phenomena. In the case of the infrared signals, the data is produced by natural surfaces, which vary from the idealistic model. The signals do not retain their characteristics over all scales, having lower and upper limits determined by the grain size and region size of the surface under interrogation. The range of scales under which the fLsm model holds is also a characteristic of the model. Techniques need to be investigated that characterize diverse behavior over different scaling regions (i.e., multiscaling).

## REFERENCES


[1] J. M. Chambers, C. L. Mallows, and B. W. Stuck, "A method for simulating stable random variables," *J. Amer. Statistic. Assoc.*, vol. 71, no. 354, pp. 340–344, June 1976.

[2] A. Fournier, D. Fussell, and L. Carpenter, "Computer rendering of stochastic models," *Commun. ACM*, vol. 25, no. 6, pp. 371–384, June 1982.

[3] T. Higuchi, "Approach to an irregular time series on the basis of fractal theory," *Physica D.*, Vol. 31, pp. 277–283, 1988.

[4] S. M. Kogon and D. G. Manlokis, "Linear parametric models for signals with long-range dependence and infinite variance," in *Proc. ICASSP-95*, Detroit, 1995, pp. 1597–1600.

[5] S. M. Kogon and D. B. Williams, "On the characterization of impulsive noise with $\alpha$-stable distributions using Fourier techniques," in *Proc. 29th Asilomar Conf. Signals, Systems, Comput.*, 1995.

[6] B. B. Mandelbrot and B. J. Van Ness, "Fractional Brownian motion, fractional Gaussian noises, and applications," *SIAM Review*, vol. 10, no. 4, pp. 422–438, 1968.

[7] B. B. Mandelbrot, *The Fractal Geometry of Nature.* San Francisco, CA: W. H. Freeman, 1982.

[8] C.-K. Peng *et al.*, "Long-range anticorrelations and non-Gaussian behavior of the heartbeat," *Physical Review Lett.*, vol. 70, no. 9, pp. 1343–1346, Mar. 1993.

[9] G. Samorodnitsky and M. S. Taqqu, "Linear models with long-range dependence and with finite and infinite variance," in D. Brillinger *et al.*, Eds., *New Directions in Time Series Analysis, Part II.* New York: Springer-Verlag, 1992, pp. 325–340.

[10] G. Samorodnitsky and M. S. Taqqu, *Stable Non-Gaussian Random Processes: Stochastic Models With Infinite Variance.* New York: Chapman & Hall, 1994.

[11] M. Shao and C. L. Nikias, "Signal processing with fractional lower order moments: Stable processes and their applications," *Proc. IEEE*, vol. 81, no. 7, pp. 986–1010, July 1993.

[12] M. S. Taqqu and R. Wolpert, "Infinite variance self-similar processes subordinate to a Poisson measure," *Zeitsch. für Wahrscheinlichkeitstheorie und verwandte Gebiete*, vol. 62, pp. 53–72, 1983.

[13] V. M. Zolotarev, *One-Dimensional Stable Distributions.* Providence, RI: American Mathematical Society, 1986.


# Statistical Analysis of Space-Varying Morphological Openings with Flat Structuring Elements

Chu-Song Chen, Ja-Ling Wu, and Yi-Ping Hung

*Abstract*— **In this work, the statistical behavior of a special type of morphological operation, the space-varying opening, is analyzed and investigated. The major distinction between the space-invariant morphological operation and the space-varying one is that the structuring element corresponding to each position in the space-invariant operation is the same, but the structuring element corresponding to each position in the space-varying operation may be different. In this paper, a regular statistical analysis method, which was originally designed for the analysis of the space-invariant opening, is applied and generalized to the analysis of the space-varying one. For the filtering of a noisy step edge, some comparative analytic results are obtained revealing that edge-preserving ability can be improved by the use of the space-varying opening.**

## I. INTRODUCTION

Usually, the morphological operations applied in most image processing applications are the *space-invariant* (SI) ones. One of the most important SI morphological operations is the SI opening [5]. If a smooth *structuring element* (SE) is used in the SI opening, the convex acute parts of a signal can be replaced by smooth curves. Hence, the SI opening can be used for signal smoothing and noise filtering. To sufficiently understand the filtering process, both syntactical and statistical analysis are needed. As a nonlinear filter based on geometric concepts, standard statistical methods cannot easily be applied to the analysis of the SI opening. In [2], Stevenson and Arce found the output distribution function of the SI opening with flat SE's by carrying out the threshold decomposition of them. In [3] and [8], an elegant approach has been proposed by Morales and Acharya, which provides general solutions for the statistical analysis of the SI openings of any compact, convex, and homothetic SE's.

