



# On an elliptic crack embedded in an anisotropic material

Kuang-Chong Wu

*Institute of Applied Mechanics, National Taiwan University, Taipei, Taiwan*

Received 6 February 1999; in revised form 9 June 1999

---

## Abstract

A generalized Stroh's formalism for three-dimensional anisotropic elasticity is applied to study the elliptic crack problem. The traction on the crack plane is expressed in a simple one-dimensional integral. The integrand contains one of the Barnett–Lothe tensors which can be calculated directly from the elastic constants. It is shown that with respect to a local coordinate system, the traction on the crack plane and relative crack face displacement in the vicinity of the crack edge have the same form as their two-dimensional counterparts. A systematic method to derive the stress intensity factors for polynomial loadings is discussed. Explicit results are given for constant, linear and quadratic loadings. © 2000 Elsevier Science Ltd. All rights reserved.

*Keywords:* Elliptic crack; Stress intensity factor; Stroh formalism; 3-D anisotropic elasticity

---

## 1. Introduction

The two-dimensional stress and displacement fields near the crack tip in a linear isotropic or anisotropic elastic material are well developed (Irwin, 1957; Sih and Liebowitz, 1968; Wu, 1989). The crack-tip stress field exhibits a square-root singularity with the amplitudes measured by the stress intensity factors (SIF's). The SIF's are dependent on the applied loading, geometry and possibly the elastic constants. In linear elastic fracture mechanics, the SIF's play a crucial role as the parameters in fracture criteria. The SIF's for a variety of two-dimensional configurations have been obtained and collected in handbooks.

Because of the mathematical complexity, three-dimensional crack analyses have been limited. The basic shapes which have been amenable to analysis are the penny-shaped and elliptic cracks. The type of loading which has received most attention is in the form of polynomials of coordinates of the crack plane. This class of loading includes such cases as uniform loading (Green and Sneddon, 1950), bending (Smith et al., 1967) and torsion (Sneddon and Lowengrub, 1969). It also serves as a useful approximation for arbitrary loading.

Sneddon (1946) appears to be the first one to study the penny-shaped crack problem. Green and Sneddon (1950) solved the problem where an elliptic crack is opened up by constant internal pressure.

Kassir and Sih (1966) showed that with respect to a local coordinate system the stress and displacement fields near the elliptic crack border are the same as the two-dimensional crack-tip fields. Results for pressure in the form of polynomials up to the sixth degree were given by Kassir and Sih (1975). The case when the elliptic crack is subjected to constant shear was analyzed by Kassir and Sih (1966). The aforementioned works are for isotropic solids. Kassir and Sih (1968) obtained the solution for linearly varying shear loading in a transversely isotropic solid. They also showed that although the angular distribution of the near-border fields are highly distorted by the elastic constants, the square-root stress singularity is the same as that associated with the isotropic case. Willis (1968) developed a method to study the stress field around an elliptic crack in general anisotropic elastic media. In Willis' method the roots of a sextic equation must be solved to obtain the elastic field.

Kassir and Sih (1967) established a theorem which states that if the displacement discontinuity normal to the elliptic crack plane is given by  $Q_n(x_1^2, x_3^2)\sqrt{1 - x_1^2/a_1^2 - x_3^2/a_3^2}$ , where  $Q_n(x_1^2, x_3^2)$  is a polynomial of degree  $n$  in the second power of the coordinates  $x_1$  and  $x_3$  of the crack plane, and  $a_1$  and  $a_3$  are the semi-axes of the ellipse, the normal stress acting on the crack surfaces is also a polynomial of degree  $n$  in  $x_1^2$  and  $x_3^2$ . Willis (1968) further found that the theorem remains true even when  $Q_n(x_1, x_3)$  is a polynomial of degree  $n$  in  $x_1$  and  $x_3$  and the displacement discontinuity is parallel to the crack plane. Sekine and Mura (1979) modified Willis' method and showed that if the displacement discontinuity is a homogeneous polynomial of degree  $n$ , the resulting stresses on the crack surfaces are inhomogeneous of degree  $n$ , whose terms are of the degree  $n, (n-2), (n-4), \dots$ .

In this paper a generalized Stroh's formalism for three-dimensional anisotropic elasticity recently developed in (Wu, 1998) is applied to study the elliptic crack problem. In the generalized Stroh's formalism the Radon transform (Deans, 1993) is first used to reduce a three-dimensional problem to a two-dimensional problem. The two-dimensional problem is then treated by the original two-dimensional Stroh's formalism (Stroh, 1958) as a six-dimensional eigenvalue problem. The orthogonality and closure relations of the eigenvectors greatly simplify the solution procedure and, in many cases including the one under consideration, enable the solutions to be expressible in terms of three real matrices, called Barnett–Lothe tensors (Ting, 1996) which can be calculated directly from the elastic constants (Barnett and Lothe, 1973). Finally, the inverse Radon transform is performed by an integration on a unit circle. An extension of Stroh's formalism to three-dimensional deformations has also been discussed by Ting (1996). In (Ting, 1996) the solutions are in terms of the three-dimensional or two-dimensional Fourier transform and connection with the corresponding two-dimensional solutions is less apparent.

In the present study the elastic field for an elliptic crack is derived from that for a two-dimensional slit crack. The traction on the crack plane is expressed in terms of a line integral involving the displacement discontinuity and one of the Barnett–Lothe tensors. It is shown that with respect to a local coordinate system, the traction on the crack plane and relative crack face displacement in the vicinity of the crack edge have the same form as their two-dimensional counterparts. A systematic method to derive the SIF's for arbitrary polynomial loadings is developed. Explicit results are obtained for constant, linear and quadratic loadings.

## 2. Formulation

A formulation recently developed by Wu (1998) for three-dimensional anisotropic elasticity is introduced in this section.

Let  $\hat{\mathbf{u}}$  be the two-dimensional Radon transform of  $\mathbf{u}$  defined as (Deans, 1993)

$$\hat{\mathbf{u}}(\xi_1, x_2, \phi) = \mathbf{R}(\mathbf{u}(x_1, x_2, x_3)) = \int_{-\infty}^{\infty} \mathbf{u}(\xi_1 \cos \phi + \xi_3 \sin \phi, x_2, -\xi_1 \sin \phi + \xi_3 \cos \phi) d\xi_3 \quad (1)$$

where  $\xi_1$  and  $\xi_3$  are the coordinates with respect to the axes obtained by rotating the  $x_1$ - and  $x_3$ -axes about the  $x_2$ -axis by  $\phi$ , respectively (see Fig. 1). The transform may also be expressed as

$$\hat{\mathbf{u}}(\xi_1, x_2, \phi) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathbf{u}(x_1, x_2, x_3) \delta(\xi_1 - \mathbf{n} \cdot \mathbf{x}) dx_1 dx_3 \quad (2)$$

where  $\delta$  is the Dirac delta function and  $\mathbf{n} = (\cos \phi, 0, -\sin \phi)^T$  is the unit vector along the  $\xi_1$ -axis. The general solution for  $\hat{\mathbf{u}}$  can be expressed as

$$\hat{\mathbf{u}} = 2 \operatorname{Re}[\mathbf{A}(\phi) \hat{\mathbf{f}}(\mathbf{z})] \quad (3)$$

where  $\operatorname{Re}$  denotes ‘the real part of’. In Eq. (3),  $\mathbf{A} = (\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ ,  $\hat{\mathbf{f}} = (\hat{f}_1(z_1), \hat{f}_2(z_2), \hat{f}_3(z_2))^T$ ,  $z_\alpha = x_1 + p_\alpha x_2$ . Here  $p_\alpha$  and  $\mathbf{a}_\alpha$ ,  $\alpha = 1, 2, 3$ , are the eigenvalues and the corresponding eigenvectors, respectively, determined by

$$[\mathbf{Q}(\phi) + (\mathbf{R}(\phi) + \mathbf{R}(\phi)^T)p + \mathbf{T}(\phi)p^2] \mathbf{a} = 0 \quad (4)$$

