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Time-optimal control strategy for saturating linear discrete systems

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An easily implemented strategy for obtaining the required minimal-time and time-optimal control sequence for a given initial state of an nth-order linear discrete system with constraints on the input is described. This strategy uses the boundary faces of the reachable set for the determination of the minimal time. The time-optimal control sequence is achieved by dividing the reachable set into linear and saturated parts.

1. Introduction

Most control system applications encounter the problem of input saturation owing to limited power. When the control input is constrained, an optimal strategy based on an unsaturating situation may not apply. The same thing happens in the time-optimal control of saturating linear discrete systems. However, optimal switching strategies can be obtained for the time-optimal control of linear continuous systems (Ryan 1982). Subject to fixed sampling interval constraint, arbitrary switching is not possible in the case of discrete systems. Therefore, a polygonal switching curve has been constructed by Desoer and Wing (1981) to find a time-optimal strategy for a second-order discrete system. Tou (1964) stated that there might exist an infinity of time-optimal solutions for a given initial state of an nth-order discrete system. Also, one of the many possible sets of optimal control sequences has n control signals staying within the saturation limit with the remaining control signals being able to take the value at the saturation limit.

In this paper, it is shown that by dividing the reachable set into linear and saturated parts, a time-optimal control strategy can be obtained for nth-order saturating linear discrete systems. The strategy is always to give the control signals their extreme values for states staying in the saturated part, and carefully to select control signals for states in the linear part. Although only scalar input systems are considered, extensions to multiple input systems are possible.

2. Time-optimal problem of saturating linear discrete systems

Let an nth-order linear time-invariant discrete system with scalar input be governed by the following state equation:

$$X_{k+1} = AX_k + Bu_k \tag{1}$$

where k is a non-negative integer, X_k is an n-dimensional state vector at the kth iteration, u_k is a scalar control input at the kth iteration, A is an $n \times n$ non-singular system matrix, B is an $n \times 1$ vector, and the sampling interval, T, in the equation has been omitted for simplicity. The control input is subject to saturating constraint and

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has the following properties:

$$|u_k| \leqslant L, \quad \text{for all } k \tag{2}$$

$$u_t = u_k, \quad \text{for } k \leqslant t < (k+1) \tag{3}$$

where L is the saturation limit.

In general, the equilibrium state can be assumed to be $X_e = 0$. Then, given an initial state X_0, if the system can be driven to equilibrium after m sampling intervals, (1) can be solved as

$$A^m X_0 + \sum_{i=0}^{m-1} A^{m-i-1} B u_i = 0 \tag{4}$$

Since A is non-singular, after multiplying by A^{-m} on both sides and rearranging, (4) becomes

$$X_0 = -\sum_{i=0}^{m-1} A^{-i-1} B u_i \tag{5}$$

The time-optimal control problem is the following: given the completely state controllable system described by (1) and initial state X_0, find the minimal number of control periods, m, and the sequence of control signals u_i, $i = 0, 1, \ldots, m-1$ such that (5) can be satisfied.

3. Determination of minimal time

3.1. *Reachable set*

Let a reachable set, R_m, represent the states that can reach equilibrium in m intervals or less. Define the frame-vectors of R_m as $F_i = -A^{-i-1}B$, $i = 0, 1, 2, \ldots, m-1$ and, if $m < n$, define the dummy frame-vectors of R_m as $F_i = -A^{-i-1}B$, $i = m, m+1, \ldots, n-1$. Then a reachable set, R_m, in the n-dimensional state space S^n is

$$R_m \triangleq \left\{ X \mid X \in S^n \quad \text{and} \quad X = \sum_{i=0}^{m-1} F_i u_i \right. \\ \left. \text{where } |u_i| \leqslant L, \quad i = 0, 1, \ldots, m-1 \right\} \tag{6}$$

From the definition of a reachable set, it can be proved that R_m is convex closed and bounded, and is symmetrical with respect to the equilibrium and $R_0 \triangleq 0$ (Desoer and Wing 1961).

