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Singularities in Social Choice Theory
—Democracy is not Mathematically Unsound

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Abstract

Mathematical foundations of social choice theory are reconstructed. Some significant previous contributions on the impossibility theorems of rational social choice are thoroughly reviewed. For continuum alternatives sets, the zero order and the first order topologies of preference spaces are studied and introduced respectively to equip the spaces of social and individual preferences in accordance with the empirical world. Stratification structure is also defined and used to modify the rationality principles of social welfare functions. It is pointed out that the avoidance of singularities of social phenomena in the frameworks of previous contributors causes the impossibility of rational social choice. With the revisions and the new theory containing the singularities, the existence theorems of rational social choice are established.

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Introduction

Kenneth Arrow [Ak] proved in 1951 a striking theorem that a social choice with minimum rational principles ~~are~~ is logically ~~inconsistent~~ impossible. For decades after Arrow, the impossibility of a rational social choice has been widely accepted in the community of social science and enormous related contributions have appeared in literatures. This paper is to reexamine the essence of social choice mechanism in reality as well as the frameworks of some significant contributors. A concept of singularity from the viewpoints of structuralism is introduced. Contrary to the common belief in the impossibility of a rational social choice, we establish a new theory to contain singularities, with which possibility theorems are proved both in the continuum and in the finite discrete cases. The infinite discrete case is treated in a paper by Chen [Ch].

More precisely, we start with the continuum case of alternatives and define relative to a given structure of the alternative space a notion of singularity for preferences (see § 2.2 for definitions). Chichilnisky [C] proved in 1982 that a continuous rational social choice on a continuum alternative space is impossible. However, we find that the impossibility arises essentially from her formulation which totally separates singular preferences from regular ones — called *separation property* (§ 2.3). By extending the notion of singularity to the discrete case, we state that Arrow's formulation (§ 4.1) also excludes the singularities from consideration (Theorems 7 and 7$'$). It is the exclusion that leads to the impossibility of a rational social choice.

The remaining question is whether or not social scientists should avoid singular phenomena in their analysis, while the investigation into singular phenomena in the nature, such as poles, δ-functions, turbulences, non-linearity and black holes, have long been recognized as the significant motivation of the research in natural science.

As our approach inevitably involves rigorous modern mathematical concepts and language, which seem not easily accessible for the general readers, we try to outline the rough idea in the following paragraphs.

(i) Let X be an alternative space and P the totality of the preferences on X, respecting the given structure of X. For example, when X is a continuum space, P is the set of all continuous preferences on X. A social welfare function is conventionally defined by a map $F : P^N \to P$. In proving the existence theorems of continuous rational social welfare functions, we establish a sequence of maps,

$$P^N \xrightarrow{\xi^N} (C^o(X))^N \xrightarrow{G_\nu} C^o(X) \xrightarrow{\pi} C^o(X)/_\sim \xrightarrow{\varphi} P,$$

where P^N denotes the product $P \times P \times \cdots \times P$ with N copies of P and $C^o(X)$ is the space of all continuous functions of X with compact-open topology. The continuity of $\xi : P \to C^o(X)$ is proved in the lifting theorem (§ 3.4) while the topological equivalence of $\varphi : C^o(X)/_\sim \to P$ is shown in the quotient theorem (§ 3.2). As for the symmetric function G_ν and the quotient map π, their continuity is obvious. By letting $F : P^N \to P$ to be the composition map $\varphi \circ \pi \circ G_\nu \circ \xi^N$, we finally obtain the continuous rational social welfare functions (Theorem 4).

(ii) For the continuum case, the controversial points of the above existence theorem are probably the adopted topologies of P which makes ξ and φ continuous. The lifting map ξ defines a universal way (see § 4.1 for definition) of constructing utility functions to represent preferences in P continuously. Here we choose the *first order* topology $\Im$ for the set P_{ind} of all the individual preferences under social choice devices and call $\Im$ the *stratification topology* of P (§ 2.1). One characteristic of the first order topology $\Im$ is the *separation property* that Chichilnisky has assumed to show the nonexistence of a continuous rational social welfare function. The only difference at this point between her formulation and ours is that she sets up her arguments in the category of differentiable structure which seems not self-consistent as criticized in § 2.2 of this paper, while we rebuild a framework in the category of topological structure, that might be more rational and compatible with the real world (§ 2.1).

(iii) As the notion of singularity plays crucial role in the later exposition, we give in this paragraph a quick illustration about its meaning. *Relative to the topological structure* of X a preference on X is *singular* if some indifferent set I_x , i.e. the set of y in X with y indifferent to x in the preference (see § 1.1 for definitions), has non-empty interior; otherwise, it is called *regular*. For example, letting $X =$ the unit square $[0,1] \times [0,1]$, Figure A shows a singular preference, while Figure B shows regularity:

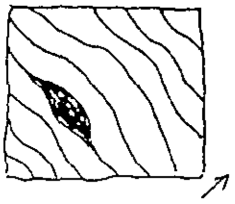

Figure A

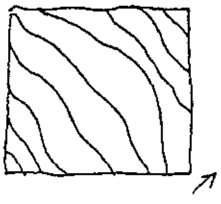

Figure B

Here the marked curves or regions in the figures denote the indifferent sets of the given preference and the arrows show the preferred direction. However, *relative to the differentiable structure* of X, a preference is *singular* if there is some points at which the preference vector vanishes. The preferences illustrated in Figures C, D, E and A are all singular relative to the differentiable structure, although they are regular relative to the topological structure, except the one in Figure A.

(iv) On the other hand, any preference on X corresponds to a *stratification* (§ 1.1) which naturally stratifies X into indifferent sets with linear order. The stratification structure has long been neglected. For the finite discrete case, it is used to redefine Arrow's rationality principle (§ 4.2) with the viewpoint that any rational social welfare function *should respect* the stratification structure, since the latter is innately built in the preferences under consideration. Using the same sequence of maps given in the paragraph (i), except replacing $C^o(X)$ by the function space $\mathfrak{F}(X)$, we show the existence theorem of rational social welfare functions for the finite discrete case (Theorem 8).

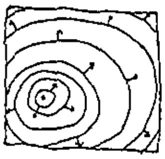
Figure C

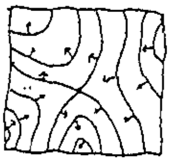
Figure D

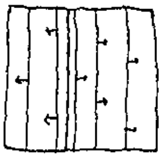
Figure E

(v) For the continuum case, the stratification structure also begets the first order topology $\Im$ for P, in the sense that the topology $\Im$ respects the stratification structure (see Stratification Approximation Theorem in § 2.3). This motivates us to call $\Im$ also the *stratification topology.* On the other hand, the set P of preferences is basically derived from the set $\mathfrak{F}(X)$ of real valued functions by neglecting cardinality yet preserving ordinality (see § 3.1). For the function space $\mathfrak{F}(X)$, various topologies have long been investigated in the modern analysis. Among them, the meaning of zero order topologies, e.g. compact-open topology or L^2-topology, and that of first order topologies are clear to mathematicians. The first order topologies may be defined by the convergence of the function values together with their derivatives. For example, in the compact-open sense or in the Sobolev 1-norm sense[1]. More precisely, the convergence of a sequences of functions in zero order topologies means the values of the functions at a given point, say, get arbitrarily closed to each other eventually, and the convergence in first order topologies requires not only the function values but also their derivatives do. It is even of mathematical interest to investigate the related zero order and first order topologies now for the set P of preferences. We introduce $\Im$ the first order topology in the compact-open sense for the set P_{ind} of individual preferences considered in a social choice apparatus, while for the set P_{soc} of social preferences, we adopt $\Im_o$ the zero order topology also in the compact open sense.

(vi) Disregarding the essential difference between individual preferences and social preferences as involved in a social choice apparatus $F : P_{ind}^N \to P_{soc}$ made Chichilnisky conclude that any social choice is discontinuous. The separation property about singularities holds in the first order topology $\Im$. However, there is no reason to extend it for P_{soc} of social preferences, although we accept $\Im$ as the topology for P_{ind} of individual preferences. In a social choice apparatus such as in a poll of election, a society is required to have a sophisticated computing ability to detect a very slight difference between distinct alternatives for social preferences, even though the difference eventually vanishes into indifference.

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[1] See Adams: Functional Analysis.

This simply means that a society allows its preference varying from regular to singular continuously. Thus, unlike P_{ind} of individual preferences having $\Im$ as its topology, P_{soc} of social preferences should be equipped with the zero order topology $\Im_o$, in which the separation property no more holds and a singular preference may therefore be a continuation of regular preferences. An example to show this is given as follows.

Example A Let $X = [-1, 1]$. Define for each $n = 1, 2, 3, \dots$, a social preference $p_n \in P_{soc}$ by
(1) $z \succ w$ in p_n, for $-1 \leq z \leq w \leq 0$,
(2) $x \prec y$ in p_n, for $0 \leq x \leq y \leq 0$, and
(3) x is indifferent to $-x/n$ in p_n, for each $x \in [0, 1]$.
Also define $p \in P_{soc}$ by
(1′) $z \succ w$ in p, for $-1 \leq z < w \leq 0$, and
(2′) x is indifferent to 0 in p, for each $x \in [0, 1]$.
Given any two alternatives x and y in $[0, 1]$, let $x_n = -x/n$ and $y_n = -y/n$, then we have
(1″) x and y are indifferent respectively to x_n and y_n in p_n, and
(2″) both x_n and y_n tend to 0 as $n \to \infty$.