The matrices  $\mathbf{Q}(\phi)$ ,  $\mathbf{R}(\phi)$  and  $\mathbf{T}(\phi)$  are given by

$$\mathbf{Q} = \Omega^T \mathbf{Q}^* \Omega, \quad \mathbf{R} = \Omega^T \mathbf{R}^* \Omega, \quad \mathbf{T} = \Omega^T \mathbf{T}^* \Omega \quad (5)$$

where

$$Q_{ik}^*(\phi) = C_{i1k1}^*(\phi), \quad R_{ik}^*(\phi) = C_{i1k2}^*(\phi), \quad T_{ik}^*(\phi) = C_{i2k2}^*(\phi) \quad (6)$$

and

$$\Omega(\phi) = \begin{pmatrix} \cos \phi & 0 & -\sin \phi \\ 0 & 1 & 0 \\ \sin \phi & 0 & \cos \phi \end{pmatrix}$$

Here  $C_{ijk_s}^*$  is the elastic constant with respect to the coordinate system  $(\xi_1, x_2, \xi_3)$ . The matrix  $\Omega(\phi)$

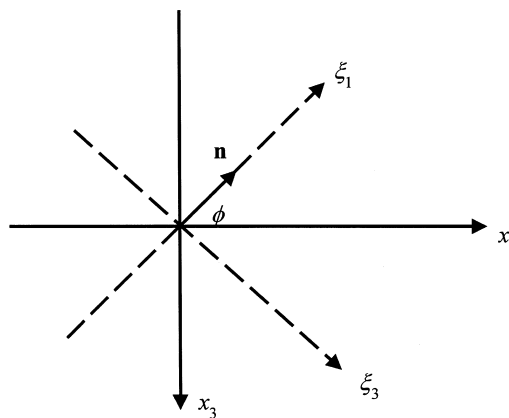


Fig. 1. The rotated coordinates  $(\xi_1, \xi_3)$  in Radon transform.

represents a rotation by  $\phi$  about the  $x_2$ -axis. Substitution of Eq. (5) into (4) yields

$$[\mathbf{Q}^*(\phi) + (\mathbf{R}^*(\phi) + \mathbf{R}^*(\phi)^T)p + \mathbf{T}^*(\phi)p^2]\mathbf{a}^* = 0 \quad (7)$$

where

$$\mathbf{a}^*(\phi) = \Omega(\phi)\mathbf{a}(\phi) \quad (8)$$

Eq. (7) shows that  $p(\phi)$  and  $\mathbf{a}^*(\phi)$  are Stroh's eigenvalue and eigenvector, respectively, for two-dimensional deformations in the  $(\zeta_1, x_2)$ -plane (Stroh, 1958). The eigenvalue  $p_\alpha$  is complex if the strain energy is positive definite. Since the eigenvalue  $p$  in Eq. (7) appear as three complex conjugate pairs, we can let  $\text{Im}[p_\alpha] > 0$ ,  $\alpha = 1, 2, 3$ , where  $\text{Im}$  denotes 'the imaginary part of'. From Eq. (8) the matrix  $\mathbf{A}$  can be expressed as

$$\mathbf{A}(\phi) = \Omega^T(\phi)\mathbf{A}^*(\phi) \quad (9)$$

where  $\mathbf{A}^* = (\mathbf{a}_1^*, \mathbf{a}_2^*, \mathbf{a}_3^*)$ .

Let  $\mathbf{t} = (\sigma_{21}, \sigma_{22}, \sigma_{23})^T$ . The Radon transform of  $\mathbf{t}$ ,  $\hat{\mathbf{t}}$ , can be expressed as

$$\hat{\mathbf{t}} = 2 \text{Re}[\mathbf{B}(\phi)\hat{\mathbf{f}}'] \quad (10)$$

where  $\mathbf{B}$  is given by

$$\mathbf{B} = \mathbf{R}^T\mathbf{A} + \mathbf{T}\mathbf{A}\mathbf{P} \quad (11)$$

$\mathbf{P} = \text{diag}(p_1, p_2, p_3)$  and prime denotes differentiation with respect to  $\zeta_1$ . With Eqs. (5) and (8), Eq. (11) can be expressed as

$$\mathbf{B}(\phi) = \Omega^T(\phi)\mathbf{B}^*(\phi) \quad (12)$$

where  $\mathbf{B}^* = \mathbf{R}^{*T}\mathbf{A}^* + \mathbf{T}^*\mathbf{A}^*\mathbf{P}$ .

A matrix  $\mathbf{L}^*$  which will play an important role is given by (Stroh, 1958)

$$\mathbf{L}^*(\phi) = -2i\mathbf{B}^*(\phi)\mathbf{B}^*(\phi)^T \quad (13)$$

where  $i = \sqrt{-1}$ . The matrix  $\mathbf{L}^*$  is real, symmetric and positive definite. The matrix  $\mathbf{L}^*$  can also be calculated directly from the elastic constants by the following integral (Barnett and Lothe, 1973)

$$\mathbf{L}^*(\phi) = -\frac{1}{\pi} \int_0^\pi \mathbf{N}_3(\theta, \phi) d\theta \quad (14)$$

where

$$\mathbf{N}_3(\theta, \phi) = \mathbf{R}^*(\theta, \phi)(\mathbf{T}^*(\theta, \phi))^{-1}\mathbf{R}^{*T}(\theta, \phi) - \mathbf{Q}^*(\theta, \phi)$$

and

$$\mathbf{Q}^*(\theta, \phi) = \mathbf{Q}^*(\phi) \cos^2 \theta + (\mathbf{R}^*(\phi) + \mathbf{R}^{*T}(\phi)) \cos \theta \sin \theta + \mathbf{T}^*(\phi) \sin^2 \theta$$

$$\mathbf{R}^*(\theta, \phi) = \mathbf{R}^*(\phi) \cos^2 \theta + (\mathbf{T}^*(\phi) - \mathbf{Q}^*(\phi)) \cos \theta \sin \theta - \mathbf{R}^{*T}(\phi) \sin^2 \theta$$

$$\mathbf{T}^*(\theta, \phi) = \mathbf{T}^*(\phi) \cos^2 \theta - (\mathbf{R}^*(\phi) + \mathbf{R}^{*T}(\phi)) \cos \theta \sin \theta + \mathbf{Q}^*(\phi) \sin^2 \theta$$

For transversely isotropic material with the  $x_2$ -axis as the symmetry axis,  $\mathbf{L}^*$  is a diagonal matrix given by

$$\mathbf{L}^* = \begin{pmatrix} L_{11}^* & 0 & 0 \\ 0 & L_{22}^* & 0 \\ 0 & 0 & L_{33}^* \end{pmatrix} \tag{15}$$

where  $L_{11}^*$ ,  $L_{22}^*$  and  $L_{33}^*$  are (Dongye and Ting, 1989)

$$L_{11}^* = (\sqrt{C_{11}C_{22}} + C_{12}) \left( \frac{C_{66}(\sqrt{C_{11}C_{22}} - C_{12})}{C_{22}(2C_{66} + \sqrt{C_{11}C_{22}} + C_{12})} \right)^{1/2}$$

$$L_{22}^* = \sqrt{\frac{C_{22}}{C_{11}}} L_{11}^*, \quad L_{33}^* = \sqrt{C_{44}C_{55}}$$