3.2. *Boundary of reachable set*

A boundary face of R_m is defined as the intersection of a support hyperplane to R_m with R_m. The largest dimension for a hyperplane in S^n is $n-1$, and thus the largest dimension for a boundary face of R_m is $n-1$. It can be observed that a boundary face of R_m with dimension less than $n-1$ is contained in some $(n-1)$-dimensional boundary faces (Lin 1970). An $(n-1)$-dimensional hyperplane containing a boundary face of R_m can be determined uniquely from

$$V'_a(X - X_b) = 0 \tag{7}$$

where V'_a is a unit vector normal to $n-1$ linearly independent frame-vectors of R_m and X_b is a state on the boundary face corresponding to the selected $n-1$ linearly independent frame-vectors (Lin 1970). Obviously, for $m < n$, there will be only m frame-vectors for R_m and it will then not be possible to determine a unique solution from (7) for the boundary face of R_m. In this case, V_α will be chosen as a unit vector normal to $n-1$ linearly independent vectors from frame-vectors $F_0, \ldots, F_{m-1}$ and dummy frame-vectors $F_m, \ldots, F_{n-1}$ of R_m. Consequently, the same number of boundary faces can be obtained for the case $m < n$ and $m = n$. Since the reachable set is symmetrical with respect to the equilibrium, if a boundary face of R_m has V_α as its normal vector then $V_\beta = -V_\alpha$ is the normal vector of its symmetrical boundary face. As a result $2h$, $h \leqslant \binom{m}{n-1}$ boundary faces can be obtained for each R_m.

Let

$$\Phi = \{0, 1, \ldots, q\}, \quad \text{where } q = m-1 \text{ for } m \geqslant n \text{ and } q = n-1 \text{ for } m < n$$

$$\theta_\alpha = \{\alpha_1, \alpha_2, \ldots, \alpha_i, \ldots, \alpha_{n-1}\},$$

where

$$\alpha_i \in \Phi, \quad i = 1, 2, \ldots, n-1, \ \alpha = 1, 2, \ldots, h$$

and $F_{\alpha i}$ for all $\alpha_i \in \theta_\alpha$ are linearly independent frame-vectors or dummy frame-vectors.

$$\bar{\theta}_\alpha = \Phi - \theta_\alpha$$

$$\theta_{\alpha 0} = \theta_\alpha \quad \text{and} \quad 0 \in \theta_{\alpha_0}$$

$$\bar{\theta}_{\alpha_0} = \Phi - \theta_\alpha \quad \text{and} \quad 0 \in \bar{\theta}_{\alpha_0}$$

Clearly, for a system given by (1) and h,

$$h \leqslant \binom{m}{n-1}$$

sets of θ_α can be obtained from Φ. Each set of θ_α corresponds to two mutually symmetrical boundary faces of R_m.

Theorem 1

A boundary state, X_b, on the αth boundary face of R_m with V_α as its unit normal vector can be described by

$$X_b = \sum_{i \in \theta_\alpha} F_i u_i + \sum_{j \in \bar{\theta}_\alpha} F_j L_j^* \tag{8}$$

where

$$|u_i| \leqslant L \quad \text{and} \quad L_j^* = L \ \operatorname{sgn}(V'_\alpha F_j)$$

and the projection of X_b in the direction of V_α is

$$\sum_{j \in \bar{\theta}_\alpha} V'_\alpha F_j L_j^*$$

Proof

The αth boundary face of R_m with V_α as its unit normal vector is the intersection of R_m and the hyperplane defined by (7) with $V'_\alpha F_{\alpha_i} = 0$ for all $\alpha_i \in \theta_\alpha$. The boundary state

X_b on the αth boundary face of R_m can be described as

$$X_b = \sum_{i=1}^{m-1} F_i u_i = \sum_{i \in \theta_\alpha} F_i u_i + \sum_{j \in \bar{\theta}_\alpha} F_j u_j \tag{9}$$

The projection of X_b in the direction of V_α is

$$V'_\alpha X_b = V'_\alpha \left[\sum_{i \in \theta_\alpha} F_i u_i \right] + V'_\alpha \left[\sum_{j \in \bar{\theta}_\alpha} F_j u_j \right] \tag{10}$$