The SI opening can remove noises smaller than the size of the SE. However, some important structures of the signal may be removed as well. For example, if we use the SI opening for filtering a noisy step edge in Fig. 1(a), the mean of the statistical analysis result can be obtained as shown in Fig. 1(b) (which is adopted from the experimental results shown in [3]). In this figure, the noise step edge model is defined as

$$f(x) = \begin{cases} N_x, & \text{if } x < 0 \\ h + N_x, & \text{if } x \geq 0 \end{cases} \tag{1.1}$$

where $h$ is the strength of the edge (here we choose $h = 6$), $N_x$ is an independent identically distributed (i.i.d.) Gaussian random noise with zero-mean and unity variance, and the SE used is a flat one with domain $\{-1, 0, 1\}$. It can be observed that the noise variance has been considerably reduced, but the sharp variation part of the step edge (e.g., at $x = 0$ or $x = -1$) has also been blurred, as shown in Fig. 1(b), by using the SI opening. Notice that the SI opening can be viewed as a window operator. To filter a signal such that the sharp variation parts are preserved and the noises are considerably

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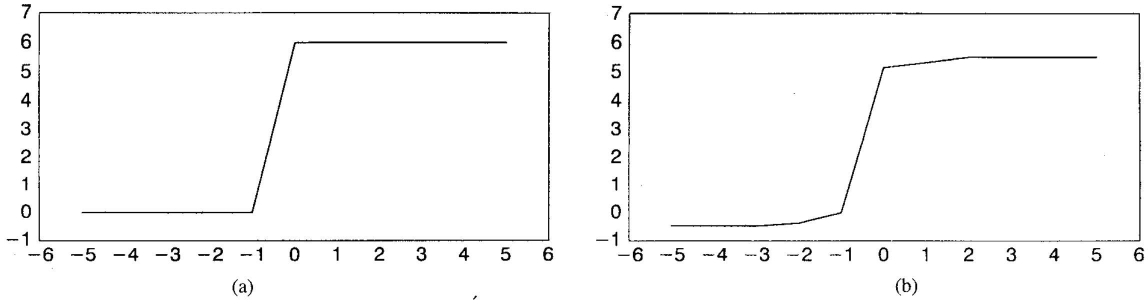


Fig. 1. (a) A step edge model with height 6; (b) the mean of the output distribution using the SI opening.

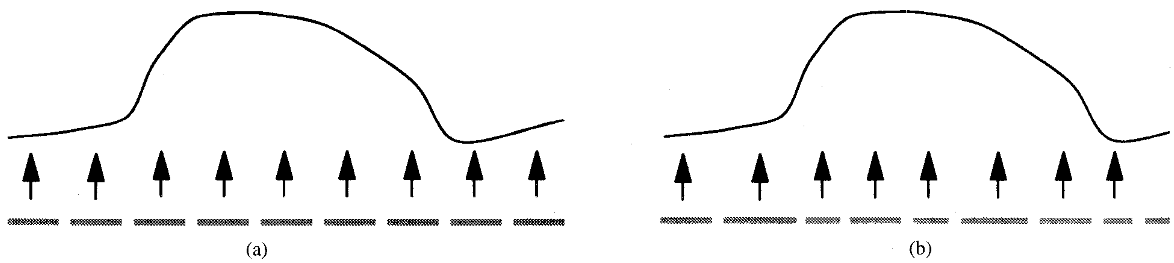


Fig. 2. (a) SI operation: using the same SE in each position; (b) SV operation: using distinct SE's in each position.

removed, some researchers [6] have shown that the window operators may change their corresponding sizes such that short windows are assigned to the sharp variation parts and long windows are assigned to the others. For the morphological opening operation this means that smaller SE's should be used for the sharp variation parts and larger SE's should be used for the other parts. In general, such an operation needs the SE to be changed with respect to spatial positions, which cannot be achieved using SI openings. Hence, the *space-varying* (SV) opening introduced in the next section should be applied.

## II. Space-Varying Opening

The major difference between the SI morphological operator and the SV morphological operator is that the SE corresponding to each position in the SI operation is the same, but the SE corresponding to each position in the SV operation may be different, as shown in Fig. 2 (assume flat SE's are used). Let $f\colon F \to R$ be a 1-D signal, and $k_x\colon K_x \to R$ be the SE function with respect to the position $x$, where $F \subseteq R$ and $K_x \subseteq R$ are the domains of $f$ and $k_x$, respectively. Let $\mathbf{k} = \{k_x\colon K_x \to R \mid x \in R\}$. The SV erosion [1, Theorem 9.3, Remark (2)(i)] of $f$ by $\mathbf{k}$ is defined as

$$(f \,\bar{\ominus}\, \mathbf{k})(x) = \min\{f(x+z) - k_x(z) \mid z \in K_x \quad \text{and} \quad x+z \in F\}. \tag{2.1}$$