In particular for isotropic material

$$L_{11}^* = L_{22}^* = \frac{\mu}{1-\nu}, \quad L_{33}^* = \mu \tag{16}$$

where  $\mu$  is the shear modulus and  $\nu$  is the Poisson’s ratio. From (12), the matrix  $\mathbf{L}$  defined as

$$\mathbf{L}(\phi) = -2i\mathbf{B}(\phi)\mathbf{B}(\phi)^T \tag{17}$$

is related to  $\mathbf{L}^*$  by

$$\mathbf{L}(\phi) = \mathbf{\Omega}(\phi)^T \mathbf{L}^*(\phi) \mathbf{\Omega}(\phi) \tag{18}$$

The displacements and the stress vector are obtained by inverting the Radon transform. The result is

$$\mathbf{u}(\mathbf{x}) = \frac{\text{sgn}(x_2)}{2\pi} \text{Im} \left[ \int_0^{2\pi} \mathbf{A}(\phi) \hat{\mathbf{f}}'(\mathbf{z}) |_{\xi_1=\mathbf{n}\cdot\mathbf{x}} d\phi \right] \tag{19}$$

$$\mathbf{t}(\mathbf{x}) = \frac{\text{sgn}(x_2)}{2\pi} \text{Im} \left[ \int_0^{2\pi} \mathbf{B}(\phi) \hat{\mathbf{f}}''(\mathbf{z}) |_{\xi_1=\mathbf{n}\cdot\mathbf{x}} d\phi \right] \tag{20}$$

provided that  $\hat{f}'_\alpha(\xi_1 + p_\alpha x_2) \rightarrow 0$  as  $\xi_1 \rightarrow \infty$ .

### 3. Stresses on the plane of crack

Consider first the problem of a Somigliana’s dislocation with  $\mathbf{b}(x_1, x_3) = \mathbf{u}^+ - \mathbf{u}^-$  on the  $(x_1, x_3)$ -plane. The corresponding two-dimensional problem in the Radon transform domain  $(\xi_1, x_2, \phi)$  is that of a Somigliana’s dislocation with  $\hat{\mathbf{b}}(\xi_1, \phi) = \hat{\mathbf{u}}^+ - \hat{\mathbf{u}}^-$  on the  $\xi_1$ -axis, where  $\hat{\mathbf{b}}$  is the Radon transform of  $\mathbf{b}$ . The analytic functions  $\hat{f}'_\alpha(z_\alpha)$  for the two-dimensional problem is given by (Stroh, 1958)

$$\hat{f}'_\alpha(z_\alpha) = \frac{1}{2\pi i} B_{k\alpha} \int_{-\infty}^{\infty} \frac{1}{s - z_\alpha} \frac{\partial \hat{b}_k(s, \phi)}{\partial s} ds \tag{21}$$

By the Plemelj formulae,  $\hat{\mathbf{f}}'^+$ , the limiting value of  $\hat{\mathbf{f}}'$  as  $x_2 \rightarrow 0^+$ , is given by

$$\hat{\mathbf{f}}'^+ = \mathbf{B}^T \left( \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{1}{s - \xi_1} \frac{\partial \hat{\mathbf{b}}(s, \phi)}{\partial s} ds + \frac{1}{2} \frac{\partial \hat{\mathbf{b}}(\xi_1, \phi)}{\partial \xi_1} \right) \quad (22)$$

where the principal value of the integral is taken. Substitution of Eq. (22) into Eq. (20) yields the following real-form expression for  $\mathbf{t}$  at  $x_2=0$

$$\mathbf{t}(x_1, x_3) = \frac{1}{4\pi} \int_0^\pi \mathbf{L}(\phi) \frac{\partial^2 \hat{\mathbf{b}}(\xi_1, \phi)}{\partial \xi_1^2} \Big|_{\xi_1 = \mathbf{n} \cdot \mathbf{x}} d\phi \quad (23)$$

where  $\mathbf{L}$  is given by Eq. (17). Note that Eq. (23) is also valid for isotropic material.

An elliptic crack can be simulated by a Somigliana's dislocation  $\mathbf{b}$  over the region  $D: (x_1/a_1)^2 + (x_3/a_3)^2 \leq 1$ . The corresponding  $\hat{\mathbf{b}}$  in the Radon transform domain is given by

$$\hat{\mathbf{b}}(\xi_1, \phi) = \int_D \mathbf{b} \delta(\xi_1 - x_1 \cos \phi + x_3 \sin \phi) dx_1 dx_3 \quad (24)$$

By introducing  $y_1 = (x_1/a_1)$ ,  $y_3 = (x_3/a_3)$ , Eq. (24) can be rewritten as (Deans, 1993)

$$\hat{\mathbf{b}}(\xi_1, \phi) = \frac{a_1 a_3}{R(\phi)} \hat{\beta}(\eta_1, \psi) \quad (25)$$

where  $\hat{\beta}(\eta_1, \psi)$  is the Radon transform of  $\mathbf{b}$  over the unit disk  $D': y_1^2 + y_3^2 \leq 1$ , i.e.

$$\hat{\beta}(\eta_1, \psi) = \int_{D'} \mathbf{b} \delta(\eta_1 - y_1 \cos \psi + y_3 \sin \psi) dy_1 dy_3 \quad (26)$$

In Eq. (26),  $\eta_1 = [\xi_1/R(\phi)]$  and

$$\cos \psi = \frac{a_1 \cos \phi}{R(\phi)}, \quad \sin \psi = \frac{a_3 \sin \phi}{R(\phi)}, \quad R(\phi) = \sqrt{a_1^2 \cos^2 \phi + a_3^2 \sin^2 \phi} \quad (27)$$

Eq. (27) can also be rewritten as

$$\cos \phi = \frac{a_3 \cos \psi}{N(\psi)}, \quad \sin \phi = \frac{a_1 \sin \psi}{N(\psi)}, \quad N(\psi) = \sqrt{a_3^2 \cos^2 \psi + a_1^2 \sin^2 \psi} \quad (28)$$

The functions  $\hat{\beta}(\eta_1, \psi)$  can be expressed as (Deans, 1993)

$$\hat{\beta}(\eta_1, \psi) = \hat{\mathbf{g}}(\eta_1, \psi) H(1 - \eta_1^2) \quad (29)$$

where  $H$  is the unit step function and  $\hat{\mathbf{g}}(\eta_1, \psi)$  is a Radon transform which satisfies

$$\hat{\mathbf{g}}(\eta_1, \psi) = \hat{\mathbf{g}}(-\eta_1, \psi + \pi) \quad (30)$$

$$\hat{\mathbf{g}}(\pm 1, \psi) = 0 \quad (31)$$

$$\int_{-1}^1 \eta_1^k \hat{\mathbf{g}}(\eta_1, \psi) d\eta_1 = \mathbf{P}_k(\cos \psi, \sin \psi) \tag{32}$$

Here  $\mathbf{P}_k(\cos \psi, \sin \psi)$  is a polynomial of degree  $k$  in  $\cos \psi$  and  $\sin \psi$ .

