Vector

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1. Representation of vector $\vec{A}$ by basis vector $\hat{e}_i$

$$\vec{A} = \sum_{i=1}^{3} A_i \cdot \hat{e}_i \quad , \quad \hat{e}_i : \text{basis vector}, \ |\hat{e}_i| = 1$$

if $\hat{e}_1 = \hat{\imath}, \hat{e}_2 = \hat{\jmath}, \hat{e}_3 = \hat{k}$ along the x, y, and z axis respectively.

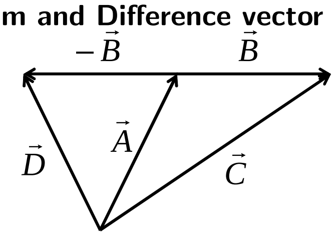
then

$$\vec{A} = A_x \cdot \hat{\imath} + A_y \cdot \hat{\jmath} + A_z \cdot \hat{k}$$

A_x : projection or component of $\vec{A}$ on x-axis, etc.

$$\left|\vec{A}\right| \equiv A = \left(A_x^2 + A_y^2 + A_z^2\right)^{\frac{1}{2}} \quad \text{magnitude of the vector}$$

2. Sum and Difference vector

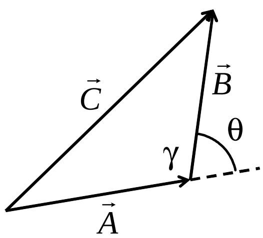


$$\vec{C} = \vec{A} + \vec{B} \ , \ \ \vec{D} = \vec{A} - \vec{B}$$

$$\vec{C} = \sum_{i=1}^{3} C_i \cdot \hat{e}_i = \sum A_i \cdot \hat{e}_i + \sum B_i \cdot \hat{e}_i = \sum_{i=1}^{3} (A_i + B_i) \cdot \hat{e}_i \qquad \therefore C_i = A_i + B_i$$

$$\vec{D} = \sum_i (A_i - B_i) \cdot \hat{e}_i \ , \qquad \therefore D_i = A_i - B_i$$

3. Scalar product



$$\vec{A} \cdot \vec{B} = A \cdot B \cdot \cos\theta$$

$$\left|\vec{A}\right| = \left(\vec{A} \cdot \vec{A}\right)^{\frac{1}{2}}$$

$$\left(\vec{A} + \vec{B}\right)^2 = \vec{C}^2 = C^2 = \left(\vec{A} + \vec{B}\right) \cdot \left(\vec{A} + \vec{B}\right) = A^2 + B^2 + 2\vec{A} \cdot \vec{B}$$

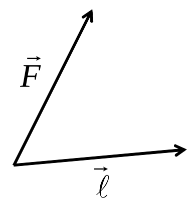
By cosine theorem $\quad A^2 + B^2 - 2A \cdot B \cdot \cos\gamma = C^2$

$$\because \theta = \pi - \gamma \qquad \therefore \vec{A} \cdot \vec{B} = -A \cdot B \cos\gamma = A \cdot B \cos\theta$$

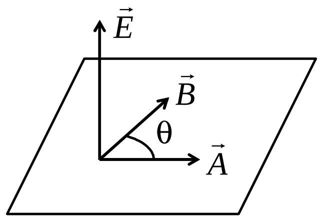
$$\begin{aligned}
\vec{A} \cdot \vec{B} &= \left(\sum_i A_i \cdot \hat{e}_i \right) \cdot \left(\sum_j B_j \cdot \hat{e}_j \right) \\
&= \sum_i \sum_j A_i \cdot B_j \hat{e}_i \cdot \hat{e}_j \\
&= \sum_i A_i \cdot B_i \qquad (\because \hat{e}_i \cdot \hat{e}_j = 1 \ \text{ if } \ i = j \ , \ \text{otherwise} = 0)
\end{aligned}$$

