

## ON PROJECTIVE THREEFOLDS OF GENERAL TYPE

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ABSTRACT. Let  $Y$  be a complex nonsingular projective 3-fold of general type. We show that  $P_{12}(Y) \geq 1$ ,  $P_{24}(Y) \geq 2$ , the canonical volume  $\text{Vol}(Y) \geq 1/2660$  and that the pluricanonical map  $\varphi_m$  is birational for all  $m \geq 77$ . Details will appear in [2].

### 1. INTRODUCTION AND KNOWN RESULTS

Let  $Y$  be a non-singular complex projective variety of dimension  $n$ . It is said to be of general type if the pluricanonical map  $\varphi_m := \Phi_{|mK_Y|}$  corresponding to the linear system  $|mK_Y|$  is birational into a projective space for all  $m \gg 0$ . It is thus natural and important to ask whether one can find a practical constant  $c(n)$  so that  $\varphi_m$  is birational for all  $m \geq c(n)$  and for all varieties of general type of dimension  $n$ .

When  $\dim Y = 1$ , it was classically known that  $|mK_Y|$  gives an embedding of  $Y$  into a projective space if  $m \geq 3$ . When  $\dim Y = 2$ , Bombieri's theorem [1] says that  $\varphi_m$  gives a birational map onto the image for  $m \geq 5$ . When  $\dim Y \geq 3$ , a recent remarkable result of Hacon and McKernan [7], Takayama [16] and Tsuji [17] affirms the existence of  $c(n)$ . However no explicit numerical bound for  $c(3)$  was known. Kollár [11] studied both irregular threefolds and threefolds with  $P_k \geq 2$  for a certain positive integer  $k$  and found a constant  $c(3)$  for those threefolds. Kollár's results have been considerably improved respectively in [5] and [4] where nearly sharp  $c(3)$  were found. For regular threefolds of general type, Luo [13] has proved some partial bounds for  $c(3)$ .

We are interested in finding a concrete  $c(3)$  for all threefolds of general type. A possible approach is to use the cohomological method via vanishing theorems. This requires estimates on the positivity of the canonical divisor which is usually measured in terms of the singularities of any minimal model and the canonical volume

$$\text{Vol}(Y) := \limsup_{\{m \in \mathbb{Z}^+\}} \left( \frac{n!}{m^n} \dim_{\mathbb{C}} H^0(Y, \mathcal{O}_Y(mK_Y)) \right).$$

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Lower bounds for the canonical volume are useful in producing lower bounds for the plurigenus  $P_m(Y) := \dim_{\mathbb{C}} H^0(Y, mK_Y)$ . The volume is an integer when  $\dim Y \leq 2$ . However it is only a rational number in higher dimensional situations.

The purpose of this article is to give an outline of [2], where we prove the following:

**Theorem 1.** *Let  $Y$  be a nonsingular projective threefold of general type. Then*

1.  $\text{Vol}(Y) \geq \frac{1}{2660}$ ,
2.  $P_{12}(Y) \geq 1$ ,
3.  $P_{24}(Y) \geq 2$ ,
4.  $\varphi_m$  is birational for all  $m \geq 77$ .

We remark that by [9] there exists a projective threefold of general type  $X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23)$  with  $\text{Vol}(X_{46}) = \frac{1}{420}$  and  $\varphi_{26}$  is not birational.

We begin by recalling some relevant known results for threefolds of general type. In [11, Corollary 4.8], using results on the semipositivity of the relative dualizing sheaves, Kollár shows that if  $P_k \geq 2$  for some  $k > 0$ , then  $\varphi_{11k+5}$  is birational. Improving on the methods of [11], the second author has shown in [6, Theorem 0.1] that if  $P_k \geq 2$  for some  $k > 0$ , then  $\varphi_{5k+6}$  is birational and in [5] that if  $p_g(Y) \geq 2$ , then  $\varphi_8$  is birational.

When  $Y$  is irregular, i.e.  $q(Y) > 0$ , there is a non-trivial Albanese map. One can effectively measure the positivity of pluricanonical sheaves via the Fourier-Mukai transform. This program was initiated by Hacon and the first author. In particular, it is shown in [4] that  $|7K_Y|$  gives a birational map.