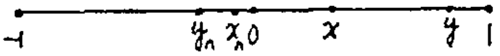

Figure F

It means that x_n and y_n should eventually become indifferent if the social aggregation apparatus is able to compute delicate difference. By transitivity, x and y should also become indifferent as $n \to \infty$, i.e. p_n tends to p in P_{soc}. Therefore, the separation property should not be accepted for P_{soc}.

(vii) In many occasions of reality, e.g. in an election poll, social preferences attached to a social choice apparatus are obtained through quantified functions with cardinalities. In terms of mathematics, a social welfare function F could be factorized into the composition $q \circ \tilde{F}$ as

$$F : P_{ind}^N \xrightarrow{\tilde{F}} \mathfrak{F}(X) \xrightarrow{q} P_{soc}.$$

For X a continuum space, the factorization is through $C^o(X)$:

$$F : P_{ind}^N \xrightarrow{\tilde{F}} C^o(X) \xrightarrow{q} P_{soc}$$

where $q = \varphi \circ \pi$ with

$$q : C^o(X) \xrightarrow{\pi} C^o(X)/_\sim \xrightarrow{\varphi} P_{soc}$$

(see § 3.1, § 3.2 for details). A topological equivalence between $C^o(X)/_\sim$ and P_{soc} under φ is proved in the quotient theorem of § 3.2, where P_{soc} is equipped with the first order topology $\Im_o$. This theorem also justifies the adequacy of adopting $\Im_o$ as the topology at P_{soc}. In this light, we also call the zero order topology $\Im_o$ the *social aggregation topology.*

(viii) The social aggregation topology $\Im_o$ is not a T_1-space, i.e. $\Im_o$ is not a space in which each set consisting of a single point is closed. In particular a sequence of preferences may have more than one limit. Although it reflects the reality of social preferences, one may feel uneasy because of the non-conventional uncertainty of limits. As a matter of fact, we adopt $\Im_o$ as the topology for P_{soc}, since we have to consider the standard model of a social welfare function. which has long been formulated as a function

$$F : P^N \to P$$

defined on P^N *into* P. However, it is possibly more reasonable to regard

$$\widetilde{F} : P^N \to C^o(X)$$

as a model of social welfare function, for $\widetilde{F}$ is in fact a *quantified* social welfare function in the sense hat the social aggregation of individual preferences could even be quantified, although only the ordinality of individual preferences are given. To support the possibility of rational social choices, the existence of $\widetilde{F}$ is stronger than that of F, if both satisfy the same required rationality. In Theorem 5, we prove that there exist quantified social welfare functions which are continuous, anonymous and unanimous, if P in the domain P^N is equipped with the stratification topology $\Im$. With this formulation, we avoid the unimaginable non-T_1 space $\Im_o$.

(ix) The lifting $\xi : P_{ind} \to C^o(X)$ and the quotient $\varphi : C^o(X)/_\sim \to P_{soc}$ are *dual* to each other. If P_{ind} is equipped with the first order topology $\Im$ and P_{soc} with the zero order topology $\Im_o$, then both of the dual maps ξ and φ are continuous, which yields the existence of continuous rational social welfare functions. One may now come back to question why P_{ind} should have the separation property and therefore be equipped with $\Im$, yet not with $\Im_o$ as P_{soc} is. A topological equivalence between $C^o(X)/_\sim$ and P_{soc} under φ is proved in the quotient theorem of § 3.2, where P_{soc} is equipped with the first order topology $\Im_o$. This theorem also justified the adequacy of adopting $\Im_o$ as the topology of P_{soc}. Besides that individuals are not as sophisticated as a society is, an individual when asked to describe his small changes of mind before the society, he has to present it in an articulate form. Suppose at the beginning the individual has indifferent preference on two alternatives x and y, and after a little while he starts to prefer y to x "very slightly" as in Example A, say. If now he was polled to express his change of preference, he would be hesitate to poll his preference from p_s to p_r as in the figure G, since the change from p_s to p_r seems drastic to him. The drastic change from p_s to p_r in P_{ind} could be continuated gradually from the boundaries such as the process in the figure H.

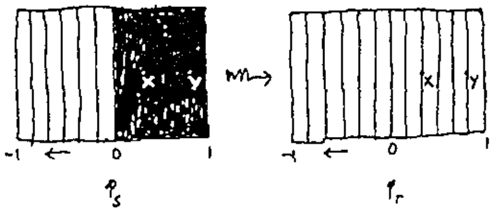

Figure G

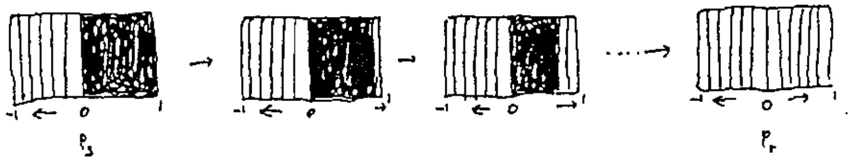

Figure H

This is consistent with the first order topology $\Im$, where the separation property is maintained.

(x) For the discrete case, we extend the concept of singularity to the continuum case. A preference is called *singular* if there exists two distinct alternatives x and y which are indifferent to each other. We show in § 4.2 that Arrow's rational principles *implicitly* excludes the singularities from consideration, and thereby we question whether the exclusion of singularities should be regarded rational.

(xi) On the other hand, we quantify Arrow's independency by defining the degree $\delta_I(F)$ of Arrow's independency of a social welfare function F (See Definition 4.1.1 (a)). Also we introduce the degree $\delta_D(F)$ of dominance which measures the power of a dominator (See Definition 4.1.1 (b)). In particular, if $\delta_I(F) = n$, where n is the number of X, F satisfies Arrow's independency; if $\delta_D(F) = n$, it means that F has a dictator. By revising Arrow's argument, we prove (Theorem 6) that $\delta_D(F) \geq \delta_I(F)$ if $\delta_I(F) \geq 3$. Also $1 \leq \delta_D(F) \leq \delta_I(F) \leq n$ in general. This means that Arrow's independency is innately connected with the existence of a dictator, especially when $\delta_I(F) \geq 3$. In fact, as $n \geq 3$, $\delta_I(F) = n$ then implies $\delta_D(F) = n$, i.e. a dictator exists. So Theorem 6 generalized Arrow's theorem. But it is a quantified theorem which detects the innate bond between the independency and the dominance in a scale gradually from small to large that makes Arrow's inconsistency theorem less striking. This leads us to reconsider the acceptance of independency as a *rational* principle. Another important aspect of the quantified theorem is that it gives a constructive research line of social choices. Theorem 6 suggests a study focussed in the interesting cases where $\delta_I(F) < 3$, since $\delta_I(F) = 3$ implies the existence of a dictator, which should be ruled out. The existing voting rule of reality, such as simple majority, arithmetic mean of inferiority and voting amendment bills, belong to the case of $\delta_I(F) = 1$ or 2 (See examples in § 4.1).

(xii) To set up a qualitative theorem on a rational social choice for the discrete case, we claim that an independence condition is acceptable only when it does not destroy the stratification structure which has been inherited from the given preferences (Proposition 1.2.2.) and should not be ignored. This is based on the viewpoint of structuralism.. We then establish an existence theorem of social welfare functions satisfying the independence condition respecting stratification structure, citizen sovereignty, strong Pareto condition and anonymous. Clearly, anonymity implies nondictatorship.

(xiii) Finally, we select Baigent formulation of impossibility of a rational social choice and also review its rationality. Baigent claimed the impossibility on the basis that the proximity property, the unanimity and the anonymity are inconsistent. We prove in Theorem 9 that proximity property is *geometrically* impossible, merely under a very simple requirement of diagonal injectivity. Thus the rationality of proximity property should also be questioned, and his theorem does not imply the impossibility of a *rational* social choice.

(xiv) The case of infinite discrete alternatives is considered by Chen [Ch] with an existence theorem of continuous rational social welfare functions. The case is interesting, because it is a joint point of the continuum case, which we treated in § 2 and § 3, and the finite discrete case handled in § 4. In other words, the totality P of preferences on an infinite discrete X has non-discrete topology, while X itself is discrete. Thus, both of the stratification topology (first order topology) and the social aggregation topology (zero order topology) make sense. On the other hand, as X itself is discrete, one may avoid the complicate topological formulation on X and still understand the context more intuitively. For details, we refer the readers to the paper by [Ch].