For SV erosion, the corresponding adjunctive dilation [4, Proposition 2.10] can be written as follows:

$$(f \,\underline{\oplus}\, \mathbf{k})(x) = \max\{f(x-z) + k_{x-z}(z) \mid z \in K_{x-z} \text{ and } x-z \in F\} \tag{2.2}$$

The SV opening (denoted by $f \,\bar{\circ}\, \mathbf{k}$) is then defined to be the composition of the SV erosion and its corresponding adjunctive dilation

$$f \,\bar{\circ}\, \mathbf{k} = (f \,\bar{\ominus}\, \mathbf{k}) \,\underline{\oplus}\, \mathbf{k}. \tag{2.3}$$

The SV opening is an increasing, anti-extensive, and idempotent morphological operation that belongs to the class of the generalized opening. The geometric interpretation of the SV opening can be explained as follows:

*Property 1—Geometric Interpretation of the SV Opening:* The SV opening of $f$ by $\mathbf{k}$ can be visualized as sliding $k_x$ under $f$, with the shape of $k_x$ varying with position $x$, and the locus of all the highest points reached by some part of $k_x (x \in R)$ during the slide then constitutes the opening result.

It can be observed that the SV opening and the SI opening have similar geometric interpretations, except that the SE can be different with each spatial position in the SV case.

## III. Statistical Analysis of Space-Varying Opening With Flat SE's

In this section, the method presented in [3] is adopted and generalized for the statistical analysis of the SV opening. In this work, we only consider rectangular sampling grids and flat SE's. Thus, a SE can be expressed by its support domain $K \subseteq Z$. Let $w_i$ be a flat SE with domain $\{-i, -i+1, \ldots, 0, \ldots, i\}$. We will start the analysis for the case of a 1-D discrete time input signal $f\colon Z \to R$, filtered with a set of flat SE's $\mathbf{k} = \{k_x \mid x \in Z\}$, by the SV opening. Let $\mathbf{k}$ be defined as follows (see Fig. 3).

$$k_x = \begin{cases} w_0, & \text{if } x = 0 \\ w_1, & \text{if } x \neq 0. \end{cases} \tag{3.1}$$

Thus, the domain of $k_x$ is as follows:

$$K_x = \begin{cases} \{0\}, & \text{if } x = 0 \\ \{-1, 0, 1\}, & \text{if } x \neq 0. \end{cases}$$

When the input $f$ is a stochastic function with a given probability distribution for each position $x$, it is useful to find the output distributions of $f \,\bar{\circ}\, \mathbf{k}(x)$, for all $x \in Z$. According to the definition as well as the geometric interpretation of the SV opening, $f \,\bar{\circ}\, \mathbf{k}(x) (x \in Z)$ can be decomposed into the following max-min structures:

$$\begin{aligned} f \,\bar{\circ}\, \mathbf{k}(-1) &= ((f \,\bar{\ominus}\, \mathbf{k}) \,\underline{\oplus}\, \mathbf{k})(-1) \\ &= \max[f \,\bar{\ominus}\, \mathbf{k}(-1-z) \mid z \in K_{-1-z}] \\ &= \max[f \,\bar{\ominus}\, \mathbf{k}(-2), f \,\bar{\ominus}\, \mathbf{k}(-1)] \\ &= \max[\min[f(-3), f(-2), f(-1)], \\ &\qquad \min[f(-2), f(-1), f(0)]]. \end{aligned} \tag{3.2}$$

Similarly,


Flat structuring element $w_0$ Flat structuring element $w_1$

(a) (b)

Fig. 3. The flat structuring elements $\geq 0$ and $\geq 1\ 0\ 1\ 0\ -1$ flat structuring element $w_0$ flat structuring element $w_1$.

$$\begin{aligned} f \bar{\circ} \mathbf{k}(0) &= ((f \bar{\ominus} \mathbf{k}) \underline{\oplus} \mathbf{k})(0) \\ &= \max[f \bar{\ominus} \mathbf{k}(-z) \mid z \in K_{-z}] \\ &= \max[f \bar{\ominus} \mathbf{k}(-1), f \bar{\ominus} \mathbf{k}(0), f \bar{\ominus} \mathbf{k}(1)] \\ &= \max[\min[f(-2), f(-1), f(0)], f(0), \\ &\qquad \min[f(0), f(1), f(2)]] \end{aligned} \tag{3.3}$$

$$\begin{aligned} f \bar{\circ} k(1) &= ((f \bar{\ominus} \mathbf{k}) \underline{\oplus} \mathbf{k})(1) \\ &= \max[f \bar{\ominus} \mathbf{k}(1-z) \mid z \in K_{1-z}] \\ &= \max[f \bar{\ominus} \mathbf{k}(2), f \bar{\ominus} \mathbf{k}(1)] \\ &= \max[\min[f(3), f(2), f(1)], \\ &\qquad \min[f(2), f(1), f(0)]]. \end{aligned} \tag{3.4}$$