Since $V'_\alpha F_i = 0$ for all $i \in \theta_\alpha$, in order to give the maximal projection in the direction of V_α, u_j, $j \in \bar{\theta}_\alpha$, of (10) has to be

$$u_j = L_j^* = L\text{sgn}\,(V'_\alpha F_j) \quad \text{for all } j \in \bar{\theta}_\alpha \tag{11}$$

or

$$\max\,(V'_\alpha X_b) = \sum_{j \in \bar{\theta}_\alpha} V'_\alpha F_j L_j^* \tag{12}$$

Substituting (11) into (9), the theorem is proved.

Thus the αth boundary face of R_m and its symmetrical face can be defined as

$$\Omega_\alpha \triangleq \left\{ X \,|\, X \in S^n \quad \text{and} \quad X = \sum_{i \in \theta_\alpha} F_i u_i + \sum_{j \in \bar{\theta}_\alpha} F_j L_j^* \right\} \tag{13}$$

The boundary of R_m is the union of Ω_α, $\alpha = 1, 2, \ldots, h$, and their symmetrical boundary faces. That is

$$\Omega \triangleq \left\{ X \,|\, X \in S^n \text{ and } X = \sum_{i \in \theta_\alpha} F_i u_i + \sum_{j \in \bar{\theta}_\alpha} F_j L_j^* \text{ for all } \quad \alpha = 1, 2, \ldots, h \right\} \tag{14}$$

3.3. *Minimal time required*

The minimal number of control iterations, m, required to drive an initial state X_0 to equilibrium can be determined as the minimal value of m such that $X_0 \in R_m$. The following theorem shows how to do this.

Theorem 2

A state X_0 is in R_m if and only if the following conditions are satisfied

$$|V'_\alpha X_0| \leqslant \sum_{j \in \bar{\theta}_\alpha} V'_\alpha F_j L_j^* \quad \text{for all } \alpha = 1, 2, \ldots, h \tag{15}$$

Proof

Since the reachable set is convex and bounded, if state X_0 is inside R_m, then the projections of X_0 in every direction of V_α, $\alpha = 1, 2, \ldots, h$ will be less than or equal to the projection of a corresponding boundary state in the same direction. According to Theorem 1, the proof is obvious.

In order to obtain the minimal m, an algorithm starting from $m = 1$ is used and the conditions of (15) are checked. If X_0 is not in R_m then m is increased by one. The same process can be carried out repeatedly until a proper value of m is found.

4. Time-optimal control sequence

Define the minimal reachable set R_m for a given state X as the reachable set with minimal m such that $X \in R_m$. Then a minimal-time control effort can be defined as a driving signal that can force the system from a given state in the minimal reachable set R_m to R_{m-1}. A sequence of such control efforts will drive the system from the initial state to equilibrium in the minimal time. When the minimal time required has been discovered, the minimal-time control sequence can be found by solving (5). However, there is no direct method for solving (5) since the control signal is subject to saturating limitation.

Therefore, a systematic strategy will be introduced, as follows. Always force the system from its minimal reachable set R_m to R_{m-1} with the extreme value of the control effort in one sampling interval, except when an extreme value of the control effort will not be a minimal-time control effort.

The states in the minimal reachable set R_{m+1} can be expressed as an extension from the boundary of R_m by a frame-vector F_m. That is

$$X = \sum_{i \in \theta_\alpha} F_i u_i + \sum_{j \in \bar{\theta}_\alpha} F_j L_j^* + F_m u_m \tag{16}$$

where

$$u_m = \operatorname{sgn}(V'_\alpha F_m) u_m^* \quad 0 < u_m^* \leqslant L$$

or

$$R_{m+1} - R_m \triangleq \left\{ X \mid X \in S^n, \text{ and } X = \sum_{i \in \theta_\alpha} F_i u_i + \sum_{j \in \bar{\theta}_\alpha} F_j L_j^* + F_m u_m \right.$$
$$\left. \text{for all } \alpha = 0, 1, \ldots, h \right\} \tag{17}$$

Given an initial state X_0, $X_0 \in \{R_{m+1} - R_m\}$, one of the minimal-time control sequences can be found by solving (16) for u_i, $i \in \theta_\alpha$; $L_j^*, j \in \bar{\theta}_\alpha$; and u_m. However, it is not easy to do the calculation.