The subtitle of this paper responds to the article "Is democracy mathematically unsound?" by Crypton appeared in Science Digest (1985) [Cr], although we do not prove that democracy is sound in reality. Indeed, social choices or democracy processes in particular are the processes more complicate than the conventional model

$$F_* : P^N \xrightarrow{F} P \longrightarrow \cancel{X}.2^{X}$$

Even a democracy process itself involves usually an iteration of maps from $P^N \to P$ which is a dynamic process rather than the static F as conventionally formulated. A more accurate model of a social choice contains a map

$$F : P^N \times T \longrightarrow P,$$

with $T \subset R$ and $F(\mathrm{p}, t) \in P$, $\forall \mathrm{p} \in P^N$, $t \in T$, attached by a nonparametric continuous profile curve $\mathrm{p} = \mathrm{p}(t)$ in P^N, where t denotes the time. The outcome $F(\mathrm{p}, s)$ of the previous stages (i.e. $s < t$) may affect, among other factors, the individual preferences $\mathrm{p}(t)$ at a given time t. As an example, we consider the Condorcet paradox (See § 4.1 for details), and let $T = \{0, 1, 2, \cdots\}$, $\mathrm{p}(0) =$ Condorcet triple. Define the social preference $F(\mathrm{p}, t)$ by the preference, corresponding to the sum of numbers of inferior alternative sets of individual preferences (See Example 4.1.2). Then the social preference $F(\mathrm{p}(0), 0)$ which is the singular preference $x \sim y \sim z$ for the three alternatives x, y and z may affect the individuals v_1, v_2 and v_3 to reconsider their preference orders in $\mathrm{p}(1)$. If $F(\mathrm{p}(1), 1)$ is regular, or at least the top alternative exists, then the social choice process is finished. Otherwise, it goes to $t = 2$ and runs iteratively until the top alternative appears. A second interesting example occurs in the two-period elections in which the two or more most preferred alternatives are selected in the first vote and listed as candidates in the second vote. For this case the individual preference $\mathrm{p}(1)$ may vary from $\mathrm{p}(0)$, even among the selected alternatives of the first vote, in the form depending also on the outcome $F(\mathrm{p}(0), 0)$.

The dynamic framework reminds us that the conventional static model $F : P^N \longrightarrow P$ is over-simplified and inapplicable for a general social choice process. Although it is not attempted in this paper to investigate the dynamic model, the establishment of the fundamental notions, such as the singularity of preferences, its stratification structure and especially the zero order and the first order topologies of the preference space which describe the different features of continuous changes of a profile, may provide basic tools to approach the general dynamic model of the social choice process.

The author would like to express his deep gratitude to Ms. Hsiang-Ju Chen for her remarkable suggestions from the viewpoint of economists and her constant assistance with carefulness and consideration during the two years of the preparation of this paper.

§ 1 Preferences and stratifications

§ 1.1 Definition of preferences

Let X be a *space of alternatives*, which mathematically is a set or more generally a topological space — a pure set is regarded as a discrete topological space. A *preference order* p on X is a binary relation $\succsim$ defined on some pairs of elements of X such that

(i) (Completeness); $\forall\, x, y$ in X, either $x \succsim y$ or $y \succsim x$, or both,

(ii) (Transitivity); $\forall\, x, y, z$ in X, $x \succsim y$ and $y \succsim z \Rightarrow x \succsim z$,

where x, y, z may be distinct or non-distinct. The set of all preference orders on X is denoted by P_*.

Let $p \in P_*$ and $\succsim$ be the binary relation defined by p. If specification is necessary, we write "$x \succsim y$ in p", otherwise "$x \succsim y$" is written for simplicity. By x is *preferred to* y (*in* p), we mean $x \succsim y$ but not $y \succsim x$, for which $x \succ y$ is denoted. On the other hand, when $x \succsim y$ and $y \succsim x$, x is called *indifferent to* y (*in* p), denoted by $x \sim y$. Given U, V subsets of X, if $\forall\, x \in U$, $y \in V$, $x \succsim y$, then we write

$$U \succsim V.$$

Also by $x \succsim V$, we mean $x \succsim y$, $\forall\, y \in V$.

It is evident that both of the binary relations $\succ$ and $\sim$ are also transitive. Furthermore, we see

$$x \succ y \text{ and } y \sim z \Rightarrow x \succ z$$

and

$$x \sim y \text{ and } y \succ z \Rightarrow x \succ z.$$

A preference order p on X *can* also be regarded as a subset G of $X \times X$, where $(x, y) \in G$ iff $x \succsim y$. This subset G is called *the graph corresponding to* p. For a topological space X, a preference order p *respects the topology of* X if the corresponding graph G is closed in $X \times X$ with product topology. Respecting the topology of X is clearly equivalent to the "*continuity* of the preference order on X" in economics terminology[2], which means that for two sequences x_n, y_n with $x_n \to x$, $y_n \to y$ in X, $x_n \succsim y_n$ in p implies $x \succsim y$ in p.

A preference order on X respecting the topology of X is called a *preference on the topological space* X. Let P denote the totality of preferences on X. In a brief statement, for given $p \in P$ and $x \in X$ we call

$$R_x(p) \equiv \{y \in X;\ y \succsim x \ \text{ in } p\}$$

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[2] See e.g. [MC-W-G] Mas-Colell, Whinston and Green "Microeconomic Theory" (1995), Oxford Univ. Press.

by the *superpreference set* of p at x, and

$$Q_x(p) \equiv \{y \in X;\ y \preceq x \ \text{ in } p\}$$

by the *subpreference set* of p at x, both of which are evidently closed in X. Similarly, we call

$$R_x^{'}(p) \equiv \{y \in X;\ y \succ x \ \text{ in } p\}$$

by the *strictly superpreference set* of p at x, while it is called the *preferred set* in the common use and

$$Q_x^{'}(p) \equiv \{y \in X;\ y \prec x \ \text{ in } p\}$$

by the *strictly subpreference set* of p at x or the *inferior set*. Clearly, $R_x^{'}(p)$ and $Q_x^{'}(p)$ are open in X. We also define the *indifferent set* by

$$I_x(p) \equiv \{y \in X;\ y \sim x \ \text{ in } p\} = R_x(p) \cap Q_x(p),$$

which is closed in X. When there arises no confusion, the abbreviation notations R_x, Q_x, $R_x^{'}$, $Q_x^{'}$ and I_x would be used.

By the notation "$\equiv$" as appeared in the last paragraph, we mean "denoting" or "defined by". This convention will be repeatedly used in the paper. In addition, we introduce several topological notations. Given a set A in X, the *closure* $\overline{A}$ of A is defined by the minimal closed set containing A, while the *interior* $Int\, A$ of A is the maximal open set contained in A. We remark that $\overline{R_x^{'}} \subset R_x$ and $R_x^{'} \subset Int\, R_x$

, but it may happen that $\overline{R_x^{'}} \neq R_x$ or $R_x^{'} \neq Int\, R_x$, unless p is "regular" in a topological structure (see Definition 2.2.1). The *boundary* ∂A of A is the difference $\overline{A} - Int\, A$. It is clear that ∂A consists of the points, of which each neighborhood intersects both A and $X - A$. Therefore $\overline{A} = \partial A \cup Int\, A$. Also notice that for a topological singular preference p in P, it is false that $\partial R_x = R_x - R_x^{'}$ at singular points $x \in \Lambda(p)$. (For the singular set $\Lambda(p)$, see Definition 1.3.1). The singular sets play a crucial role in our framework.

Throughout this paper, we assume X a general topological space which includes both the cases of continuum space and pure set. A pure set X of alternatives can be regarded as a topological space with discrete topology, i.e. each point in the space is itself an open set.

In §1 and §2, we consider X a topological space with conditions such as locally-compact, T_3, locally connected, and connected which would be specified when required. For those who are not familiar with topological language X may be thought as a square in $\mathbb{R}^2$, a quadrant of $\mathbb{R}^2$ or a topological manifolds. If connectedness is not assumed, X could also be regarded as a pure set.

In §3, we concentrate to consider X a continuum space (for definition see §3.1). Again, the square, the quadrant and connected manifolds are examples of continuum spaces. The discrete case, i.e. X is a pure set, is handled however in the last section §4.

§ 1.2 Stratification structures

It is assumed in this section that X is a pure set or more generally an arbitrarily given locally compact T_3-space[3]. We introduce the notion of the stratification structure of an alternative space X, and show the equivalence relationship between a preference on X and a stratification of X. Basically, a stratification of X is a decomposition of X into $\{X_\alpha; \alpha \in A\}$ which is linear ordered with the order respecting the topology of X. Here is the rigorous definition.

Definition 1.2.1 Given an alternative space X, a *stratification* σ is a decomposition of X into non-empty closed subsets X_α, $\alpha \in A$, associated with an injective assignment of a non-empty subindex set A_α of A to each $\alpha \in A$, where A is an index set, such that

(S1) $X = \bigcup_{\alpha\in A} X_\alpha$ and $X_\alpha \cap X_\beta = \phi, \forall\, \alpha,\ \beta \in A$ with $\alpha \neq \beta$, (Decomposition)

(S2) R_α and Q_α are closed in X, where (Topological coherence)

$$R_\alpha \equiv \bigcup_{\alpha\in A_\alpha} X_\alpha,\ Q_\alpha \equiv (X - R_\alpha) \cup X_\alpha,$$

(S3) $A_\alpha \cap A_\beta = A_\alpha$ or A_β, (Linear ordering of inclusion)

(S4) $\alpha \in A_\alpha,\ \forall\, \alpha \in A$; $A_\beta \subset A_\alpha$, whenever $\beta \in A_\alpha$.

(Minimality of α in A_α)

We denote the stratification σ by $(X,\ X_\alpha;\ A,\ A_\alpha)$.

The following proposition illustrates as an example how the topology of X is reflected on the index set A.