Example: work W done by a force $\vec{F}$ along a displacement $\vec{\ell}$

$$W = \vec{F} \cdot \vec{\ell}$$



4. Vector product



$\vec{A} \times \vec{B} = \vec{E}$ then $E = A \cdot B \cdot \sin\theta$

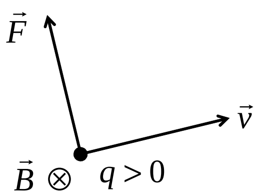
and $\vec{E} \perp \left(\text{coplane of } \vec{A} \ \& \ \vec{B}\right)$ as shown

$$\begin{aligned}
\vec{E} &= \left(\sum_i A_i \cdot \hat{e}_i \right) \times \left(\sum_j B_j \cdot \hat{e}_j \right) = \sum_i \sum_j A_i \cdot B_j \hat{e}_i \times \hat{e}_j \\
&= (A_1 \cdot B_2 - A_2 \cdot B_1) \cdot \hat{e}_3 + (A_2 \cdot B_3 - A_3 \cdot B_2) \cdot \hat{e}_1 + (A_3 \cdot B_1 - A_1 \cdot B_3) \cdot \hat{e}_2
\end{aligned}$$

$$(\because \hat{e}_1 \times \hat{e}_2 = \hat{e}_3 \quad , \quad \hat{e}_2 \times \hat{e}_1 = -\hat{e}_3 \quad , \quad \hat{e}_1 \times \hat{e}_1 = 0 \quad , \quad \text{etc.})$$

Example 1 *Force $\vec{F}$ on a q charged particle moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$*

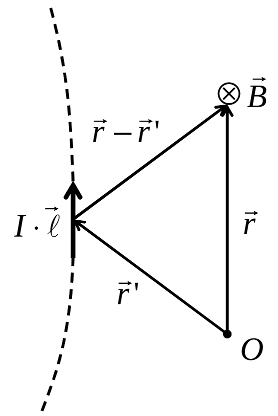
$$\vec{F} = q\vec{v} \times \vec{B}$$



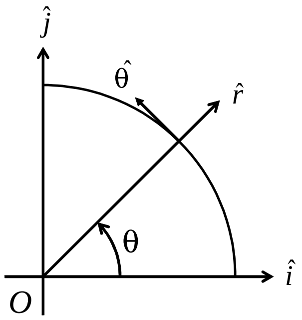
Example 2 *Magnetic field $\vec{B}$ at $\vec{r}$ generated by a current element $I \cdot \vec{\ell}$ at $\vec{r}'$*

$$\vec{B}\left(\vec{r}\right) = \frac{\mu I \cdot \vec{\ell} \times \left(\vec{r} - \vec{r}'\right)}{4\pi \left(\vec{r} - \vec{r}'\right)^3}$$

(Biot-Savart law)



5. Change of basis or rotation of the basis vectors



$\vec{A} = A_x \cdot \hat{i} + A_y \cdot \hat{j}$ basis $\hat{i}, \hat{j}$

$\vec{A} = A_r \cdot \hat{r} + A_\theta \cdot \hat{\theta}$ basis $\hat{r}, \hat{\theta}$

$\hat{r}$: radical vector

$\hat{\theta}$: tangent vector

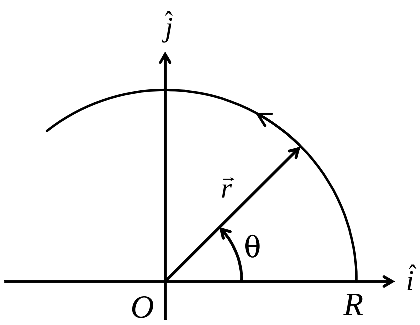
$\hat{r} = \cos\theta \cdot \hat{i} + \sin\theta \cdot \hat{j}$, $\hat{\theta} = -\sin\theta \cdot \hat{i} + \cos\theta \cdot \hat{j}$

$A_r = \vec{A} \cdot \hat{r} = A_x \cdot \cos\theta + A_y \cdot \sin\theta$, $A_\theta = \vec{A} \cdot \hat{\theta} = -A_x \cdot \sin\theta + A_y \cdot \cos\theta$

$\therefore$ Both of A_r and A_θ are linear combination of A_x , A_y

$A_r^2 + A_\theta^2 = A_x^2 + A_y^2$

Example: Uniform circular motion with radius R and angular frequency ω



position vector of the particle at time t

$\vec{r}\,(t) = R \cdot \hat{r}$ position vector

$\hat{r}\,(t) = \cos\theta\,(t) \cdot \hat{i} + \sin\theta\,(t) \cdot \hat{j}$

velocity $\vec{v} = R \cdot \omega \cdot \hat{\theta}$, centripetal acceleration $\vec{a} = -R \cdot \omega^2 \cdot \hat{r}$

$\hat{\theta}\,(t) = -\sin\theta\,(t) \cdot \hat{i} + \cos\theta\,(t) \cdot \hat{j}$