Eq. (23) can be expressed in terms of  $\psi$  as

$$\mathbf{t}(y_1, y_3) = \frac{1}{4\pi a_3} \int_0^\pi \mathbf{M}(\psi) \frac{\partial^2}{\partial \eta_1^2} \hat{\beta}(\eta_1, \psi) |_{\eta_1 = \mathbf{n}' \cdot \mathbf{y}} d\psi \tag{33}$$

where  $\mathbf{n}' = (\cos \psi, -\sin \psi)^T$ ,  $\mathbf{M}(\psi) = (1/a_1)N(\psi)\mathbf{L}(\phi(\psi))$ ,  $\mathbf{y} = (y_1, y_3)^T$  and the following identity has been used

$$N(\psi)R(\phi) = a_1 a_3$$

From Eq. (29),

$$\begin{aligned} \frac{\partial^2}{\partial \eta_1^2} \hat{\beta}(\eta_1, \psi) &= \left( \frac{\partial^2}{\partial \eta_1^2} \hat{\mathbf{g}}(\eta_1, \psi) \right) H(1 - \eta_1^2) + 2 \left( \frac{\partial}{\partial \eta_1} \hat{\mathbf{g}}(\eta_1, \psi) \right) (\delta(\eta_1 + 1) - \delta(\eta_1 - 1)) + \hat{\mathbf{g}}(\eta_1, \\ &\psi) (\delta'(\eta_1 + 1) - \delta'(\eta_1 - 1)) \end{aligned} \tag{34}$$

Let  $y_1 = y \cos \psi_0$ ,  $y_2 = -y \sin \psi_0$ ,  $y = \sqrt{y_1^2 + y_2^2}$ . The variable  $\eta_1$  can be replaced by

$$\eta_1 = y \cos(\psi - \psi_0)$$

For  $y < 1$ ,  $|\eta_1| < 1$  and Eq. (34) becomes

$$\frac{\partial^2}{\partial \eta_1^2} \hat{\beta}(\eta_1, \psi) = \left( \frac{\partial^2}{\partial \eta_1^2} \hat{\mathbf{g}}(\eta_1, \psi) \right) H(1 - \eta_1^2)$$

and the stress vector inside the crack is given by

$$\mathbf{t}(y, \psi_0) = \frac{1}{4\pi a_3} \int_0^\pi \mathbf{M}(\psi) \frac{\partial^2}{\partial \eta_1^2} \hat{\mathbf{g}}(\eta_1, \psi) |_{\eta_1 = y \cos(\psi - \psi_0)} d\psi \tag{35}$$

Eq. (35) is an integral equation of  $\hat{\mathbf{g}}(\eta_1, \psi)$  for a given function of  $\mathbf{t}$ .

To derive  $\mathbf{t}$  for  $y > 1$ , it is more convenient to change the integration variable for  $\psi$  to  $\eta_1$  for Eq. (33). The result is

$$\mathbf{t}(y, \psi_0) = \frac{1}{4\pi a_3} \int_{-y \cos \psi_0}^{y \cos \psi_0} \frac{1}{\sqrt{y^2 - \eta_1^2}} \mathbf{M}(\psi) \frac{\partial^2}{\partial \eta_1^2} \hat{\beta}(\eta_1, \psi) d\eta_1 \tag{36}$$

Substitution of Eq. (34) into Eq. (36) leads to

$$\begin{aligned} \mathbf{t}(y, \psi_0) &= \frac{1}{4\pi a_3} \int_{-1}^1 \frac{1}{\sqrt{y^2 - \eta_1^2}} \mathbf{M}(\psi) \frac{\partial^2}{\partial \eta_1^2} \hat{\mathbf{g}}(\eta_1, \psi) d\eta_1 + \frac{1}{4\pi a_3} \frac{1}{\sqrt{y^2 - 1}} \mathbf{M}(\psi_{-1}) \frac{\partial}{\partial \eta_1} \hat{\mathbf{g}}(\eta_1, \\ &\psi_{-1}) |_{\eta_1 = -1} - \frac{1}{4\pi a_3} \frac{1}{\sqrt{y^2 - 1}} \mathbf{M}(\psi_1) \frac{\partial}{\partial \eta_1} \hat{\mathbf{g}}(\eta_1, \psi_1) |_{\eta_1 = 1} \end{aligned} \tag{37}$$

where

$$\psi_{-1} = \psi_0 + \pi - \cos^{-1} \frac{1}{y}, \quad \psi_1 = \psi_0 + \cos^{-1} \frac{1}{y}$$

and Eq. (31) has been used. As  $y \rightarrow 1^+$  the integral on the right side of Eq. (37) is non-singular and the remaining two terms have square-root singularity. Thus, the asymptotic stress near the crack edge at  $\psi = \psi_0$  is given by

$$\mathbf{t}(y, \psi_0) = -\frac{1}{2\sqrt{2}\pi a_3} \frac{1}{\sqrt{y-1}} \mathbf{M}(\psi_0) \frac{\partial}{\partial \eta_1} \hat{\mathbf{g}}(\eta_1, \psi_0) |_{\eta_1=1} \tag{38}$$

where the identity  $\mathbf{M}(\psi + \pi) = \mathbf{M}(\psi)$  (Wu, 1998), and Eq. (30) have been effected. With respect to the local coordinate system  $(n, x_2, t)$  with origin at  $(a_1 \cos \psi_0, 0, -a_3 \sin \psi_0)$  on the crack edge as shown in Fig. 2, the stress intensity factor (SIF) is defined as

$$\mathbf{K}(\psi_0) = [K_{II}, K_I, K_{III}]^T = \sqrt{2\pi r} \Omega(\phi_0) \lim_{r \rightarrow 0} \mathbf{t}(r, \psi_0) \tag{39}$$

where  $\phi_0$  is the angle between the normal at the point and the  $x_1$ -axis and  $r$  is the normal distance from the crack edge. It can be easily shown that  $\phi_0$  is related to  $\psi_0$  in exactly the same way as  $\phi$  is to  $\psi$  through Eq. (28) and that the normal distance  $r$  is given by

$$r = \frac{a_1 a_3}{N(\psi_0)} (y - 1) \tag{40}$$

where  $N(\psi_0)$  is given in Eq. (28)<sub>3</sub>. Substitution of Eqs. (38) and (40) yields

$$\mathbf{K}(\psi_0) = -\frac{1}{2} \sqrt{\frac{N(\psi_0)}{\pi a_1 a_3}} \mathbf{L}^*(\phi_0) \Omega(\phi_0) \frac{\partial}{\partial \eta_1} \hat{\mathbf{g}}(\eta_1, \psi_0) |_{\eta_1=1} \tag{41}$$

where Eq. (18) has been used.

For transversely isotropic material, with Eq. (15), Eq. (35) can be decoupled into two independent sets of equations as

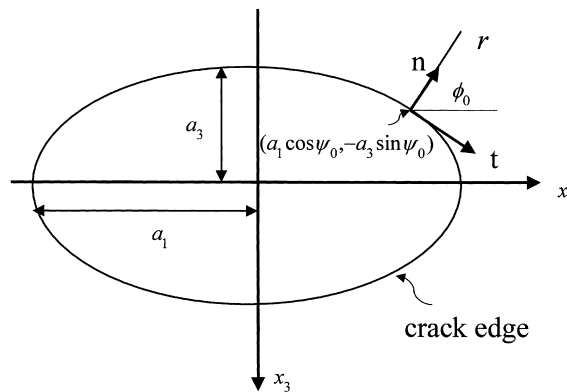


Fig. 2. The local coordinate system  $(n, t)$  with origin at  $(a_1 \cos \psi_0, -a_3 \sin \psi_0)$  on the crack edge.

$$\sigma_{22}(y_1, y_3) = \frac{1}{4\pi a_3} \int_0^\pi M_{22}(\psi) \frac{\partial^2}{\partial \eta_1^2} \hat{g}_2(\eta_1, \psi) \Big|_{\eta_1=\mathbf{n}'\cdot\mathbf{y}} d\psi \tag{42}$$

$$\begin{pmatrix} \sigma_{21}(y_1, y_3) \\ \sigma_{23}(y_1, y_3) \end{pmatrix} = \frac{1}{4\pi a_3} \int_0^\pi \mathbf{m}(\psi) \frac{\partial^2}{\partial \eta_1^2} \begin{pmatrix} \hat{g}_1(\eta_1, \psi) \\ \hat{g}_3(\eta_1, \psi) \end{pmatrix} \Big|_{\eta_1=\mathbf{n}'\cdot\mathbf{y}} d\psi \tag{43}$$

where

$$M_{22}(\psi) = \frac{L_{22}^*}{a_1} N(\psi)$$

$$\mathbf{m}(\psi) = \frac{a_1 L_{11}^*}{N(\psi)} \begin{pmatrix} (k'^2 - \rho) \cos^2 \psi + \rho & -k'(1 - \rho) \cos \psi \sin \psi \\ -k'(1 - \rho) \cos \psi \sin \psi & (k'^2 \rho - 1) \cos^2 \psi + 1 \end{pmatrix}$$