According to the geometric interpretation, the SV opening performs the same for $|x| \geq 2$. The max-min structure for $|x| \geq 2$ is shown as follows:

$$\begin{aligned} f \bar{\circ} \mathbf{k}(x) = \max[&\min[f(x-2), f(x-1), f(x)], \\ &\min[f(x-1), f(x), f(x+1)], \\ &\min[f(x), f(x+1), f(x+2)]], \quad |x| \geq 2. \end{aligned} \tag{3.5}$$

Let $p_x(y)$ be the probability density function of $f \bar{\circ} \mathbf{k}(x)$. Now let's consider the main problem of statistical analysis, that is, representing the output distribution $p_x(y)$ as a function of the input distributions. Notice that $p_x(y)$ can be written as

$$\begin{aligned} p_x(y) &= \frac{d}{dy} \text{pr}(f \bar{\circ} \mathbf{k}(x) \leq y) \\ &= \lim_{dy \to 0} \frac{1}{dy} \text{pr}(y < f \bar{\circ} \mathbf{k}(x) \leq y + dy) \\ &= \lim_{dy \to 0} \frac{1}{dy} \sum_{i=-2}^{2} \text{pr}(f \bar{\circ} \mathbf{k}(x) = f(x+i), \\ &\qquad y < f(x+i) \leq y + dy) \\ &= \lim_{dy \to 0} \frac{1}{dy} \sum_{i=-2}^{2} \text{pr}(y < f(x+i) \leq y + dy) \\ &\qquad \times \text{pr}(f \bar{\circ} \mathbf{k}(x) = f(x+i) \mid y < f(x+i) \leq y + dy). \end{aligned} \tag{3.6}$$

where pr$(A)$ is the probability of event $A$.

Let $r_x(y)$ be the probability density function of $f(x)$, and $R_x(y) =$ pr$(f(x) \leq y)$ be the probability distribution function of $f(x)$. Then (3.6) can be rewritten as

$$p_x(y) = \sum_{i=-2}^{i=2} r_{x+i}(y) \lim_{dy \to 0} \Phi_{x,i}(y, dy) \tag{3.7}$$

where

$$\Phi_{x,i}(y, dy) = \text{pr}(f \bar{\circ} \mathbf{k}(x) = f(x+i) \mid y < f(x+i) \leq y + dy). \tag{3.8}$$

We want to represent $p_x(y)$ in terms of the input statistics $r_x(y)$ and $R_x(y)$ for each $x \in Z$. From (3.7), the only unknowns are $\Phi_{x,i}(y, dy), i = -2, -1, \ldots, 2$. Now, let's first compute $p_{-1}(y)$, the output distribution at $x = -1$. In this case, the terms that need to be obtained are $\Phi_{-1,i}(y, dy)$ for $i = -2, -1, \ldots, 2$. From (3.2), it can be observed that in this case only $f(-3), f(-2), f(-1)$, and $f(0)$ are the possible outputs, while $f(1)$ is impossible to be the output. As a consequence, $\Phi_{-1,2}(y, dy) = 0$. Other terms can be obtained by combining several statistical events as shown in the following:

$$\begin{aligned} &\Phi_{-1,-2}(y, dy) \\ &\quad = \text{pr}(f \bar{\circ} \mathbf{k}(-1) = f(-3) \mid y < f(-3) \leq y + dy) \\ &\quad = \text{pr}(f(0) \leq y, y + dy < f(-2), f(-1)) \\ &\quad = R_0(y)(1 - R_{-2}(y + dy))(1 - R_{-1}(y + dy)) \end{aligned} \tag{3.9}$$

$$\begin{aligned} &\Phi_{-1,-1}(y, dy) \\ &\quad = \text{pr}(f \bar{\circ} \mathbf{k}(-1) = f(-2) \mid y < f(-2) \leq y + dy) \\ &\quad = \text{pr}(y + dy < f(-3), f(-1), f(0)) \\ &\qquad + \text{pr}(f(0) \leq y, y + dy < f(-3), f(-1)) \\ &\qquad + \text{pr}(f(-3) \leq y, y + dy < f(-1), f(0)) \\ &\quad = (1 - R_{-3}(y + dy))(1 - R_{-1}(y + dy))(1 - R_0(y + dy)) \\ &\qquad + R_0(y)(1 - R_{-3}(y + dy))(1 - R_{-1}(y + dy)) \\ &\qquad + R_{-3}(y)(1 - R_{-1}(y + dy))(1 - R_0(y + dy)). \end{aligned} \tag{3.10}$$