According to (16), the current minimal-time control effort is u_0. Therefore, if $0 \in \bar{\theta}_\alpha$ then $u_0 = L_0^*$, an extreme value of control can be a minimal-time control effort. On the contrary, if $0 \in \theta_\alpha$, then an unsaturating control effort should be used. Based on this, the reachable set can be divided into two parts. They are P_s: the saturated part; and P_L: the linear part. P_s is that part of the reachable set in which an extreme value of control is a suitable minimal-time control effort. P_L is that part in which a minimal-time control effort is within the saturating limits. Therefore, P_s and P_L can be described as

$$P_s \triangleq \left\{ X \mid X \in S^n, \text{and } X = \sum_{j \in \theta_\alpha} F_i u_i + \sum_{j \in \bar{\theta}_{\alpha_0}} F_j L_j^* + F_m u_m \right.$$
$$\left. \text{for all } m = 1, 2, \ldots, \text{ and } \alpha = 1, 2, \ldots, h \right\} \tag{18}$$

and

$$P_L \triangleq \left\{ X \mid X \in S^n, \text{and } X = \sum_{i \in \theta_{\alpha_0}} F_i u_i + \sum_{j \in \bar{\theta}_\alpha} F_j L_j^* + F_m u_m \right.$$
$$\left. \text{for all } m = 1, 2, \ldots, \text{and } \alpha = 1, 2, \ldots, h \right\} \tag{19}$$

where θ_{α_0} and $\bar{\theta}_{\alpha_0}$ are defined as before.

If the system comes within the linear part and the minimal time, m, is known, then the minimal-time control sequence can be obtained by solving the expression in (19) for some θ_{α_0}. That is, by solving

$$\sum_{i \in \theta_{\alpha_0}} F_i u_i + F_m u_m = X_0 - \sum_{j \in \bar{\theta}_\alpha} F_j L_j^* \tag{20}$$

or the vector–matrix equation

$$FU = Y \tag{21}$$

where

$$F = [F_{\alpha_1}, F_{\alpha_2}, \ldots, F_{\alpha_{n-1}}, F_m]$$
$$U = [u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_{n-1}}, u_m]$$

and

$$Y = \left[X_0 - \sum_{j \in \bar{\theta}_\alpha} F_j L_j^* \right]$$

Since $L_j^* = L \operatorname{sgn}(V'_\alpha F_j), j \in \bar{\theta}_\alpha$ can be determined in advance if θ_{α_0} is certain. Therefore, (21) can easily be solved by elementary matrix operations.

Case 1 F_m is independent of F_i, $i \in \theta_{\alpha_0}$.

F^{-1} exists and the solution is uniquely determined by

$$U = F^{-1} Y \tag{22}$$

Case 2 F_m is dependent on F_i, $i \in \theta_{\alpha_0}$.

It is possible to find a number of elementary row operations,

$$E = E_q, \ldots, E_1$$

such that

$$EFU = EY \tag{23}$$

where

$$EF = \begin{bmatrix} F_e \\ 0_{1 \times n} \end{bmatrix}$$

and

$$EY = \begin{bmatrix} Y_e \\ 0 \end{bmatrix}$$

Then U can be obtained by solving

$$F_e U = Y_e \tag{24}$$

A general solution to (24) is

$$U = F'_e (F_e F'_e)^{-1} Y_e + \{ I_n - F'_e (F_e F'_e)^{-1} F_e \} W \tag{25}$$

where I_n is an $n \times n$ identity matrix and W is any $n \times 1$ column matrix (Lin *et al.* 1983). Among the solutions of (25), only those satisfying the saturating limitation can be used.

As a result, the remaining time-optimal control sequence is

$$u_i,\ i \in \theta_{\alpha 0};\quad u_j = L_j^*,\ j \in \bar{\theta}_\alpha \quad \text{and} \quad u_m$$

Therefore, the time-optimal control strategy can be stated as the following steps.