$k' = (a_3/a_1)$ , and  $\rho$  is defined as

$$\rho = \frac{L_{33}^*}{L_{11}^*} = \left( \frac{C_{55}C_{22}(2C_{66} + \sqrt{C_{11}C_{22}} + C_{12})}{(C_{11}C_{22} - C_{12}^2)(\sqrt{C_{11}C_{22}} + C_{12})} \right)^{1/2}$$

The dimensionless parameter  $\rho$  reduced to  $1-\nu$  for isotropic material. Similarly SIF of Eq. (41) can also be decomposed into

$$K_I(\psi_0) = -\frac{1}{2} \sqrt{\frac{N(\psi_0)}{\pi a_1 a_3}} L_{22}^* \frac{\partial}{\partial \eta_1} \hat{g}_2(\eta_1, \psi_0) \Big|_{\eta_1=1} \tag{44}$$

$$\begin{pmatrix} K_{II}(\psi_0) \\ K_{III}(\psi_0) \end{pmatrix} = -\frac{1}{2} \frac{L_{11}^*}{\sqrt{\pi k' N(\psi_0)}} \begin{pmatrix} k' \cos \psi_0 & -\sin \psi_0 \\ \rho \sin \psi_0 & \rho k' \cos \psi_0 \end{pmatrix} \tag{45}$$

$$\frac{\partial}{\partial \eta_1} \begin{pmatrix} \hat{g}_1(\eta_1, \psi_0) \\ \hat{g}_3(\eta_1, \psi_0) \end{pmatrix} \Big|_{\eta_1=1} \tag{46}$$

Since from Eq. (42)  $L_{22}^* \hat{g}_2$  does not depend on the material constants, Eq. (44) indicates that  $K_I$  is independent of the elastic constants. Likewise Eqs. (43) and (46) show that  $K_{II}$  and  $K_{III}$  depend on the elastic constants only through the dimensionless parameter  $\rho$ . The conclusion has been noted in Kassir and Sih (1975). However, their expression for  $\rho$  is incorrect.

#### 4. Relative crack face displacement

Without loss of generality, let the relative crack face displacement be expressed as

$$\mathbf{b}(y_1, y_3) = \sqrt{1 - y^2} \mathbf{Q}(y_1, y_2) H(1 - y^2) \tag{47}$$

where  $\mathbf{Q}(y_1, y_2)$  is an arbitrary function of  $y_1$  and  $y_2$ . By Eq. (1), the Radon transform  $\hat{\beta}(\eta_1, \psi)$  of  $\mathbf{b}$  defined by Eq. (26) can be expressed as

$$\hat{\beta}(\eta_1, \psi) = \int_{-\infty}^{\infty} \mathbf{b}(\eta_1 \cos \psi + \eta_3 \sin \psi, -\eta_1 \sin \psi + \eta_3 \cos \psi) d\eta_3 \quad (48)$$

where  $\eta_1$  and  $\eta_3$  are the coordinates with respect to the axes obtained by rotating the  $y_1$ - and  $y_3$ -axes about the  $x_2$ -axis by  $\psi$ , respectively. Substitution of Eq. (47) into Eq. (48) with a change of variable  $\eta_3 = c\lambda$  leads to

$$\hat{\mathbf{g}}(\eta_1, \psi) = (1 - \eta_1^2) \int_{-1}^1 \sqrt{1 - \lambda^2} \mathbf{Q}(\eta_1 \cos \psi + c\lambda \sin \psi, -\eta_1 \sin \psi + c\lambda \cos \psi) d\lambda \quad (49)$$

where  $c = \sqrt{1 - \eta_1^2}$ . Differentiation of Eq. (49) with respect to  $\eta_1$  at  $\eta_1 = 1$  yields

$$\left. \frac{\partial \hat{\mathbf{g}}(\eta_1, \psi)}{\partial \eta_1} \right|_{\eta_1=1} = -\pi \mathbf{Q}(\cos \psi, -\sin \psi) \quad (50)$$

With respect to the local coordinate system  $(n, x_2, t)$  shown in Fig. 2, the asymptotic expression for Eq. (47) as the point is approached is

$$\lim_{y \rightarrow 1^-} \frac{\mathbf{b}^*(\cos \psi_0, -\sin \psi_0)}{\sqrt{(1-y)}} = \sqrt{2} \Omega(\phi_0) \mathbf{Q}(\cos \psi_0, -\sin \psi_0) \quad (51)$$

where  $\mathbf{b}^*(y_1, y_3) = \Omega(\phi_0) \mathbf{b}(y_1, y_3)$ . From Eqs. (41) and (50), Eq. (51) can be expressed as

$$\lim_{y \rightarrow 1^-} \frac{\mathbf{b}^*(\cos \psi_0, -\sin \psi_0)}{\sqrt{r}} = 2 \sqrt{\frac{2}{\pi}} (\mathbf{L}^*(\phi_0))^{-1} \mathbf{K}(\psi_0) \quad (52)$$

Eq. (52) is the same as the two-dimensional expression (Wu, 1989).

## 5. Polynomial loadings

Let  $\mathbf{Q}(y_1, y_2)$  in Eq. (47) be given by (Willis, 1968)

$$\mathbf{Q}(y_1, y_2) = \text{Re}[(y_1 + iy_2)^l (y_1 - iy_2)^{n-l} \mathbf{q}] \quad (53)$$

where  $\mathbf{q}$  is a constant vector of a complex number,  $n$  and  $l$  are positive integers. For each  $l \leq n$ , Eq. (53) represents a homogeneous polynomial of degree  $n$  in  $y_1$  and  $y_2$ . Substitution of Eq. (53) into Eq. (49) yields

$$\hat{\mathbf{g}}(\eta_1, \psi) = (1 - \eta_1^2) \text{Re}[e^{i(n-2l)\psi} h(\eta_1) \mathbf{q}]$$

where

$$h(\eta_1) = \int_{-1}^1 \sqrt{1 - \lambda^2} (\eta_1 + ic\lambda)^l (\eta_1 - ic\lambda)^{n-l} d\lambda \quad (54)$$