Proposition 1.2.1 Let $x_n \to x$ in X, $x_n \in X_{\alpha_n}$, $x \in X_\alpha$ and

$$A_{\alpha_n} \supset A_{\alpha_{n+1}} \supset A_\alpha,\ \forall n.\ \text{Then}\ \bigcap_{n=1}^{\infty} A_{\alpha_n} = A_\alpha.$$

Proof. $\bigcap_{n=1}^{\infty} A_{\alpha_n} \supset A_\alpha$ is clear since $A_{\alpha_n} \supset A_\alpha,\ \forall n$. It suffices to show that $\bigcap_{n=1}^{\infty} A_{\alpha_n} \subset A_\alpha$. Suppose the contrary, i.e. $\exists \gamma \in \bigcap_{n=1}^{\infty} A_{\alpha_n} - A_\alpha$. Since $\gamma \in \bigcap_{n=1}^{\infty} A_{\alpha_n}$, we have $A_\gamma \subset A_{\alpha_n},\ \forall n$, by the minimality S4. Given n, we have

---

[3] A domain in the Euclidean space $\mathbb{R}^n$ which is often considered in welfare economics is an example of locally compact T_3-*space*. A topological space X is called *locally compact*, if $\forall x \in X$, there exists a compact neighborhood V of x. A *neighborhood of* x may not be open in general, it is a set containing an open set W with $x \in W$. A topological space X is called a T_3-*space*, if any single point is closed in X, and given a point x in X as well as a closed set A of X with $x \notin A$, there exists two disjoint open sets V and W such that $x \in V$ and $A \subset W$. Since we have assumed X is of T_3 and is locally compact, it is easy to see that $\forall\, x \in X$, and $\forall$ open neighborhood U of x, there exists a compact neighborhood V of x such that $V \subset U$. (see also the footnote in § 2.3)

Case 1: $\alpha_n \in A_\gamma$, which implies $A_{\alpha_n} \subset A_\gamma$, i.e. $A_{\alpha_n} = A_\gamma$. By the injectivity, $\alpha_n = \gamma$, and $X_{\alpha_n} = X_\gamma$. *Case 2*: $\alpha_n \notin A_\gamma$, which implies

$$X_{\alpha_n} \subset X - \bigcup_{\beta \in A_\gamma} X_\beta.$$

In either case, we have $x_n \in Q_\gamma$, since

$$x_n \in X_{\alpha_n} \subset (X - \bigcup_{\beta \in A_\gamma} X_\beta) \cup X_\gamma \equiv Q_\gamma.$$

On the other hand, $\gamma \notin A_\alpha$ and $\gamma \in A_\gamma$. Hence, A_γ is not contained in A_α. By the linear ordering S3, $A_\gamma \supset A_\alpha$ with $\gamma \neq \alpha$. This together with the definition of Q_γ yields that $X_\alpha \cap Q_\gamma = \phi$. But $x \in X_\alpha$; we have

$$x \notin Q_\gamma.$$

However, $x_n \in Q_\gamma$ and $x_n \to x$ in X, $\forall\ n$. By Q_γ is closed (due to S2), we would have $x \in Q_\gamma$. This is a contradiction. ■

Now we show the equivalence relationship between preferences and stratifications.

Proposition 1.2.2 Let p be a preference on X. Then p defines a stratification $\sigma \equiv (X, X_\alpha; A, A_\alpha)$, where for each $x \in X_\alpha$, we have $X_\alpha = I_x$, $R_\alpha = R_x$ and $Q_\alpha = Q_x$. Conversely, given a stratification $\sigma \equiv (X, X_\alpha; A, A_\alpha)$, there defines a preference p on X such that

$$x \preceq y \text{ iff } A_\alpha \supset A_\beta$$

with $x \in X_\alpha$, $y \in X_\beta$ and $\alpha, \beta \in A$.

Proof. Given $p \in P$, let X be decomposed into indifferent sets. The decomposition is well-defined by the transitivity of $\sim$ of p. Let the index set A be the quotient set[4] corresponding to the decomposition. We have $X = \bigcup_{\alpha \in A} X_\alpha$, where $X_\alpha = I_x$, $\forall x \in X$ with $x \in X_\alpha$. Hence S1 is satisfied. By $X_\alpha = I_x = R_x \cap Q_x$, each X_α is closed. For $x \in X_\alpha$, we define

$$A_\alpha \equiv \{\beta \in A;\ \exists y \in X_\beta \text{ with } y \succsim x \text{ (in } p)\},$$

then by transitivity,we have $\beta \in A_\alpha$ iff $\forall z \in X_\beta$, $z \succsim x$. Hence,

$$\begin{aligned} R_\alpha &\equiv \bigcup_{\beta \in A_\alpha} X_\beta = \{y \in X;\ y \succsim x\} = R_x \\ Q_\alpha &\equiv (X - \bigcup_{\beta \in A_\alpha} X_\beta) \cup X_\alpha = \{y \in X;\ y \preceq x\} = Q_x, \end{aligned}$$

---

[4] We may even equipe the index set A a natural topology by assuming A the topological quotient space $X/\sim$, where $\sim$ is the indifference relation given by p.

and therefore R_α and Q_α are closed in X. This is S2. To show S3, let $X_\alpha = I_x$, $X_\beta = I_y$ with $x, y \in X$. Assume without loss of generality that $x \succsim y$. It holds that

$$\begin{aligned} A_\alpha \cap A_\beta &= \{\gamma \in A;\ \forall z \in X_\gamma,\ z \succsim x \text{ and } z \succsim y\} \\ &= \{\gamma \in A;\ \forall z \in X_\gamma,\ z \succsim x\} \qquad \text{(by transitivity of } p) \\ &= A_\alpha \end{aligned}$$

For S4, we have $\alpha \in A_\alpha$, $\forall \alpha \in A$, since if $X_\alpha = I_x$ for some x then $y \succsim x$ $\forall y \in X_\alpha = I_x$. To claim that $A_\beta \subset A_\alpha$, $\forall \beta \in A_\alpha$, we let $\gamma \in A_\beta$. Let $X_\alpha = I_x$ and $y \in X_\beta$, we have $y \succsim x$. However, $\forall z \in X_\gamma$, we have $z \succsim y$. Hence $z \succsim x$ by the transitivity of p, and therefore $\gamma \in A_\alpha$.

Conversely, given $\sigma \equiv (X,\ X_\alpha;\ A,\ A_\alpha)$ we define a preference p on X by

$$x \preceq y \text{ in } p \text{ iff } A_\alpha \supset A_\beta \tag{1.2a}$$

where $x \in X_\alpha$ and $y \in X_\beta$. Evidently, p is a preference order on X. In fact, by S3, we have $A_\alpha \supset A_\beta$ or $A_\beta \supset A_\alpha$ or both. The previous two cases correspond to $x \preceq y$ and $y \preceq x$ respectively. When $A_\alpha = A_\beta$, we have $\alpha = \beta$ by the injectivity. Hence $X_\alpha = X_\beta$ and $x, y \in X_\alpha$, i.e. $x \sim y$. Finally we have to show that p respects the topology of X, in order that p becomes a preference on X. By S3 and S4, it is clear that $\beta \in A_\alpha$ iff $A_\alpha \supset A_\beta$, and therefore iff $y \succsim x$ by (1.2a). Hence

$$\{y \in X;\ y \succsim x\} = \bigcup_{\beta \in A_\alpha} X_\beta,$$

i.e. $R_x = R_\alpha$. Thus R_x is closed in X, by S2. Similarly,

$$\{y \in X;\ y \preceq x\} = (X - \bigcup_{\beta \in A_\alpha} X_\beta) \cup X_\alpha$$

i.e. $Q_x = Q_\alpha$. We also have Q_x closed in X. By the following Proposition 1.2.3, the graph G defining the preference order p is closed in $X \times X$. The proof would then be completed. ■

By Proposition 1.2.2, we see that the stratification structure can completely depict a preference.

The sets X_α, $\alpha \in A$, or equivalently, the sets I_x, $x \in X$, are also called the *stratification sets* of a preference.

We have defined in §1.1 that a preference on a topological space X is a preference order on X with the corresponding graph G closed in $X \times X$. In general, a set S in $X \times X$ with S_x and T_x closed in X, $\forall x \in X$, is not necessarily closed in $X \times X$, where for a given $x \in X$, S_x and T_x are defined by

$$\begin{aligned} S_x &\equiv \{(x, y) \in S;\ y \in X\} \\ T_x &\equiv \{(y, x) \in S;\ y \in X\}. \end{aligned}$$

However, for the graph G corresponding to a preference order p we will show that R_x and Q_x closed in X, $\forall\, x \in X$, implies G closed in $X \times X$.

Proposition 1.2.3 Let p be as preference order on X with each R_x and Q_x closed in X, $\forall\, x \in X$, where

$$R_x \equiv \{y \in X;\ y \succsim x \ \text{ in } p\}\,,\ Q_x \equiv \{y \in X;\ y \preceq x \ \text{ in } p\}\,.$$

Then the graph $G \equiv \{(x,\, y) \in X \times X;\ x \succsim y \ \text{ in } p\}$ is closed in $X \times X$.

Proof.