Since the problems are birational in nature, it is equivalent to study minimal threefolds of general type. The existence of minimal models is guaranteed by the 3-dimensional MMP (see for instance [10, 12, 14]). Pick a minimal model  $X$  of  $Y$ . One knows that  $X$  has at worst  $\mathbb{Q}$ -factorial terminal singularities.

If  $X$  is Gorenstein, then by taking a special resolution according to Reid and by the Miyaoka-Yau inequality, one sees  $\chi(X, \mathcal{O}_X) < 0$ . Hence either  $q(X) > 0$  or  $p_g(X) \geq 2$ . In fact,  $\varphi_m$  is birational for all  $m \geq 5$  as shown in [3]. Therefore it remains to study those minimal threefolds  $X$  with Cartier index  $r(X) > 1$ , i.e., those having at least one terminal singularity with canonical index  $> 1$ . It is now natural to utilize Reid's Riemann-Roch formula [15] that plays a fundamental role in our work.

## 2. BASKET OF SINGULARITIES AND PACKING OF FORMAL BASKETS

Let  $X$  be a minimal threefold of general type with non-Gorenstein singularities. According to Reid (see last section of [15]), there is a "basket" of pairs of integers  $\mathcal{B}(X) := \{(b_i, r_i)\}$  such that

$$\chi(\mathcal{O}_X(mK_X)) = \frac{1}{12}m(m-1)(2m-1)K_X^3 - (2m-1)\chi(\mathcal{O}_X) + l(m),$$

where the correction term  $l(m)$  can be computed as:

$$l(m) := \sum_{Q_i \in \mathcal{B}(X)} l_{Q_i}(m) := \sum_{Q_i \in \mathcal{B}(X)} \sum_{j=1}^{m-1} \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i}.$$

Here,  $b_i$  is coprime to  $r_i$  and  $0 < b_i < r_i$ . The ratio  $\frac{b_i}{r_i}$  is called the *slope* of  $(b_i, r_i)$ .

Because  $X$  is minimal and is of general type, vanishing theorems (see, for instance, [10]) yield that the plurigenus  $P_m(X) = \chi(\mathcal{O}_X(mK_X))$  for all  $m > 1$  and so  $\text{Vol}(Y) = \text{Vol}(X) = K_X^3$ .

One can thus rewrite the Riemann-Roch formulae inductively as:

$$P_2 = \frac{1}{2}(K_X^3 - \sum \frac{b_i^2}{r_i}) + \frac{1}{2} \sum b_i - 3\chi,$$

$$P_{m+1} - P_m = \frac{m^2}{2}(K_X^3 - \sum \frac{b_i^2}{r_i}) + \frac{m}{2} \sum b_i - 2\chi + \Delta^m,$$

where  $\Delta^m = \sum_i (\frac{\overline{mb_i(r_i - mb_i)}}{2r_i} - \frac{mb_i(r_i - mb_i)}{2r_i})$ . We see that Reid's basket  $\mathcal{B}(X)$ ,  $\chi(\mathcal{O}_X)$  and  $P_2$  actually determines the canonical volume  $K_X^3$  and the plurigenera  $P_m$  for all  $m \geq 3$ . Because of this reason, we introduce the notion "formal basket" which is a triple  $(B, \tilde{\chi}, \tilde{P}_2)$  where  $B$  is a basket,  $\tilde{\chi}$  is an integer and  $\tilde{P}_2$  is a non-negative integer. Then we define the volume  $\tilde{K}^3$  and the plurigenus  $\tilde{P}_m$  of a given formal basket via the above Riemann-Roch formulae. As we have seen, a variety gives a formal basket  $(\mathcal{B}(X), \chi(\mathcal{O}_X), P_2(X))$ <sup>1</sup> and all plurigenera and the canonical volume are determined by this formal basket.

The new ingredient of our paper [2] is that we define a partial ordering, which we call a *packing*, among the set of all formal baskets. The notion of packings is closely related to that of partial resolutions for canonical quotient singularities.