By using the binomial theorem and the integral formula

$$\int_{-1}^1 \sqrt{1 - \lambda^2} \lambda^j = 0, \quad (j \text{ odd})$$

the function  $h(\eta_1)$  can be shown to be an inhomogeneous polynomial in  $\eta_1$ , consisting of terms of degree  $n, n - 2, n - 4, \dots$ . The corresponding stress vector  $\mathbf{t}$  obtained by substituting Eq. (55) into Eq. (35) is in the form of polynomials in  $y_1$  and  $dy_2$ , whose terms are of degree  $n, (n - 2), (n - 4), \dots$  (Sekine and Mura, 1979). Thus, for an arbitrary inhomogeneous polynomial of  $\mathbf{Q}(y_1, y_2)$  in  $y_1$  and  $y_2$  with terms of degree  $n, n - 2, n - 4, \dots$ , the corresponding  $\hat{\mathbf{g}}(\eta_1, \psi)$  in Eq. (49) can be expressed as

$$\hat{\mathbf{g}}(\eta_1, \psi) = (1 - \eta_1^2) \text{Re} \left[ \sum_{l=0}^{[n/2]} e^{i(n-2l)\psi} \sum_{m=0}^{[n/2]} \eta_1^{n-2m} \mathbf{q}_{n-2m}^{(n-2l)} \right] \quad (55)$$

where  $[n/2]$  denotes the integer part of  $n/2$ ,  $\mathbf{q}_{n-2m}^{(0)}$  are real constant vectors and  $\mathbf{q}_{n-2m}^{(2l-n)}, 2l-n \neq 0$ , are complex constant vectors. The constant vectors  $\mathbf{q}_{n-2m}^{(2l-n)}, 2l-n \neq 0$  are not all independent. In fact from Eq. (32), they are related by

$$\int_{-1}^1 \eta_1^k \left( (1 - \eta_1^2) \sum_{m=0}^{[n/2]} \eta_1^{n-2m} \mathbf{q}_{n-2m}^{(n-2l)} \right) d\eta_1 = 0, \quad (k = 0, 1, \dots, n - 2l - 1) \quad (56)$$

for  $l = 0, 1, \dots, [n/2]-1$ . For  $\mathbf{t}$  in the form of an arbitrary homogeneous polynomial of degree  $n$ , the constant vectors  $\mathbf{q}_{n-2m}^{(n-2l)}$  can be determined by Eqs. (35) and (56).

In enforcing Eq. (35) the following matrices are crucial

$$\mathbf{M}^{(2k)} = \frac{1}{\pi} \int_0^\pi e^{-i2k\psi} \mathbf{M}(\psi) d\psi, \quad k = 0, 1, 2, \dots \quad (57)$$

These matrices are actually the coefficients in the Fourier series of  $\mathbf{M}(\psi)$ , i.e.

$$\mathbf{M}(\psi) = \mathbf{M}^{(0)} + \text{Re} \left[ \sum_{k=1}^\infty e^{i2k\psi} \mathbf{M}^{(2k)} \right]$$

Once the constant vectors  $\mathbf{q}_{n-2m}^{(n-2l)}$  are determined, the stress intensity factor can be calculated by

$$\mathbf{K}(\psi_0) = \sqrt{\frac{N(\psi_0)}{\pi a_1 a_3}} \mathbf{L}^*(\phi_0) \Omega(\phi_0) \text{Re} \left[ \sum_{l=0}^{[n/2]} e^{i(n-2l)\psi} \sum_{m=0}^{[n/2]} \mathbf{q}_{n-2m}^{(n-2l)} \right] \quad (58)$$

The procedure is illustrated by the following examples.

### 5.1. Uniform loading

For uniform loading  $\mathbf{t} = -\mathbf{t}^{(0)}$ , Eq. (55) with  $n = 0$  yields

$$\hat{\mathbf{g}}(\eta_1, \psi) = (1 - \eta_1^2) \mathbf{q}_0^{(0)}$$

From Eq. (35)

$$\mathbf{q}_0^{(0)} = 2a_3 (\mathbf{M}^{(0)})^{-1} \mathbf{t}^{(0)}$$

The SIF is obtained from Eq. (58) as

$$\mathbf{K}(\psi_0) = 2\sqrt{\frac{a_3 N(\psi_0)}{\pi a_1}} \mathbf{L}^*(\phi_0) \mathbf{\Omega}(\phi_0) (\mathbf{M}^{(0)})^{-1} \mathbf{t}^{(0)} \quad (59)$$

For transversely isotropic material, with  $\mathbf{L}^*$  given by Eq. (15),  $\mathbf{M}^{(0)}$  is a diagonal matrix with elements given by

$$M_{11}^{(0)} = \frac{2L_{11}^*}{\pi} [(k'^2 - \rho)I_{20}(k) + \rho I_{00}(k)]$$

$$M_{22}^{(0)} = \frac{2L_{22}^* J_{00}(k)}{\pi}$$

$$M_{33}^{(0)} = \frac{2L_{11}^*}{\pi} [(k'^2 \rho - 1)I_{20}(k) + I_{00}(k)]$$

where  $k^2 = 1 - k'^2$  and expressions for  $J_{00}$ ,  $I_{00}$  and  $I_{20}$  in terms of the complete elliptic integrals are given in the Appendix. The explicit expression for the SIF of Eq. (59) is

$$K_I(\psi_0) = \sqrt{\frac{\pi a_3 N(\psi_0)}{a_1}} \frac{t_2^{(0)}}{E(k)} \quad (60)$$

$$K_{II}(\psi_0) = 2\sqrt{\frac{a_3}{\pi a_1}} \frac{L_{11}^*}{\sqrt{N(\psi_0)}} \left( \frac{a_3 \cos \psi_0}{M_{11}^{(0)}} t_1^{(0)} - \frac{a_1 \sin \psi_0}{M_{33}^{(0)}} t_3^{(0)} \right) \quad (61)$$

$$K_{III}(\psi_0) = 2\sqrt{\frac{a_3}{\pi a_1}} \frac{\rho L_{11}^*}{\sqrt{N(\psi_0)}} \left( \frac{a_1 \sin \psi_0}{M_{11}^{(0)}} t_1^{(0)} + \frac{a_3 \cos \psi_0}{M_{33}^{(0)}} t_3^{(0)} \right) \quad (62)$$

Eq. (60) recovers the result derived in Green and Sneddon (1950) and Eqs. (61) and (62) agree with the result in Kassir and Sih (1975).

## 5.2. Linear loading

Let  $\mathbf{t} = -\text{Re}[\mathbf{t}^{(1)} \boldsymbol{\zeta}]$ , where  $\mathbf{t}^{(1)}$  is a complex constant and  $\boldsymbol{\zeta} = (y_1 + iy_3)$ . In this case,  $n = 1$  and Eq. (55) becomes

$$\hat{\mathbf{g}}(\eta_1, \psi) = (1 - \eta_1^2) \eta_1 \text{Re}[e^{i\psi} \mathbf{q}_1^{(1)}]$$

From Eq. (35)

$$-\text{Re}[\mathbf{t}^{(1)} \boldsymbol{\zeta}] = -\frac{3}{2\pi a_3} \text{Re} \left[ \left( \int_0^\pi e^{i\psi} \eta_1 \mathbf{M}(\psi) d\psi \right) \mathbf{q}_1^{(1)} \right] \quad (63)$$

By replacing  $\eta_1$  with

$$\eta_1 = \frac{1}{2}(e^{i\psi} \boldsymbol{\zeta} + e^{-i\psi} \bar{\boldsymbol{\zeta}})$$

Eq. (63) yields

$$\overline{\mathbf{M}}^{(2)} \mathbf{q}_1^{(1)} + \mathbf{M}^{(0)} \overline{\mathbf{q}}_1^{(1)} = \frac{4a_3}{3} \mathbf{t}^{(1)} \tag{64}$$

The vector  $\mathbf{q}_1^{(1)}$  can be solved from Eq. (64) and the corresponding SIF is given by Eq. (58) as

$$\mathbf{K}(\psi_0) = \sqrt{\frac{N(\psi_0)}{\pi a_1 a_3}} \mathbf{L}^*(\phi_0) \Omega(\phi_0) \text{Re}[e^{i\psi_0} \mathbf{q}_1^{(1)}] \tag{65}$$