Step 1 For K a compact set in X, we first claim $\exists\, x_o \in K$ such that $x_o \succsim K$. Suppose there exists no such x_o in K, then given $x \in K$, we have $y_x \in K$ with $y_x \succ x$. By R_{y_x} closed in X, $Q'_{y_x} \equiv \{z \in X;\ y_x \succ z\}$ is open in X. There exists an open neighborhood U_x of x such that $U_x \subset Q'_{y_x}$, i.e.

$$y_x \succ U_x.$$

Let $\{U_{x_1},\, \cdots,\, U_{x_N}\}$ be a finite subcover of $\{U_x;\ x \in K\}$. Let $y_o \in K$ be an element in $\{y_{x_1},\, \cdots,\, y_{x_N}\}$ such that

$$y_o \succsim y_{x_i},\ \forall\, i = 1, 2, \cdots, N.$$

Then $y_o \succ U_{x_i}$ $\forall\ i = 1, 2, \cdots, N$. But $\{U_{x_i};\ i = 1, 2, \cdots, N\}$ covers K. We have $y_o \succ K$, contradicting to the hypothesis.

Step 2 Let G^c be the complement $X \times X - G$. We show that G^c is open in $X \times X$, i.e. given $(x,\, y) \in G^c$, $\exists$ neighborhoods V, W of x, y respectively, such that $V \times W \subset G^c$. First, we remark that $(x,\, y) \in G^c$ iff $x \prec y$ in p. Since R_y is closed in X, Q'_y is open, there exists a neighborhood U of x such that $U \subset Q'_y$. By the assumption that X is locally compact, and of T_3, there exists a compact neighborhood V of x with $V \subset U$. ¿From Step 1, $\exists\ z \in V$ such that

$$z \succsim V.$$

On the other hand, Q_z is closed in X. Hence R'_z is open. But $y \succ U$, $z \in V \subset U$, we have $y \succ z$, i.e. $y \in R'_z$. There exists a neighborhood W of y such that $W \subset R'_z$, i.e.

$$W \succ z.$$

The transitivity of p implies that $W \succ V$, i.e.

$$V \times W \subset G^c.$$

The proof is completed. ■

Proposition 1.2.4 Given $p \in P$ and two points x, y in X with $x \prec y$ in p, there exist neighborhoods U, V of x, y respectively such that

$$U \prec V \ \text{ in } p.$$

Proof. Q_y' is open and $x \in Q_y'$. Since X is a locally compact T_3-space, there exists a compact neighborhood U of x with $U \subset Q_y'$. Let $z \in U$ be such $z \succsim U$, by Step 1 of Proposition 1.2.3. Notice that $z \in U \subset Q_y'$ implies $z \prec y$ in p. Consider now the open set R_z'. By $y \in R_z'$, $\exists$ a compact neighborhood V of y with $V \subset R_z'$. In addition, let $w \in V$ be such $w \preceq V$. Similarly, $w \in V \subset R_z'$ also implies that $z \prec w$. Therefore, $U \preceq z \prec w \preceq V$, which gives the required property. ∎

Clearly, we have

Proposition 1.2.5 Given $p \in P$ and $x, y \in X$. Suppose $x_n \to x$ and $y_n \to y$ in X with $x_n \succsim y_n$ in p, $\forall n$. Then $x \succsim y$ in p.

In economics terminology, we say that a preference order p is *continuous* if the property in Proposition 1.2.5 holds, which is equivalent to that p respects the topology of X.

§ 1.3 Examples of preferences

We give some examples to illustrate certain notions defined in the previous sections.

Example 1.3.1 (See Figure 1.3.1)

Let $X = \mathbb{R}^2$ where $\mathbb{R}$ is the line of real numbers, i.e. X is the real plane. Let (x_1, x_2) be the Euclidean coordinate system of X and let $\lambda \equiv (\lambda_1, \lambda_2)$ be a unit vector. We define a *linear preference* p on X by

$$x \equiv (x_1, x_2) \precsim y \equiv (y_1, y_2) \text{ in } p$$

iff

$$\langle \lambda, x \rangle \leq \langle \lambda, y \rangle ,$$

where $\langle \lambda, x \rangle \equiv \lambda_1 x_1 + \lambda_2 x_2$ is the inner product of λ and x. Then the indifferent sets are the straight lines:

$$\langle \lambda, x \rangle = \alpha; \ \alpha \in A, \text{with } A \equiv (-\infty, \infty) \text{ the index set.}$$

Given a point $x = (x_1, x_2)$ in X, we have

$$\begin{aligned} I_x &= \{y \in X;\ \langle \lambda, y \rangle = \langle \lambda, x \rangle\} \\ R_x &= \{y \in X;\ \langle \lambda, y \rangle \geq \langle \lambda, x \rangle\} \\ Q_x &= \{y \in X;\ \langle \lambda, y \rangle \leq \langle \lambda, x \rangle\} \end{aligned}$$

where R_x and Q_x are closed ($I_x = R_x \cap Q_x$ is consequently closed). Therefore p respects the topology of $X = \mathbb{R}^2$ and $p \in P$. On the other hand, for each α,

$$\begin{aligned} X_\alpha &= \{x \in X;\ \langle \lambda, x \rangle = \alpha\} \\ R_\alpha &= \{x \in X;\ \langle \lambda, x \rangle \geq \alpha\} \\ Q_\alpha &= \{x \in X;\ \langle \lambda, x \rangle \leq \alpha\} \end{aligned}$$

with $I_x = X_\alpha$, $R_x = R_\alpha$ and $Q_x = Q_\alpha$ where $\alpha = \langle \lambda, x \rangle$. Clearly,

$$A_\alpha = [\alpha, \infty)$$

and $\sigma \equiv (X, X_\alpha; A, A_\alpha)$ is the stratification defined by p as in Proposition 1.2.2.

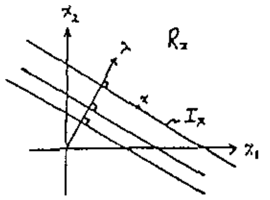

Fig. 1.3.1

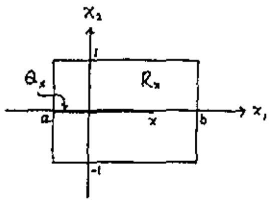

Fig. 1.3.2

Example 1.3.2 (See Figure 1.3.2)

Let $X = [a, b] \times [-1, 1]$. We define a preference order p by $x \equiv (x_1, x_2) \precsim y \equiv (y_1, y_2)$ in p, iff (i) $x_1 \leq y_1$, when $x_2 = y_2 = 0$, and (ii) $|x_2| \leq |y_2|$, otherwise. Then X is stratified into the points on x_1-axis and the horizontal line segments in X except for the one in x_1-axis. We see that p is a preference order, but p does *not* respect the topology of X, since for example, at $x = (c, 0)$ with $a < c < b$,

$$R_x = \{(x_1, 0) \in X; c \leq x_1 \leq b\} \cup \{(x_1, x_2) \in X; x_2 \neq 0\}$$

and R_x is not a closed set in X, although Q_x is. Hence

$$p \notin P$$

We see by this example that $P_* - P \neq \phi$, ϕ denoting the empty set.

Example 1.3.3 (See Fig. 1.3.3)

Let X be the torus $S^1 \times S^1$ where S^1 denotes a unit circle. Given the product coordinate (θ, φ) for each point x of X such that

$$X = \{(\theta, \varphi);\ 0 \leq \theta < 2\pi,\ 0 \leq \varphi < 2\pi\}.$$

Define a preference order p on X by

$$x \equiv (\theta_o, \varphi_o) \precsim y \equiv (\theta_1, \varphi_1) \text{ iff } \theta_o \leq \theta_1.$$

Clearly $p \in P_*$. However, p does *not* respect the topology of X, since at each $x = (\theta_o, \varphi_o)$, $R_x = \{(\theta, \varphi) \in X;\ \theta_o \leq \theta < 2\pi\}$ which is not closed in X. Hence p is not a preference, i.e. $p \notin P$.

Example 1.3.4

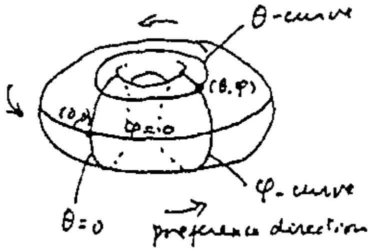

Figure 1.3.3

Let X be the surface of the Earth. Define a preference order p by the criterion: "the higher the more preferred". Then $R_x = \{y \in X$; the altitude at y is greater than or equal to that at $x\}$. It is easily seen that R_x and Q_x are closed in X, if the altitude is a continuous function on the surface of the Earth. The level curves are the indifferent sets of p. The surface of the Earth is thus stratified into level curves with their higher sets as the superpreference sets. The summits of mountains, the ridge points, the saddle points and the points on a flat plateau, or on the bottom of a basin give various singularities of which the notion would be introduced in section 2.2.

Example 1.3.5

Let X be the 2-dimensional unit sphere S^2, i.e.

$$X = \{(x_1, x_2, x_3) \in \mathbb{R}^3;\ x_1^2 + x_2^2 + x_3^2 = 1\}$$

Clearly, X is a differentiable manifold of dimension 2^5. Define a preference order p on X by

$$x \equiv (x_1, x_2, x_3) \precsim y \equiv (y_1, y_2, y_3) \text{ in } p \text{ iff } x_3 \leq y_3.$$

It is clear that $p \in P$. The indifferent sets (also called stratification sets) are the meridians

$$I_\alpha = \{x \in X;\ x_3 = \alpha\},\ \alpha = [-1, 1].$$

They are all nice differentiable curves except at the north pole and at the south pole, where $\alpha = +1$ and $\alpha = -1$ respectively. Such a preference will be called *singular in the differentiable structure* (see Section 2.2.) We note that any preference on S^2 must be singular at some points in this sense, because the singular points such as the north pole and south pole are not avoidable, according to Hopf's Index theorem [H].