Similarly, $\Phi_{-1,0}(y, dy)$ and $\Phi_{-1,1}(y, dy)$ can be obtained as follows:

$$\begin{aligned} &\Phi_{-1,0}(y, dy) \\ &\quad = (1 - R_{-3}(y + dy))(1 - R_{-2}(y + dy))(1 - R_0(y + dy)) \\ &\qquad + R_0(y)(1 - R_{-3}(y + dy))(1 - R_{-2}(y + dy)) \\ &\qquad + R_{-3}(y)(1 - R_{-2}(y + dy))(1 - R_0(y + dy)) \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\Phi_{-1,1}(y, dy) \\ &\quad = R_{-3}(y)(1 - R_{-2}(y + dy))(1 - R_{-1}(y + dy)). \end{aligned} \tag{3.12}$$

Notice that in the above, $\Phi_{-1,i}(y, dy)(i = -2, -1, \ldots, 2)$ are successfully represented in terms of the input statistics, hence $p_{-1}(y)$ can be easily obtained by substituting them into (3.7).

Following the dual steps, $p_1(y)$ can be computed by combining the following terms from (3.4):

$$\Phi_{1,2}(y, dy) = R_0(y)(1 - R_2(y + dy))(1 - R_1(y + dy)) \tag{3.13}$$

$$\begin{aligned} \Phi_{1,1}(y, dy) = &(1 - R_3(y + dy))(1 - R_1(y + dy)) \\ &\times (1 - R_0(y + dy)) + R_0(y)(1 - R_3(y + dy)) \\ &\times (1 - R_1(y + dy)) + R_3(y)(1 - R_1(y + dy)) \\ &\times (1 - R_0(y + dy)) \end{aligned} \tag{3.14}$$

$$\begin{aligned} \Phi_{1,0}(y, dy) = &(1 - R_3(y + dy))(1 - R_2(y + dy)) \\ &\times (1 - R_0(y + dy)) + R_0(y)(1 - R_3(y + dy)) \\ &\times (1 - R_2(y + dy)) + R_3(y)(1 - R_2(y + dy)) \\ &\times (1 - R_0(y + dy)) \end{aligned} \tag{3.15}$$

$$\Phi_{1,-1}(y, dy) = R_3(y)(1 - R_2(y + dy))(1 - R_1(y + dy)). \tag{3.16}$$

Notice that for $|x| \geq 2$, the SV opening performs the same as the SI one does. Since the statistical analysis of the SI opening has been implicitly given in [3], we omit the complete derivations here while only listing the results of the SI case and denoting it to be $s_x(y)$, which is shown in the Appendix. The output distribution can then be described as $p_x(y) = s_x(y)$, for $|x| \geq 2$. It can also be observed that, with the SE assigned as (3.1), $f \bar{\circ} \mathbf{k}(0)$ is always equal to $f(0)$, hence, $p_0(y) = r_0(y)$.

## IV. Statistical Analysis of the Filtering Result of a Noisy Step Edge Model

Assume a noisy step edge model is defined as in (1.1). How to assign a set of SE's to each spatial position is an important problem in

TABLE I
THE MEAN, VARIANCE, AND SKEWNESS OF THE OUTPUT DISTRIBUTION OF A NOISY STEP EDGE USING (A) SI OPENING; (B) SV OPENING

| *x* | *−3* | *−2* | *−1* | *0* | *1* | *2* |
|---|---|---|---|---|---|---|
| *mean* | *−0.4801* | *−0.3811* | *−0.0000* | *5.1537* | *5.3368* | *5.5198* |
| *variance* | *0.4938* | *0.5791* | *0.9999* | *0.5594* | *0.5601* | *0.4938* |
| *skewness* | *−0.3583* | *−0.2458* | *−0.0002* | *−0.2127* | *−0.2974* | *−0.3581* |

(a)

| *x* | *−3* | *−2* | *−1* | *0* | *1* | *2* |
|---|---|---|---|---|---|---|
| *mean* | *−0.4801* | *−0.3811* | *−0.5642* | *6.0000* | *5.3368* | *5.5198* |
| *variance* | *0.4938* | *0.5791* | *0.6817* | *1.0000* | *0.5602* | *0.4938* |
| *skewness* | *−0.3583* | *−0.2458* | *−0.1370* | *0.0000* | *−0.2976* | *−0.3581* |