**Definition 2.** Given a basket  $B = \{(b_1, r_1), (b_2, r_2), (b_3, r_3), \dots\}$  and set  $B' = \{(b_1 + b_2, r_1 + r_2), (b_3, r_3), \dots\}$ , then we say that  $B'$  is a packing of  $B$ , denoted  $B \succ B'$ .

Given formal baskets  $\mathbf{B} = (B, \tilde{\chi}, \tilde{P}_2)$  and  $\mathbf{B}' = (B', \tilde{\chi}', \tilde{P}_2')$ , we say that  $\mathbf{B} \succeq \mathbf{B}'$  if  $\tilde{\chi} = \tilde{\chi}'$ ,  $\tilde{P}_2 = \tilde{P}_2'$  and  $B \succ B'$ .

This defines a partial ordering  $\succeq$  on the set of formal baskets. Such a partial ordering has the following properties justifying our notation (see [2, Section 5 and Lemma 8.1]).

**Proposition 3.** *Given formal baskets  $\mathbf{B} \succeq \mathbf{B}'$ , then:*

1.  $\tilde{P}_m(\mathbf{B}) \geq \tilde{P}_m(\mathbf{B}')$  for all  $m$ ,
2.  $\tilde{K}^3(\mathbf{B}) \geq \tilde{K}^3(\mathbf{B}')$ .

The relation between formal baskets and plurigenera can be elucidated by further analysis on packing of baskets. Given a basket  $B = (b, r)$ , the slope  $\frac{b}{r}$  is in  $(\frac{1}{n}, \frac{1}{n+1}]$  for some integer  $n$ . One sees that  $B$  may be obtained by a finite sequence of packings starting from the basket  $B^{(0)} := \{(nb + b - r) \times (1, n), (r - nb) \times (1, n + 1)\}$ . Indeed, we define a uniquely determined sequence of baskets

$$\mathcal{B}^{(0)}(B) \succ \mathcal{B}^{(5)}(B) \succ \dots \succ \mathcal{B}^{(n)}(B) \succ \dots = B, \quad (1)$$

that realizes the process (see [2, 4.12]).

For a general basket, we just add up the sequences for each  $(b_i, r_i)$ .

Roughly speaking,  $\mathcal{B}^{(n)}(B)$  is obtained by unpacking each  $(b_i, r_i)$  in  $B$  into baskets with slope  $\frac{q}{p}$  where  $(p, q) = 1$  and either  $q = 1$  or  $p \leq n$ .

<sup>1</sup>Since  $(b_i, r_i)$  and  $(r_i - b_i, r_i)$  have the same contribution to the Riemann-Roch formula, we may normalize  $\mathcal{B}(X)$  and assume that each  $(b_i, r_i)$  has slope  $\leq \frac{1}{2}$ .

**Example 4.** Let us take, for example,  $B = \{(3, 7)\}$ .

$$\begin{aligned}\mathcal{B}^0(B) &= \{2 \times (1, 2), (1, 3)\}, \\ \mathcal{B}^{(5)}(B) &= \{(1, 2), (2, 5)\} = \mathcal{B}^{(6)}(B), \\ \mathcal{B}^{(7)}(B) &= \{(3, 7)\} = \mathcal{B}^{(8)}(B) = \dots = B.\end{aligned}$$

In [2, 4.12 and 4.13], we prove that  $\mathcal{B}^{(n-1)}(B)$  is obtained from  $\mathcal{B}^{(n)}(B)$  by unpackings those baskets of slope  $\frac{q}{n}$  with  $(q, n) = 1$ . Denote by  $\epsilon_n(B)$  the total number of such unpacking. It's clear that  $\epsilon_n \geq 0$ .

The key point is that for a given formal basket  $\mathbf{B} = (B, \tilde{\chi}, \tilde{P}_2)$ , the sequence (1) of  $B$  is almost completely determined by  $\tilde{\chi}$  and the plurigena  $\tilde{P}_m$ . In particular, for each  $n$ , the inequality  $\epsilon_n \geq 0$  gives rise to an inequality among  $\tilde{\chi}$  and the plurigena. More details can be found in Section 5 of [2].