(1) Determine the minimal time required for the current state with the method described in § 3.

(2) Compute the control sequence with (21). If any one of the solved control signals is not within the saturation limit, then the current state stays in the saturated part. An extreme control effort with proper polarity is used as the current time-optimal control effort. Then decrease the minimal time by one and repeat the computations.

(3) Otherwise, if all the computed control signals are within the saturation limit, the current state is in the linear part. The computed control signals can be ordered as a time-optimal control sequence for the remaining control intervals.

Of course, the above steps can be reorganized as a recursive procedure which calculates the required minimal time and time-optimal control effort at each interval to prevent any calculation error or system fault.

5. Example

Consider the second order plant $G(s) = \dfrac{1}{s^2}$. The sampling interval is $T = 0{\cdot}4$ s, the initial condition on the plant is $X_0 = (-6, 1)$ and the saturation limit is $L = 1{\cdot}0$. The state equation for the plant in the discrete version is shown as (1) where

$$A = \begin{bmatrix} 1 & 0{\cdot}4 \\ 0 & 1 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0{\cdot}08 \\ 0{\cdot}4 \end{bmatrix}$$

By the definition of the frame-vector, we have

$$F_i = -\begin{bmatrix} 1 & 0{\cdot}4 \\ 0 & 1 \end{bmatrix}^{-i-1} \begin{bmatrix} 0{\cdot}08 \\ 0{\cdot}4 \end{bmatrix}, \quad i = 0, 1, 2, \ldots$$

Since this is a second-order system, θ_α has only one element and $\theta_{\alpha 0} = \{0\}$. Therefore, according to (19) the linear part can be represented by

$$P_L \triangleq \left\{ X \mid X \in S^n, \text{and } X = F_0 u_0 + \sum_{j \in \bar{\theta}_\alpha} F_j L_j^* + F_m u_m \text{ for all } m = 1, 2, \ldots, \text{ and } \alpha = 1 \right\}$$

where

$$\bar{\theta}_\alpha = \{1, 2, \ldots, m-1\} \quad \text{and} \quad V_i' F_0 = 0$$

The linear part is plotted in the Figure for $m \leqslant 30$. The time-optimal control efforts for $X = (-6, 1)$ are found as follows.

Step 1. The minimal time required is determined from (15). In this case, it is found that as

$$\Phi = \{0, 1, 2, \dots, 10\}$$

the conditions

$$|V'_\alpha X_0| \leqslant \sum_{j \in \bar{\theta}_*} V'_\alpha F_j L_j^*, \quad \text{for all } \alpha = 1, 2, \dots, h$$

are satisfied. Thus $m = 11$ is the minimum time required.

Step 2. X_0 is first treated as a state in the linear part. The control sequence is then calculated from

$$X = F_0 u_0 + \sum_{j \in \bar{\theta}_*} F_j L_j^* + F_m u_m$$

or in vector matrix form

$$\begin{bmatrix} u_0 \\ u_m \end{bmatrix} = [F_0 F_m]^{-1} \left[X - \sum_{j \in \bar{\theta}_*} F_j L_j^* \right]$$

The first three computations for $m = 11$, 10 and 9 show that the system is in the saturated part and thus $u_0 = u_1 = u_2 = 1$ are used.

Step 3. As $m = 8$, the computed control sequence is $u_3 = 0{\cdot}75$, $u_4 \sim u_9 = -1$, $u_{10} = -0{\cdot}25$.

