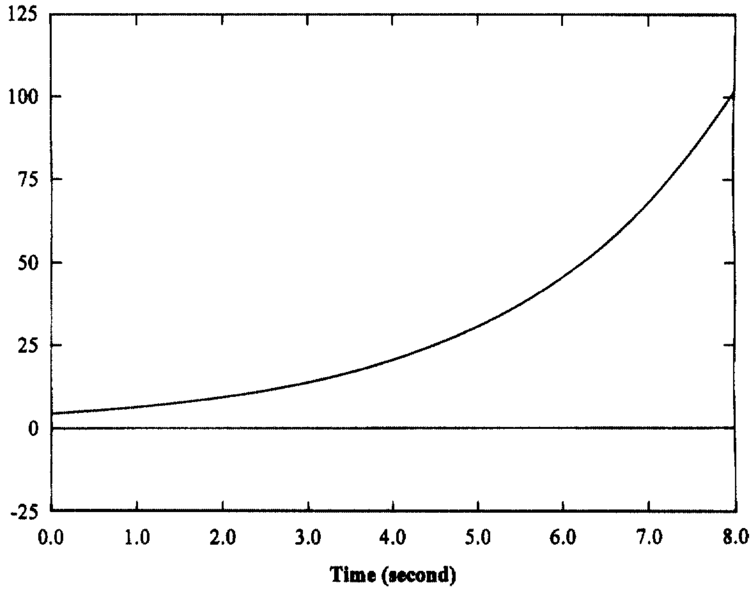

Fig. 3. Perturbed response with quadratic control.

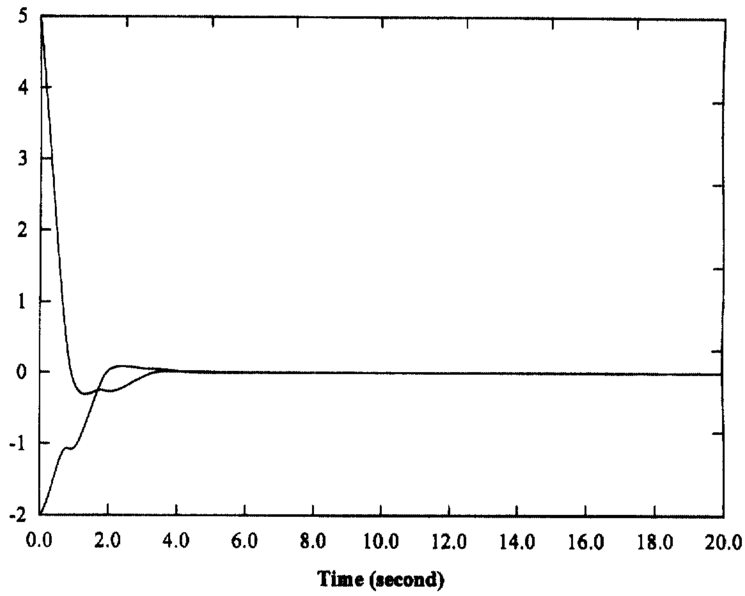

Fig. 4. Perturbed response with new nonlinear control.

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