(b)

using the SV opening. Generally, the SE's can be adaptively changed with respect to the local characteristics of the signal. For example, in our previous works [9], [7], different methods based on local residue measurement have been used for SE assignment. In [10] and [11], SE adaptation rules based on the local shape of binary images have also been developed for image enhancement or noise filtering. In this work, the adaptation rule adopted is that the SE is changed with respect to the edge strength of the signal. The edge strength used here is defined as $D(x) = f(x) - f(x-1)$. For filtering this kind of signal with the SV opening, we essentially want to adopt smaller SE's with respect to the important feature points (e.g., at $x = 0$), and larger SE's for other places, so as to remove noises in the signal while preserving the important features. Thus, the SE adaptation rule for SE assignment used is

$$k_x = \begin{cases} w_0, & \text{if } D(x) > T \\ w_1, & \text{If } D(x) \le T \end{cases} \tag{4.1}$$

where $T$ is a threshold for feature parts determination.

Using standard statistical methods, we can analyze the performance of this adaptation rule theoretically. Since $D(x)$ are also Gaussian distributions

$$D(x) = \begin{cases} N'_x, & \text{if } x \ne 0 \\ h + N'_x, & \text{if } x = 0 \end{cases}$$

where $N'_x$ is a Gaussian random variable, the mean of $N'_x$ is 0, and the variance of $N'_x$ is 2, as shown in Fig. 4. If the *a priori* probabilities are the same, the optimal value of $T$ can then be easily obtained using the minimum-error-rate discriminate function of a two-category classifier [12] as $T = h/2$. Let $n(y)$ be the probability density function (p.d.f.) of a Gaussian distribution with zero-mean and unity variance, and let $N(y)$ be the probability distribution function with respect to $n(y), y \in R$. For $h = 6$, the sum of the Type-I error (a step edge is missed to be detected) and the Type-II error (not a step edge but is detected) is equal to $2N(3/\sqrt{2}) < 3.5\%$. Hence, for most cases (over 96.5%) the SE adaptation method used here can assign smaller SEs to the step edge points (i.e., the feature parts). Notice that in this situation, the adaption rule will produce the SE assignment as the same given in (3.1). Without loss of generality, the statistical analysis of the SE assignment as (3.1) is investigated in the following.

Since the p.d.f. of $f(x)$ is $n(y)$ for $x < 0$ and is $n(y-h)$ for $x \ge 0$, we will replace the results listed in Section III according to

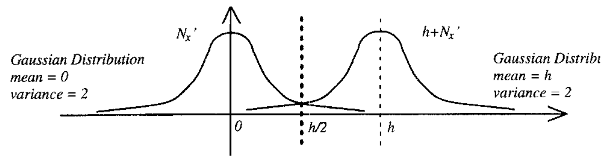


Fig. 4. Determination of the threshold $T$ of (4.1) $h0h/2N'_x h+N'_x$ Gaussian distribution mean = 0, variance = 2, Gaussian distribution mean = $h$, variance = 2.

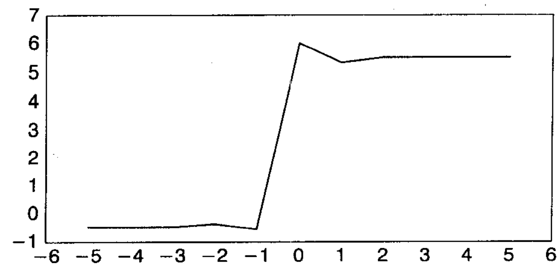


Fig. 5. Mean of the output distribution using SV opening (a) (b).

the following substitutions:

$$r_x(y) = \begin{cases} n(y), & \text{if } x < 0 \\ n(y-h), & \text{if } x \ge 0 \end{cases}$$

and

$$R_x(y) = \begin{cases} N(y), & \text{if } x < 0 \\ n(y-h), & \text{if } x \ge 0. \end{cases}$$

Then, the p.d.f. of $f \bar{\circ} \mathbf{k}(x)(-1 \le x \le 1)$ can be rewritten as follows:

$$\begin{aligned} p_{-1}(y) = {} & [n(y)N(y-h) + n(y-h)N(y)](1-N(y))^2 \\ & + 2n(y)[(1-N(y))^2 + N(y)(1-N(y)) \\ & \times (1-N(y-h))] \end{aligned} \tag{4.2}$$

$$\begin{aligned} p_1(y) = {} & 2n(y-h)N(y-h)(1-N(y-h))^2 \\ & + 2n(y-h)[(1-N(y-h))^3 \\ & + 2N(y-h)(1-N(y-h))^2] \end{aligned} \tag{4.3}$$

$$p_0(y) = n(y-h). \tag{4.4}$$

We do not list other p.d.f.'s for $|x| \ge 2$, since they can be easily obtained by applying the results given in Section III, and is identical to the case of the SI opening as shown in [3]. One can check that the p.d.f.'s given in (4.2) and (4.3) satisfy the constraint of the density function, that is,

$$1 = \int_{-\infty}^{\infty} p_{-1}(y)dy = \int_{-\infty}^{\infty} p_1(y)dy = \int_{-\infty}^{\infty} p_0(y)dy.$$