A typical inequality which we would like to mention here is the following one coming from the fact  $\epsilon_{10} + \epsilon_{12} \geq 0$  (see [2, Equation 5.3]).

$$2\tilde{P}_5 + 3\tilde{P}_6 + \tilde{P}_8 + \tilde{P}_{10} + \tilde{P}_{12} \geq \tilde{\chi} + 10\tilde{P}_2 + 4\tilde{P}_3 + \tilde{P}_7 + \tilde{P}_{11} + \tilde{P}_{13}. \quad (2)$$

Inequality (2) plays a crucial role in our argument though we have obtained many other similar inequalities. An immediate consequence is the following:

**Proposition 5.** *A formal basket with  $\tilde{P}_m \leq 1$  for all  $2 \leq m \leq 12$  must have  $\tilde{\chi} \leq 8$ . Furthermore, the set of formal baskets with  $\tilde{P}_m \leq 1$  for all  $2 \leq m \leq 12$  is finite.*

*In particular, let  $X$  be a threefold of general type with  $P_m(X) \leq 1$  for all  $2 \leq m \leq 12$ , then  $\chi(\mathcal{O}_X) \leq 8$ .*

Now we can outline the proof of Theorem 1.

*Sketch of the proof of Theorem 1.* As we have stated in the introduction, we only have to study minimal threefolds  $X$  of general type with Cartier index  $r(X) > 1$ .

**Step 1.** If  $P_{m_0} \geq 2$  for some  $m_0 \leq 12$ , we study the pluricanonical map  $\varphi_{m_0}$  by a technique developed by the second author. One obtains, for example,  $K^3 \geq \frac{11}{12m_0(m_0+1)^2}$  [2, Theorem 3.1] and  $\varphi_m$  is birational for  $m \geq 5m_0 + 6$ .

**Step 2.** If  $P_m \leq 1$  for all  $m \leq 12$ , then using equation (2) we get  $\chi(\mathcal{O}_X) \leq 8$ .

When  $\chi(\mathcal{O}_X) = 1$ , then  $P_2 = 0$  by a result of Fletcher [8]. When  $8 \geq \chi(\mathcal{O}_X) \geq 2$ , we prove  $P_2 = 0$  in Section 7 of [2]. By Section 5 of [2], there are only finitely many formal baskets with the given constraints. A complete classification can be found in Section 6 and 7 of [2].

With Step 1 and Proposition 3, we get the lower bound of  $K^3$  for all threefolds of general type.

**Step 3.** As we pointed out, there are some inequalities between  $\chi$  and the plurigenus due to  $\epsilon_n \geq 0$ . We show that there are no formal baskets satisfying those inequalities and  $P_2 = P_3 = P_4 = P_6 = P_{12} = 0$ . Hence we see that  $P_{12} > 0$  (cf. [2, Theorem 8.8]). The idea to prove  $P_{24} \geq 2$  is similar but more involved (cf. [2, Theorem 8.9]).

**Step 4.** With  $P_{24} \geq 2$ , one already has that  $\varphi_m$  is birational for all  $m \geq 126$  by Step 1. In order to obtain the announced birationality in Theorem 1, one needs to consider various situations with more careful analysis (cf. [2, 8.12-8.19]).

This completes the sketch of the proof.  $\square$

Lastly, we would like to remark that, by our classification of formal baskets with  $\tilde{\chi} = 1$ , we proved the lower bound  $\text{Vol}(Y) \geq \frac{1}{420}$  for all threefolds of general type with  $\chi(Y, \mathcal{O}_Y) = 1$  (cf. [2, Theorem 6.11]). This bound is sharp. The authors do not know any concrete example with  $\chi(Y, \mathcal{O}_Y) \geq 2$  and  $q(Y) = 0$ . As another interesting application, these methods also show that Iano-Fletcher’s computer generated list of 46 families of canonical hypersurface threefolds in weighted projective spaces is complete (cf. [9, Conjecture 15.1-15.2]).

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