Let f be a continuous real valued function on X. We define a preference order p on X by

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[5] A differentiable manifold of dimension n is a topological space, in which a point has a neighborhood homeamorphic (i.e. topologically equivalent) to the Euclidean space $\mathbb{R}^n$, and on which differentiable functions are well defined.

$$x \succsim y \text{ in } p \text{ iff } f(x) \geq f(y). \tag{1.3.1}$$

We usually call f a *utility function* of the preference p. It is evident that $R_x = \{y \in X;\ f(y) \geq f(x)\}$ and $Q_x = \{y \in X;\ f(y) \leq f(x)\}$. R_x and Q_x are closed in X for any $x \in X$, since f is continuous on X. Therefore, p respects the topology of X, and $p \in P$. Let $\sigma = (X, X_\alpha;\ A, A_\alpha)$ be the stratification defined by p as in Proposition 1.2.2 with $X_\alpha = \{x \in X;\ f(x) = \alpha\}$. The index set A is the range of the function f. In economics terminology, f is a utility function on X.

We remark that a preference p considered in social choice theory has usually no intrinsic meaning to quantify its degree of preference with a corresponding utility function f. Simply for expository convenience in illustrating examples, we use f to define p, neglecting the values of f, while preserving their inequality as in Formula (1.3.1).

We give in the following a weird example of preferences in measure theory.

Example 1.3.6 (See Figure 1.3.6) Let X be the square $I^2 = [0,1] \times [0,1]$. We may denumerate the set of all rational points of X, i.e. the points in X with both the two coordinates x_1 and x_2 rational into

$$\{r_1, r_2, r_3, \cdots, r_n, \cdots\}$$

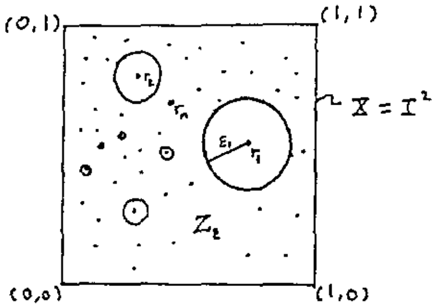


Figure 1.3.6 ($\varepsilon_1 = \sqrt{\varepsilon/8}$)

Given ε with $0 < \varepsilon < 1$, define $Z_\varepsilon \equiv X - \bigcup_{k=1}^{\infty} B\left(r_k,\ \sqrt{\varepsilon/2^{k+2}}\right)$ where $B(x_o, \rho)$ denotes the open disk $\{x \in \mathbb{R}^2;\ |x - x_o| < \rho\}$. Let μ be the "Lebesque measure" on $\mathbb{R}^2$ which is essentially the extended notion of the "area" on the nice sets in $\mathbb{R}^2$ to the relatively complicate sets whose area is not intuitively clear. Evidently, Z_ε has the Lebesque measure

$$\mu(Z_\varepsilon) = 1 - \frac{\pi}{4}\varepsilon > 1 - \varepsilon > 0,$$

although it has no interior point, i.e. no point of Z_ε with its neighborhood contained in Z_ε, because the rational points are dense everywhere. Define a preference order p on X by

$$y \succsim x \text{ in } p \quad \text{iff} \quad dist(y,\, Z_\varepsilon) \leq dist(x,\, Z_\varepsilon)$$

where $dist(x,\, A)$ denotes the Euclidean distance from a point of x to a closed set A in $\mathbb{R}^2$. Since $dist\,(x,\, Z_\varepsilon)$ is a continuous function of x, p respects the topology of X. As an example, at $x \in Z_\varepsilon$, I_x is equal to Z_ε and closed in X. However, given $x \in Z_\varepsilon$, the indifferent set I_x has $\partial I_x = I_x$, since $Int\, I_x = \phi$, and therefore $\mu\,(\partial I_x) = \mu\,(Z_\varepsilon) > 0$.

Definition 1.3.1 Given a preference $p \in P$ on X. Define the *topological singular set* $\Lambda(p)$ by $\cup\{I_x(p);\ x \in \Lambda^o(p)\}$, where the *interior singular set* $\Lambda^o(p) = \{x \in X;\ \exists \text{ neighborhood } V \text{ of } x \text{ with } V \subset I_x(p)\}$

A point $x \in \Lambda(p)$ is called a *singular point* of p. Otherwise x is *a regular point* of p. For $x \in \Lambda^o(p)$, we call x an *interior singular point* of p.

Example 1.3.7 Consider the Cantor function which is a continuous increasing function $f : [0,1] \rightarrow [0,1]$ defined by $f(x) = \frac{1}{2}$ for $\frac{1}{3} \leq x \leq \frac{2}{3}$; $f(x) = \frac{1}{4}$ for $\frac{1}{9} \leq x \leq \frac{2}{9}$, $f(x) = \frac{3}{4}$ for $\frac{7}{9} \leq x \leq \frac{8}{9}$; $f(x) = \frac{1}{8}$ for $\frac{1}{27} \leq x \leq \frac{2}{27}$, $f(x) = \frac{3}{8}$ for $\frac{7}{27} \leq x \leq \frac{8}{27}$, $f(x) = \frac{5}{8}$ for $\frac{19}{27} \leq x \leq \frac{20}{27}$, $f(x) = \frac{7}{8}$ for $\frac{25}{27} \leq x \leq \frac{26}{27}$; $\cdots$ etc. Let p be the corresponding preference on $[0,1]$ defined by f. We have

$$I_{\frac{1}{2}} = [\frac{1}{3}, \frac{2}{3}], I_{\frac{1}{4}} = [\frac{1}{9}, \frac{2}{9}], I_{\frac{3}{4}} = [\frac{7}{9}, \frac{8}{9}], \ ...$$

the inferior singular set

$$\Lambda^o(p) = (\frac{1}{3}, \frac{2}{3}) \cup (\frac{1}{9}, \frac{2}{9}) \cup (\frac{7}{9}, \frac{8}{9}) \cup \cdots,$$

with the (topological) singular set identical with the Cantor set, i.e.

$$\Lambda(p) = [\frac{1}{3}, \frac{2}{3}] \cup [\frac{1}{9}, \frac{2}{9}] \cup [\frac{7}{9}, \frac{8}{9}] \cup \ldots .$$

But the closure of $\Lambda^o(p)$ is

$$\overline{\Lambda^o(p)} = [0,1] = X,$$

which is different from $\Lambda(p)$, and therefore the singular set $\Lambda(p)$ is *not* closed. In this example, although each ∂I_x with $x \in \Lambda(p)$contains exactly two points, $\partial\Lambda(p)$ has uncountable set of points. Also we notice that

$$\mu(\partial\Lambda(p)) = 0, \ \mu(\Lambda^o(p)) = 1$$

The existence of the particular examples such as the last two examples would make our analysis of social choice theory unnecessarily complicate. To include such examples, we would need to develope the whole theory of social choice on the basis of measure theory. This is similar to the historical development that for centuries from Newton to Weierstrass, Riemann integration theory where the number of discontinuity of functions are regulated, is sufficient to describe the reality that human beings ever encounter with. After the rigorous real number system was invented in the middle 19th century, studies of Lebesque integration theory is inevitable, because more complicate functions (e.g. Cantor's function) have to be handled. At the time being, as the social choice theory is concerned we make assumptions to exclude the complicate preferences as in the example 1.3.6 and 1.3.7 which are often occurred in measure theory:

Additional Assumption: For a preference p in P we assume in the latter sections that

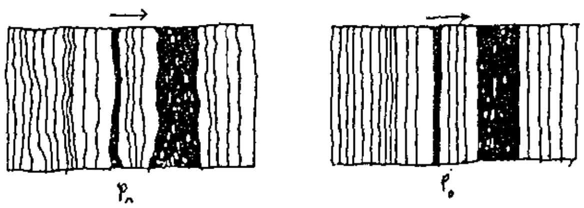
$$\mu(\partial I_x(p)) = 0, \ \forall\, x \in X.$$

Intuitively, the requirement means that the boundary of each indifferent set is "thin", so that it occupies no "area" (see Example 1.3.6). Also we assume that the topological singular set $\Lambda(p)$ is closed in X (see Example 1.3.7).

§ 2 Stratification convergence and singularities

§ 2.1 Stratification convergence

Let P be the set of all preferences on X. The equivalence of preference and stratification is established in Proposition 1.2.2. We now introduce a natural notion of convergence in P, i.e. a natural topology of P, based on this equivalence, to reflect the stratification structure of preferences, which will be called *stratification convergence* in P. The criterion of *p_n convergent to p_o in stratification structure* is that *the stratification sets of p_n tend to the corresponding stratification sets of p_o, while preserving the preference order. The convergence of stratification sets is required to be uniform in an arbitrarily given compact set of X*. The figure 2.1.1 illustrate the rough idea of the stratification convergence:



(The dark lines and regions denote the indifferent sets;
the arrows illustrate the preference directions)
Fig. 2.1.1

The convergence is indeed the "first order convergence" of which the meaning will be explained in the later sections.

To define the stratification convergence rigorously, we use the viewpoint of open sets, i.e. we regard a topology of P as a family[6] of open sets in P.