For transversely isotropic material the non-zero elements of  $\mathbf{M}^{(2)}$  are given by

$$M_{11}^{(2)} = \frac{2L_{11}^*}{\pi} [2(k'^2 - \rho)I_{40}(k) - (k'^2 - 3\rho)I_{20}(k) - \rho I_{00}]$$

$$M_{22}^{(2)} = \frac{2L_{11}^*}{\pi} [J_{20}(k) - J_{02}(k)]$$

$$M_{33}^{(2)} = \frac{2L_{11}^*}{\pi} [2(k'^2 \rho - 1)I_{40}(k) - (k'^2 \rho - 3)I_{20}(k) - I_{00}]$$

$$M_{12}^{(2)} = M_{21}^{(2)} = i \frac{4L_{11}^*}{\pi} k'(1 - \rho)I_{22}(k)$$

where the expressions for  $I_{\alpha\beta}$  and  $J_{\alpha\beta}$  in terms of the complete elliptic integrals are listed in the Appendix. For  $\mathbf{t}^{(1)} = (0, t_2^{(1)}, 0)^T$ , Eq. (64) yields

$$\mathbf{q}_1^{(1)} = \frac{\pi a_3}{3L_{22}^*} \left( 0, \frac{\text{Re}[t_2^{(1)}]}{J_{20}(k)} - i \frac{\text{Im}[t_2^{(1)}]}{J_{02}(k)}, 0 \right)^T$$

and Eq. (65) becomes

$$K_I(\psi_0) = \frac{1}{3} \sqrt{\frac{\pi a_3 N(\psi_0)}{a_1}} \left( \frac{\text{Re}[t_2^{(1)}]}{J_{20}(k)} \cos \psi_0 + \frac{\text{Im}[t_2^{(1)}]}{J_{02}(k)} \sin \psi_0 \right) \tag{66}$$

Eq. (66) is identical with the result derived in Kassir and Sih (1975). For  $\mathbf{t}^{(1)} = (t_1^{(1)}, 0, 0)^T$ , where  $t_1^{(1)}$  is real, Eq. (64) yields

$$\mathbf{q}_1^{(1)} = \frac{\pi a_3 t_1^{(1)}}{3L_{11}^*} (C, 0, -ik'D)^T$$

$$C = \frac{(1 - \rho)k'^2 I_{22}(k) - J_{02}(k)}{(1 - \rho)(J_{00}(k) - k^2 J_{20}(k))I_{22}(k) - J_{20}(k)J_{02}(k)}$$

$$D = \frac{(1 - \rho)I_{22}(k)}{(1 - \rho)(J_{00}(k) - k^2 J_{20}(k))I_{22}(k) - J_{20}(k)J_{02}(k)}$$

and the corresponding SIF's are

$$K_{II}(\psi_0) = \frac{\sqrt{\pi k'} a_3 t_1^{(1)}}{3\sqrt{N(\psi_0)}} (C \cos^2 \psi_0 - D \sin^2 \psi_0)$$

$$K_{III}(\psi_0) = \frac{\sqrt{\pi \rho} a_3 t_1^{(1)}}{3\sqrt{k' N(\psi_0)}} (C + k'^2 D) \cos \psi_0 \sin \psi_0$$

The results are the same as those obtained in Kassir and Sih (1975).

### 5.3. Quadratic loading

Let

$$\mathbf{t} = -\mathbf{t}_0^{(2)} \zeta \bar{\zeta} - \text{Re}[\mathbf{t}_2^{(2)} \zeta^2] \quad (67)$$

where  $\mathbf{t}_0^{(2)}$  is real and  $\mathbf{t}_2^{(2)}$  is complex. Setting  $n = 2$  in Eq. (55) gives

$$\hat{\mathbf{g}}(\eta_1, \psi) = (1 - \eta_1^2) \text{Re} \left[ \sum_{l=0}^1 e^{i(2-2l)\psi} \sum_{m=0}^1 \eta_1^{2-2m} \mathbf{q}_{2-2m}^{(2-2l)} \right] \quad (68)$$

From Eq. (32)  $\mathbf{q}_0^{(2)}$  and  $\mathbf{q}_2^{(2)}$  are related by

$$\mathbf{q}_2^{(2)} + 5\mathbf{q}_0^{(2)} = 0 \quad (69)$$

Substitution of Eqs. (67) and (68) into Eq. (35) leads to

$$\mathbf{M}^{(0)}(\mathbf{q}_2^{(0)} - \mathbf{q}_0^{(0)}) + \text{Re}[\overline{\mathbf{M}^{(2)}}(\mathbf{q}_2^{(2)} - \mathbf{q}_0^{(2)})] = 0 \quad (70)$$

$$\mathbf{M}^{(0)} \mathbf{q}_2^{(0)} + \text{Re}[\overline{\mathbf{M}^{(2)}} \mathbf{q}_2^{(2)}] = \frac{2a_3}{3} \mathbf{t}_0^{(2)} \quad (71)$$

$$2 \text{Re}[\mathbf{M}^{(2)}] \mathbf{q}_2^{(0)} + \overline{\mathbf{M}^{(4)}} \mathbf{q}_2^{(2)} + \mathbf{M}^{(0)} \overline{\mathbf{q}_2^{(2)}} = \frac{4a_3}{3} \mathbf{t}_2^{(2)} \quad (72)$$

The vectors  $\mathbf{q}_2^{(0)}$  and  $\mathbf{q}_2^{(2)}$  can be solved from Eqs. (71) and (72). The vector  $\mathbf{q}_0^{(2)}$  is then determined by Eq. (69) and  $\mathbf{q}_0^{(0)}$  by Eq. (70). The corresponding SIF is

$$\mathbf{K}(\psi_0) = \sqrt{\frac{N(\psi_0)}{\pi a_1 a_3}} \mathbf{L}^*(\phi_0) \Omega(\phi_0) \left\{ \mathbf{q}_2^{(0)} + \mathbf{q}_0^{(0)} + \frac{4}{5} \text{Re}[e^{i2\psi_0} \mathbf{q}_2^{(2)}] \right\} \quad (73)$$

For transversely isotropic material subjected to quadratic normal pressure, Eqs. (71) and (72) yield

$$\text{Re}[q_2^{(2)}] = \frac{4a_3}{3} \frac{m_0 \text{Re}[t_2^{(2)}] - m_2 t_0^{(2)}}{(m_4 + m_0)m_0 - 2m_2^2}$$

$$\text{Im}[q_2^{(2)}] = \frac{4a_3}{3} \frac{\text{Im}[t_2^{(2)}]}{m_4 - m_0}$$

$$q_2^{(0)} = \frac{2a_3}{3} \frac{(m_4 + m_0)t_0^{(2)} - 2m_2 \operatorname{Re}[t_2^{(2)}]}{(m_4 + m_0)m_0 - 2m_2^2}$$

$$q_0^{(0)} = q_2^{(0)} + \frac{6m_2}{5m_0} \operatorname{Re}[q_2^{(2)}]$$

where  $m_j \equiv M_{22}^{(j)}$ ,  $j = 0, 2, 4$  and  $m_4 = 2L_{22}^*(J_{00}(k) - 8J_{22}(k))/\pi$ . The corresponding  $K_I$  is given by

$$K_I(\psi_0) = \frac{2}{15} \sqrt{\pi k' N(\psi_0)} \left\{ \frac{3}{J_{00}} t_0^{(2)} + [(J_{02} - 2J_{22})t_0^{(2)} + J_{02} \operatorname{Re}[t_2^{(2)}]] \frac{\cos^2 \psi_0}{\Delta} + [(J_{20} - 2J_{22})t_0^{(2)} - J_{20} \operatorname{Re}[t_2^{(2)}]] \frac{\sin^2 \psi_0}{\Delta} + \frac{\operatorname{Im}[t_2^{(2)}]}{J_{22}} \sin \psi_0 \cos \psi_0 \right\}$$

where  $\Delta = J_{20}J_{02} - J_{00}J_{22}$ . The above expression agrees with that in Kassir and Sih (1975) except that the first term is missing in their result.