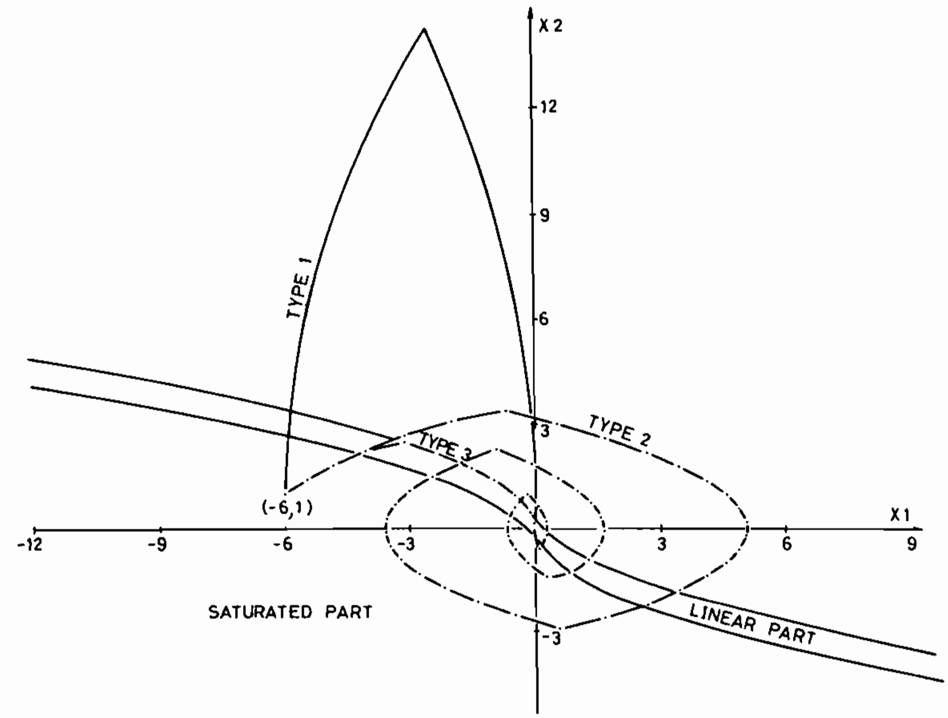
The results are shown in the Table and the Figure, labelled 'Type 3'. In comparison, if the saturating constraint is considered, but only the state feedback control law for the unsaturating system, as shown in the following equation, is applied (Kuo 1977), then

$$u_k = [1 \quad 0][F_0 \quad F_1]^{-1}[X_k]$$

Type of controller	Number of sampling periods	Control sequence		
Type 1	2	$u_0 = 33{\cdot}75$	$u_1 = -36{\cdot}25$	
Type 2	58	$u_{0\text{-}5} = 1$	$u_{6\text{-}20} = -1$	$u_{21} = -0{\cdot}750009$
		$u_{22\text{-}34} = 1$	$u_{35\text{-}43} = -1$	$u_{44} = -0{\cdot}249824$
		$u_{45\text{-}49} = 1$	$u_{50} = 0{\cdot}998987$	$u_{51\text{-}53} = -1$
		$u_{54} = -0{\cdot}996625$	$u_{55} = 1$	$u_{56} = 0{\cdot}994937$
		$u_{57} = -0{\cdot}497469$		
Type 3	11	$u_{0\text{-}2} = 1$	$u_3 = 0{\cdot}75$	$u_{4\text{-}9} = -1$
		$u_{10} = -0{\cdot}25$		

Listing of control sequences and times needed.

The resulting control sequence is listed in the Table and the trajectory is plotted in the Figure, labelled 'Type 2'. It can be observed that the result is no longer a time-optimal control.



State plane plot for $G(s) = 1/s^2$; $T = 0{\cdot}4$.

Assume that the control input is not subject to constraint. The control efforts can be obtained directly from Kuo (1977):

$$\begin{bmatrix} u_0 \\ u_1 \end{bmatrix} = [F_0 \quad F_1]^{-1}[X_0]$$

The results are presented in the Table and the Figure, labelled 'Type 1'. Obviously, the control efforts run out off the limitation.

6. Conclusions

A systematic strategy for the determination of the minimal time and the time-optimal control sequence for saturating linear discrete systems is presented. The strategy is based on the construction of linear and saturated parts in a reachable set. The resulting operation of control is much like time-optimal bang–bang control. Only when the system has a state in the linear part is it driven by a linearly adjusted control effort. Otherwise, a control effort with its extreme value is used. Although only the method for the scalar-input case is discussed in this paper, extension to a multi-input system is possible.

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