Definition 2.1.1 Let $\Im_o$ be the *zero order topology of* P defined by the subbase

$$B_o \equiv \{G(K,L);\ K,L \text{ being compact sets in } X\}$$

where $G(K,L) \equiv \{p \in P;\ K \prec L \text{ in } p\}$and let $\Im_1$ be the *first order topology of* P defined by the base

$$B_1 \equiv \{H(K,W);\ K \text{ compact, } W \text{ open in } X \text{ and } K \subset W\}$$

where $H(K,W) \equiv \{p \in P;\ K \subset \Lambda^o(p) \subset \Lambda(p) \subset W\}$.and $\Lambda(p)$ is the topological singular set (See Definition 1.3.1) with $\Lambda^o(p)$ its interior.

Definition 2.1.2 The topology $\Im$ defined by

$$\Im \equiv \Im_o \cup \Im_1$$

is called the *stratification topology* of P. Given a sequence p_n in P and $p_o \in P$, we say p_n *converges to* p_o *in stratification* if $\forall T \in \Im$ with $p_o \in T$, there exists N such that $p_n \in T, \forall n > N$. The convergence is usually denoted by

$$p_n \to p_o \text{ in } \Im.$$

Since B_o and B_1 generate $\Im$, a criterion for $p_n \to p_o$ in $\Im$ is that: $\forall G(K,L)$ and $\forall H(J,W)$ with K,L,J compact and W open in X, if $p_o \in G(K,L) \cap H(J,W)$, then there exists N such that $p_n \in G(K,L) \cap H(J,W)$, $\forall n > N$.

Proposition 2.1.1 Let $p_n \to p_o$ in $\Im$. Then

$$x \succ y \text{ in } p_o \quad \Rightarrow \quad \begin{array}{l} \exists \text{ neighborhoods } U,V \text{ of } x,y \text{ respectively} \\ \text{and } \exists N \text{ such that } U \succ V \text{ in } p_n, \forall n > N. \end{array}$$

Proof By Prop.1.2.4, $\exists$ neighborhoods U_1, V_1 of x, y respectively such that

$$U_1 \succ V_1 \text{ in } p_o.$$

Since X is locally compact, there exists $\overline{U}$ and $\overline{V}$, compact neighborhoods of x and y respectively, such that $\overline{U} \subset U_1$ and $\overline{V} \subset V_1$ with U,V open neighborhoods of x in X. Clearly $p_o \in G(\overline{V},\overline{U})$. By the definition of $\Im$, $\exists N$ such that $p_n \in G(\overline{V},\overline{U})$, $\forall n > N$. In particular,

$$U \succ V \text{ in } p_n,\ \forall n > N,$$

as required. ■

Remark 2.1.1 If the antecedent is replaced by $p_n \to p_o$ in $\Im_o$, the proposition 2.1.1 is still valid. If fact, the above proof involves only open sets generated by $G(U,V)$ in $\Im_o$.

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[6] For reference see "General Topology", by J. Kelly.

The following corollary is logically equivalent to the last proposition.

Corollary 2.1.1 Let $p_n \to p_o$ in $\Im_o$ and $x_n \succsim y_n$ in p_n with $x_n \to x$ and $y_n \to y$ in X. Then $x \succsim y$ in p_o.

Definition 2.1.3 A topology $\Im'$ of P *respects the topology of* X if given $p_n \in P$ with $p_n \to p_o$ in $\Im'$ and given $x_n, y_n \in X$ with $x_n \to x, y_n \to y$ in X and $x_n \succsim y_n$ in $p_n, \forall n$, we have $x \succsim y$ in p_o.

Evidently, both $\Im_o$ and $\Im$ respect the topology of X.

Example 2.1.1 (See Figure 2.1.2)

Consider the linear preference p_n on $\mathbb{R}^2$ defined as in Example 1.3.1 by $\lambda_n = \left(\cos\frac{\pi}{n}, \sin\frac{\pi}{n}\right)$ and consider p_o defined by $\lambda_o = (1,0)$. Then $p_n \to p_o$ in $\Im$ as well as in $\Im_o$, when $n \to \infty$.

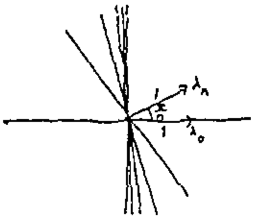

Fig. 2.1.2

Example 2.1.2 (See Figure 2.1.3)

Let $X = [0,\infty) \times [0,\infty) \subset \mathbb{R}^2$ and p_n be defined by the stratification with

(i) $X_\alpha(p_n) = \left\{(x_1, x_2) \in X;\ (x_1 - \alpha)(x_2 - \alpha) = \varepsilon_n^2\right\}$, $\varepsilon_n \geq 0$,

(ii) $x \preceq y$ in p_n iff $\alpha \leq \beta$, where $x \in X_\alpha(p_n)$, $y \in X_\beta(p_n)$..

Let $\{\varepsilon_n\}$ be a sequence of positive numbers tending to $\varepsilon_o \equiv 0$, as $n \to \infty$. Then $p_n \to p_o$ in $\Im$ and also in $\Im_o$.

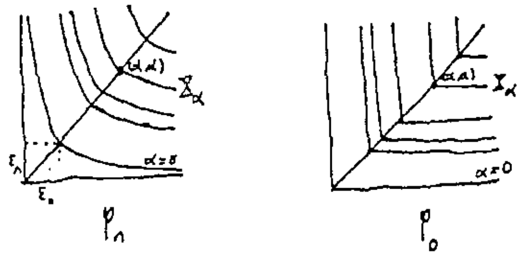

Fig. 2.1.3

Example 2.1.3 (See Figure 2.1.4)

Let p_n be the preference on $X = [-1,1] \times [-1,1]$ defined by the corresponding utility function $f_n(x) \equiv (1 - e_n^2)x_1^2 + x_2^2$, where $x = (x_1, x_2) \in X$, $0 < e_n < 1$, and $e_n \to 1$ as $n \to \infty$. Let p_o be defined by $f_o(x) \equiv x_2^2$. Then we have $p_n \to p_o$ in $\Im$ (as well as in $\Im_o$). The stratification sets of p_n and p_o are ellipses and horizontal straight lines respectively.

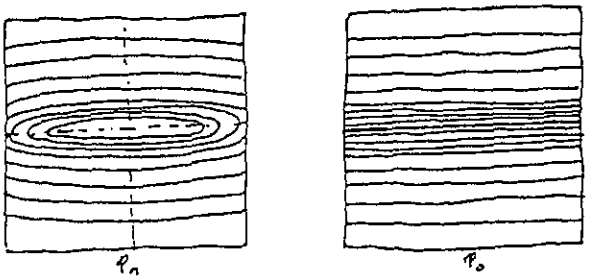

Fig. 2.1.4

Example 2.1.4 (see Figure 2.1.5) Let $X = (-1, 3)$ and let the preference p_n be defined by the utility function f_n, which is constructed as follows.

$$f_n(x) = \begin{cases} -(1-x) + x/n & , \text{ for } x \in (-1, \frac{1}{n}] \\ 0 & , \text{ for } x \in (\frac{1}{n}, 1] \\ x - 1 & , \text{ for } x \in [1, 2] \\ -x + 3 & , \text{ for } x \in [2, 3) \end{cases}$$

Let $f_o(x) = \lim_{n \to \infty} f_n(x)$ and p_o be defined by f_o. Then $\Lambda_o(p_o) = (0, 1)$ and $\Lambda_o(p_n) = (\frac{1}{n}, 1)$. Also, $\Lambda(p_o) = [0, 1]$ and $\Lambda(p_n) = [\frac{1}{n}, 1]$. If $p_o \in H(K, W)$, then $K \subset (0, 1) \subset [0, 1] \subset W$. For n sufficiently large, we see

$$K \subset (\tfrac{1}{n}, 1) \subset [\tfrac{1}{n}, 1] \subset W,$$

which yields that $p_n \in H(K, W)$, for large n. It is also clear that given $G(J, L) \in B_o$, $p_o \in G(J, L)$ implies $p_n \in G(J, L)$ for large n. Therefore,

$$p_n \to p_o \text{ in } \Im.$$

However, at $x = 0$ the subpreference set $Q_x(p_n) = (-1, 0]$, while $Q_x(p_o) = (-1, 1]$. Notice that $Q_x(p_n)$ *does not tend to* $Q_x(p_o)$.

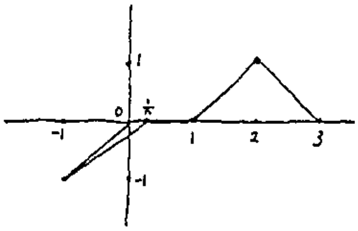

Fig. 2.1.5

Example 2.1.5 (See Figure 2.1.6) Let $X = [0, 2\pi]$ and let p_n be defined as in Example 1.3.6 by f_n, where we construct $f_n(x)$ as follows. Divide $[0, 2\pi]$ into $2m$ intervals of equal length with dividing points:

$$a_o = 0,\ a_1 = \tfrac{\pi}{m},\ a_2 = \tfrac{2\pi}{m},\ \ldots, a_k = \tfrac{k\pi}{m}, \ldots, a_{2m} = \tfrac{2m\pi}{m} = 2\pi,$$

where $m = 2^n$, $n \geq 3$, and let

$$f_n(x) \equiv \begin{cases} \sin a_k, & \text{for } x \in [a_{2k}, a_{2k+1}] \\ (1-\lambda)\sin a_{2k+1} + \ \lambda \sin a_{2k+2} & \\ & \text{for } x = (1-\lambda)\sin a_{2k+1} + (1-\lambda)\sin a_{2k+2} \in [a_{2k+1}, a_{2k+2}] \end{cases}$$

Define $f_o = \sin x$ and let p_o be defined by f_o. Then $f_n(x)$ converges to $f_o(x)$ uniformly in $[0, 2\pi]$, while p_n does not tend to p_o in $\Im$. In fact, $\Lambda(p_o) = \Lambda^o(p_n) = \phi$. Therefore, $p_o \in H(\phi, \phi)$. But for each p_n,

$$\Lambda(p_n) = \bigcup_{k=0}^{m-1} [a_{2k}, a_{2k+1}] \neq \phi$$

and here $p_n \notin H(\phi, \phi)$, $\forall\, n$.