Table I lists the statistical analysis results of both the SV and the SI openings. Three statistical parameters are listed in this table—mean, variance, and skewness. The mean and variance are computed by the first- and second-order moments, which can be used to described the overall trend and the corresponding noise levels of the output signal. The skewness [13] is computed by using at most the third-order moment, as shown in (4.5).

$$\text{skewness} = \int_{-\infty}^{\infty} (p(y) - \mu)^3/\sigma^3 dy. \tag{4.5}$$

where $\mu$ is the mean and $\sigma$ is the standard deviation with respect to the p.d.f. $p(y)$. Since the output density functions described in this paper are asymmetric functions, the skewness calculation can provide a better idea of how asymmetric these functions are, and hence can describe the overall shape of these complex p.d.f.s in more detail.

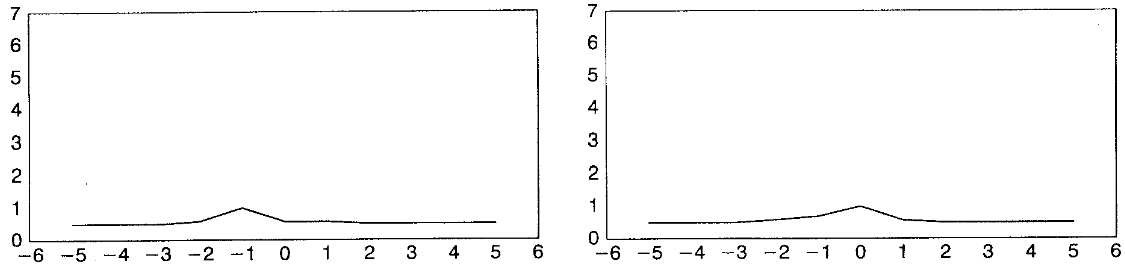


Fig. 6. The variance of the output distribution using (a) SI opening; (b) SV opening.

The resultant curves are plotted in Figs. 5 and 6, where Fig. 5 shows the mean and Fig. 6 shows the variance of the output distribution using the SV opening. The mean curves drawn in Figs. 1(b) and 5 can be viewed as the most probable shapes of the output distributions. It can be observed from these mean curves that the SV opening has better edge-preserving ability than the SI one does. The edge shape is degraded in Fig. 1(b); however, an interesting effect is that the edge shape is not only preserved, but also a little more "enhanced" as shown in Fig. 5. Essentially, the mean value of the output distribution gives evidence of the shape-preserving ability of the SV opening.

The variance curves shown in Fig. 6 characterize the corresponding noise-removing abilities. Remember that in our experiments the variance of the input signal is one, while through the nonlinear filtering of either the SI or the SV openings, most of the variances can be reduced to be smaller than one as listed in Table I. Hence, this provides evidence that morphological openings have considerable noise-removing abilities. It is observed from Fig. 6 that the variance curves of the SI and the SV openings are similar, except that the largest variance occurred at $x = -1$ for the SI case but at $x = 0$ for the SV one. Basically, since the total variance from using the SV opening is only a little larger than that of the SI one, it is reasonable to make the assertion that the SV and the SI openings possess approximately the same noise-removing ability. Hence, in general it has been shown that the sharpness of a noisy step edge is enhanced without losing the corresponding noise-removing ability when using the SV instead of the SI opening.

Generally, although only the step edge model is chosen in the analysis, it is believed that the method shown in this paper can be generalized to other more complicated cases by involving more elaborated derivations.

## Appendix

For the case of applying the SI opening with a flat SE $k = w_1$, the corresponding output p.d.f., denoted by $s_x(y)$, can be obtained as follows:

$$
\begin{aligned}
s_x(y) &= r_x(y) \\
&\times [\mathrm{pr}(y + dy < f(x-2), f(x-1), f(x+1), f(x+2)) \\
&\quad + \mathrm{pr}(f(x-2), f(x-1) \le y, y+dy < f(x+1), f(x+2)) \\
&\quad + \mathrm{pr}(f(x+2), f(x+1) \le y, y+dy < f(x-1), f(x-2)) \\
&\quad + \mathrm{pr}(f(x-2) \le y, y+dy < f(x-1), f(x+1), f(x+2)) \\
&\quad + \mathrm{pr}(f(x+2) \le y, y+dy < f(x-1), f(x+1), f(x-2)) \\
&\quad + \mathrm{pr}(f(x-1) \le y, y+dy < f(x+1), f(x+2), f(x-2)) \\
&\quad + \mathrm{pr}(f(x+1) \le y, y+dy < f(x-1), f(x+2), f(x-2)) \\
&\quad + \mathrm{pr}(f(x+2), f(x-2) \le y, y+dy < f(x+1), f(x-1))] \\
&+ r_{x+1}(y) \\
&\times [\mathrm{pr}(f(x-2), f(x-1) \le y, y+dy < f(x), f(x+2)) \\
&\quad + \mathrm{pr}(f(x-2) \le y, y+dy < f(x-1), f(x), f(x+2)) \\
&\quad + \mathrm{pr}(f(x-2), f(x+2) \le y, y+dy < f(x-1), f(x)) \\
&\quad + \mathrm{pr}(f(x-1) \le y, y+dy < f(x-2), f(x), f(x+2))] \\
&+ r_{x-1}(y) \\
&\times [\mathrm{pr}(f(x+2), f(x+1) \le y, y+dy < f(x), f(x-2)) \\
&\quad + \mathrm{pr}(f(x+2) \le y, y+dy < f(x+1), f(x), f(x-2)) \\
&\quad + \mathrm{pr}(f(x+2), f(x-2) \le y, y+dy < f(x+1), f(x)) \\
&\quad + \mathrm{pr}(f(x+1) \le y, y+dy < f(x+2), f(x), f(x-2))] \\
&+ r_{x+2}(y) \\
&\times [\mathrm{pr}(f(x-2), f(x-1) \le y, y+dy < f(x), f(x+1)) \\
&\quad + \mathrm{pr}(f(x-1) \le y, y+dy < f(x), f(x+1), f(x-2))] \\
&+ r_{x-2}(y) \\
&\times [\mathrm{pr}(f(x+2), f(x+1) \le y, y+dy < f(x), f(x-1)) \\
&\quad + \mathrm{pr}(f(x+1) \le y, y+dy < f(x), f(x-1), f(x+2))].
\end{aligned}
$$

## References


[1] J. Serra, Ed., "Image analysis and mathematical morphology," *Theoretical Advances*, vol. 2. New York: Academic, 1988.

[2] R. L. Stevenson and G. R. Arce, "Morphological filters: Statistics and further syntactic properties," *IEEE Trans. Circuits Systems*, vol. CAS-34, no. 11, pp. 1292–1305, 1987.

[3] A. Morales and R. Acharya, "Statistical analysis of morphological openings," *IEEE Trans. Signal Processing*, vol. 41, no. 10, pp. 3052–3056, 1993.

[4] H. J. A. M. Heijmans and C. Ronse, "The algebraic basis of mathematical morphology. I. dilations and erosions," *Comput. Vision, Graphics, Image Processing*, vol. 50, pp. 245–295, 1990.

[5] R. M. Haralick and L. G. Shapiro, *Computer and Robot Vision*, vol. 1. Reading, MA: Addison, 1992.

[6] W. E. L. Grimson and T. Pavlidis, "Discontinuous detection for visual surface reconstruction," *Comput. Vision, Graphics, and Image Processing*, vol. 30, pp. 316–330, 1985.

[7] C.-S. Chen *et al.*, "Space varying mathematical morphology for adaptive smoothing of 3D range data," in *Proc. Asia Conf. Comput. Vision*, Osaka, Japan, Nov. 1993, pp. 39–42.

[8] A. Morales and R. Acharya, "Non-linear multiscale filtering using mathematical morphology," in *Proc. IEEE Int. Conf. Comput. Vision Patt. Recog.*, June 1992, pp. 572–578.

[9] C.-S. Chen *et al.*, "Theoretical aspects of the gray-level morphological $V$-opening and its applications on adaptive signal filtering," in *Proc. 1995 IEEE Workshop Nonlinear Signal Image Processing*, Noes Marmaras, Halkidiki, Greece, June 1995, pp. 42–45.

[10] Y. Yao, R. Acharya, and S. Srihari, "Image enhancement using mathematical morphology with adaptive SEs," in *Proc. SPIE Nonlinear Image Processing V (1994)*, San Jose, CA, vol. pp2180, pp. 198–208.

[11] F. Cheng and A. N. Venetsanopoulos, "An adaptive morphological filter for image processing," *IEEE Trans. Image Processing*, vol. 1, no. 4, pp. 533–539, 1992.

[12] R. O. Duda and P. E. Hart, *Pattern Classification and Scene Analysis*. Stanford Research Institute, Menlo Park, CA: Stanford Research Institute, 1973.

[13] R. V. Hogg and A. T. Craig, *Introduction to Mathematical Statistics*, New York: Macmillan, 1978.