### 6. Conclusions

The traction on the plane of a elliptic crack in a general anisotropic elastic solid is expressed in terms of a simple one-dimensional integral of the Radon transform of the displacement discontinuity convoluted by a matrix connected with the elastic constants. The integral equation is used to show that with respect to a local coordinate system the traction on the crack plane and the relative crack face displacement near the crack border are the same as those in the two-dimensional case. A general form for the Radon transform of the displacement discontinuity which can be used to obtain the stress intensity factors for arbitrary polynomial loadings is proposed. Explicit results for constant, linear and quadratic loadings are derived.

### Acknowledgements

The research was supported by the National Science Council of Taiwan under Grant No. NSC-88-2212-E-002-004.

### Appendix

In this Appendix several relevant results related to the elliptic integrals for transversely isotropic material are given.

Define  $I_{\alpha\beta}(k)$  and  $J_{\alpha\beta}(k)$  as

$$I_{\alpha\beta}(k) = \int_0^{\pi/2} \frac{\sin^\alpha t \cos^\beta t}{\sqrt{1 - k^2 \sin^2 t}} dt, \quad J_{\alpha\beta}(k) = \int_0^{\pi/2} \sin^\alpha t \cos^\beta t \sqrt{1 - k^2 \sin^2 t} dt$$

For  $\alpha, \beta > -1$ ,  $|k| < 1$ , the integrals can be expressed as (Gradshteyn and Ryzhik, 1980)

$$I_{\alpha\beta}(k) = \frac{1}{2}B\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right)F\left(\frac{\alpha+1}{2}, \frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2\right)$$

$$J_{\alpha\beta}(k) = \frac{1}{2}B\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right)F\left(\frac{\alpha+1}{2}, -\frac{1}{2}; \frac{\alpha+\beta+2}{2}; k^2\right)$$

where 'B' is the beta function and  $F$  is the hypergeometric function. In particular for  $\alpha = \beta = 0$

$$I_{00}(k) = K(k) = \frac{\pi}{2}F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right)$$

$$J_{00}(k) = E(k) = \frac{\pi}{2}F\left(\frac{1}{2}, -\frac{1}{2}; 1; k^2\right)$$

where  $K(k)$  and  $E(k)$  are the complete elliptic integrals of the first and second kinds, respectively. If  $\alpha$  and  $\beta$  are even integers,  $I_{\alpha\beta}$  and  $J_{\alpha\beta}$  are also expressible in terms of the complete elliptic integrals by using the following formulas (Gradshteyn and Ryzhik, 1980):

$$F(a, b; c+1; w) = \frac{c}{(c-a)(c-b)}(1-w)^{c+1-a-b}$$

$$\frac{d}{dw}[(1-w)^{a+b-c}F(a, b; c; w)]$$

$$F(a+1, b; c+1; w) = -\frac{c}{a(c-b)}(1-w)^{1-a}\frac{d}{dw}[(1-w)^aF(a, b; c; w)]$$

$$\frac{dK}{dw} = \frac{1}{2w}\left(\frac{E}{1-w} - K\right), \quad \frac{dE}{dw} = \frac{1}{2w}(E - K)$$

where  $w = k^2$ . Some values of  $I_{\alpha\beta}$  and  $J_{\alpha\beta}$  are listed below:

$$I_{20}(k) = \frac{1}{k^2}[K(k) - E(k)]$$

$$I_{02}(k) = I_{00}(k) - I_{20}(k) = \frac{1}{k^2}[E(k) - (1-k^2)K(k)]$$

$$J_{20}(k) = \frac{1}{3k^2}[(2k^2 - 1)E(k) + (1 - k^2)K(k)]$$

$$J_{02}(k) = J_{00}(k) - J_{20}(k) = \frac{1}{3k^2}[(k^2 + 1)E(k) - (1 - k^2)K(k)]$$

$$I_{22}(k) = \frac{1}{k^2}(I_{02}(k) - J_{02}(k)) = \frac{1}{3k^4}[(2 - k^2)E(k) - 2(1 - k^2)K(k)]$$

$$J_{40}(k) = \frac{1}{15k^4}[(8k^4 - 3k^2 - 2)E(k) - (4k^4 - 2k^2 - 2)K(k)]$$

$$J_{22}(k) = J_{02}(k) - J_{40}(k) = \frac{1}{15k^4}[(k^4 - k^2 + 1)E(k) - (k^4 - 3k^2 + 2)K(k)]$$

## References

- Barnett, D.M., Lothe, J., 1973. Synthesis of the sextic and the integral formalism for dislocations, Green's function and surface waves in anisotropic elastic solids. *Phys. Norv.* 7, 13–19.
- Deans, S.R., 1993. *The Radon Transform and Some of Its Applications*. Krieger, Malabar, FL.
- Dongye, C., Ting, T.C.T., 1989. Explicit expressions of Barnett–Lothe tensors and their associated tensors for orthotropic materials. *Q. Appl. Math.* 47, 723–734.
- Gradshteyn, I.S., Ryzhik, I.M., 1980. *Table of Integrals, Series and Products*. Academic Press, New York.
- Green, A.E., Sneddon, I.N., 1950. The distribution of stress in the neighborhood of a flat elliptic crack in an elastic solid. *Proc. Cambridge Phil. Soc.* 46, 159–164.
- Irwin, G.R., 1957. Analysis of stresses and strains near the end of a crack transversing a plate. *J. Appl. Mech.* 24, 36–6364.
- Kassir, M.K., Sih, G.C., 1966. Three-dimensional stress distribution around an elliptical crack under arbitrary loadings. *J. Appl. Mech.* 33, 601–611.
- Kassir, M.K., Sih, G.C., 1967. Geometric discontinuities in elastostatics. *J. Math. Mech.* 16, 927–948.
- Kassir, M.K., Sih, G.C., 1968. Three-dimensional stress distribution around elliptical cracks in transversely isotropic solid. *Engng Fracture Mech.* 1, 327–345.
- Kassir, M.K., Sih, G.C., 1975. Sih, G.C. (Ed.), *Mechanics of Fracture, II*. Noordhoff, Leyden.
- Sih, G.C., Liebowitz, H., 1968. Liebowitz, H. (Ed.), *Fracture, II*. Academic Press, New York.
- Sekine, H., Mura, T., 1979. The elastic field around an elliptical crack in an anisotropic medium under an applied stress of polynomial forms. *Int. J. Engng Sci.* 17, 641–649.
- Smith, F.W., Kobayashi, A.S., Emery, A.F., 1967. Stress intensity factors for penny-shaped cracks, Part I — infinite solid. *J. Appl. Mech.* 34, 947–952.
- Sneddon, I.N., 1946. The distribution of stress in the neighborhood of a crack in an elastic solid. *Proc. Roy. Soc. A* 187, 229–260.
- Sneddon, I.N., Lowengrub, M., 1969. *Crack Problems in Classical Theory of Elasticity*. Wiley, New York.
- Stroh, A.N., 1958. Dislocations and cracks in anisotropic elasticity. *Phil. Mag.* 3, 625–646.
- Ting, T.C.T., 1996. *Anisotropic Elasticity — Theory and Application*. Oxford University Press, New York.
- Willis, J.R., 1968. The stress field around an elliptic crack in an anisotropic medium. *Int. J. Engng Sci.* 6, 253–263.
- Wu, K.-C., 1989. On the crack-tip fields of a dynamically propagating crack in an anisotropic elastic solid. *Int. J. Fract.* 41, 253–266.
- Wu, K.-C., 1998. Generalization of the Stroh formalism to three-dimensional anisotropic elasticity. *J. Elasticity* 51, 213–225.