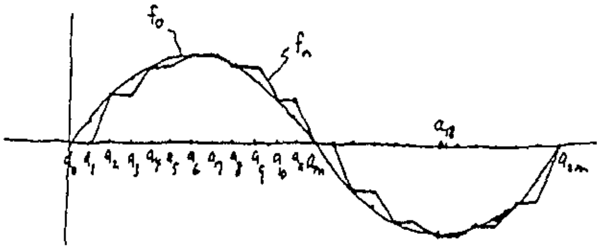

Fig. 2.1.6 $(m = 12)$

In the last example, we see that $f_n \to f_o$ in *the zero order* but *not* in *the first order*. In other words, although $f_n \to f_o$ uniformly, the derivatives f_n' of f_n' does not tend to f_o'. For preferences, it appears that the first order approximation has no clear meaning, i.e. it seems ambiguous to say that $p_n \to p_o$ in the first order. To give it a precise meaning, we need the first order topology $\Im_1$. As the cardinality discarded while ordinality maintained, derivatives are meaningless. However, the distinction between regularities and singularities in the context of topological structure (for more details, see § 2.2) still makes sense. This is the idea underlying the definition of $\Im_1$.In Example 2.1.6, the singular set $\Lambda(p_n) = \bigcup_{k=0}^{n-1} [a_{2k}, a_{2k+1}]$ occupies half of length (i.e. "measure") of the defined interval $[0, 2\pi]$, while $\Lambda(p_o)$ is empty. This makes p_n divergent in the first order topology $\Im_1$, in spite that p_n converges to p_o in the zero order topology $\Im_o$.By the abuse of language, we also call $\Im = \Im_0 \cup \Im_1$ the first order topology to mean the approximation *up to* the first order.

§ 2.2 Singularities relative to given structures

In this subsection, we let X a general connected topological space with various structures, not necessarily assuming it a continuum alternative space.

The concept of regularity and singularity of preferences corresponds to the structure of X under consideration. We will define them in topological structure, in manifold structure, in differentiable structure, as well as in measure structure. Although the notion of regularity has a different description, each formed by the concerned structure, it has a core characteristic that when the preference order gradually increases, the corresponding stratification covers the alternative space X as *evenly and smoothly* as the relative structure *could detect* and describe. Singularity occurs otherwise.

Definition 2.2.1 Let X be a connected topological space. A preference p on X is *singular in the topological structure* if there exists a point x in X and a neighborhood V_x of x such that $V_x \subset I_x(p)$. Otherwise, p is *regular in the topological structure*.

Example 2.2.1
(a) Let p be given by the utility function f with $f(x_1, x_2) = Ax_1^2+2Bx_1x_2+Cx_2^2$ for $x \in X \equiv R^2$ with $B^2 - AC \neq 0$, then p is regular in the (canonical) topological structure (although singular both in the manifold structure and in the differentiable structure to be defined later).
(b) If p is defined by the utility function (See Figure 2.2.1)

$$f(x_1, x_2) = \begin{cases} x_1, & if\ x_1 \leq 0 \\ 0, & if\ 0 < x_1 < 1 \\ x_1 - 1, & if\ x_1 \geq 1, \end{cases}$$

then p is singular in the topological structure.

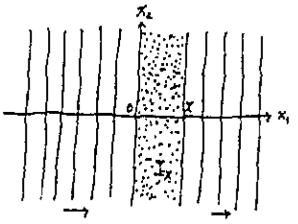

Fig. 2.2.1 $(x = (1, 0))$

Definition 2.2.2 Let X be a connected topological manifold of dimension n. A preference is *regular in the manifold structure* if each indifferent set I_x of an inferior point x in X is a topological sub-manifold of X with dimension $n-1$ and any neighborhood of x contains a smaller neighborhood V_x such that $V_x - I_x$ has exactly two components, one in the preferred set R_x' and the other in the inferior set Q_x'. Otherwise, p is *singular in the manifold structure*.

Example 2.2.2 Let p be given by the utility function

$$f(x_1, x_2) = x_1^n, \ \ x = (x_1, x_2) \in R^2 \equiv X$$

Then p is regular in the (canonical) manifold structure for n odd, and singular for n even. However, it is regular in the topological structure for any n.

Definition 2.2.3 Let X be a connected differentiable manifold of dimension n. A preference p on X is regular in the differentiable structure if p is *regular in* *the manifold structure* and each I_x is a differentiable submanifold of dimension $n-1$. Otherwise p is *singular in the differentiable structure.*

Example 2.2.3

Let p on $X = R^2$ be defined as follows (see Figure 2.2.2)

(i) $\forall\ x = (a, 0)$ with $a \in R$, $I_x = \{\,(a(1+|t|),\ t);\ t \in R\}$

(ii) For $x = (a, 0)$ and $y = (b, 0)$, $x \succ y$ in p iff $a > b$.

Then p is regular in the (canonical) manifold structure but singular in the (canonical) differentiable structure[7], since I_x is non-differentiable on $x_2 = 0$.

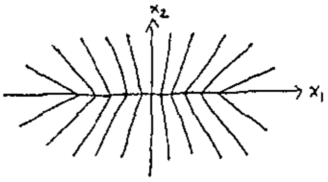

Fig. 2.2.2

It follows immediately that a preference singular in a topological structure is singular in the corresponding manifold structure. And if p is singular in a manifold structure, then it is singular in the related differentiable structures. Finally, we consider the singularities in a measure structure.

Definition 2.2.4 Let X be a locally compact Hausdorff space with a positive Borel measure μ. A preference p on X is *regular in the measure structure* if each I_x has zero measure, i.e. $\mu(I_x) = 0$. Otherwise p is *singular in the measure structure.*

It is evident that a preference singular in a topological structure is also singular in the related measure structure μ. In fact, if $\exists\, x \in X$ with a neighborhood $V_x \subset I_x$, then $\mu(I_x) \geq \mu(V_x) > 0$. Conversely, a preference singular in the measure structure is not necessarily singular in the topological structure, as Example 1.3.6 shows. (In that example, $\mu(I_x) = \mu(Z_\varepsilon) > 0$ for $x \in Z_\varepsilon$ with $1 > \varepsilon > 0$, but each indifference set has no interior point.) However, we have assumed in §1.3 that $\mu(\partial I_x) = 0$, which excludes the case like Example 1.3.6. Under this additional assumption, the converse is clearly true. If p has $\mu(I_x) > 0$ for some $x \in X$, we claim that I_x has nonempty interior points. Suppose the contrary, i.e. $Int\, I_x = \phi$, we see $I_x = (Int\, I_x) \cup (\partial I_x) = \partial I_x$ and $\mu(\partial I_x) = \mu(I_x) > 0$, against the hypothesis.

Chichilnisky [C] shows nonexistence of a continuous rational social welfare function for preferences on an n-dimensional cube I^n of R^n. We now compare her formulation of in [C] with the one defined in this paper. In [C], a preference on I^n is defined by a vector field v on I^n with $|v(x)| = 1$ or 0, such that $v(x)$ is differentiable in x as $|v(x)| = 1$. In other words, she considered preference relations relative to the canonical differentiable structure of I^n. Let the totality

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[7] The differentiable structure in the cannonical one in $\mathbb{R}^2$.

of all such preferences,i.e. *all such vector fields, be denoted by P_c*. She defined a topology $\Im_c$ on P_c by inducing the distance d, where

$$d(v,w) \equiv \max\{|v(x) - w(x)|\,;\, x \in X\}$$

for two v, $w \in P_c$. If $v(x) = 1$ everywhere, it is *called regular*, otherwise it is *singular.*

Given $v \in P_c$, we say v is *integrable* if there corresponds a preference $p_v \in P$, such that at each x with $|v(x)| = 1$, $v(x)$ is the unit normal of the indifferent hypersurface $I_x(p_v)$ pointing toward R'_x and such that at a point x where $|v(x)| = 0$, $I_x(p_v)$ is nondifferentiable. A nonintegrable example is seen in Fig.2.2.3. Let $\tilde{P}_c$ be the set of all integrable preferences v in P_c. Then the definitions of regularity and singularity of a preference coincide with those defined in this paper relative to differentiable structure (Def. 2.2.3). However, the topologies $\Im_c$ in [C] and the topology $\Im$ in this paper restricted to $\tilde{P}_c$ are not consistent.. In the category of differentiability, the example 2.1.2 shows $p_n \to p_o$ in $\Im$, which should be commonly accepted, *while in $\Im_c$ the preference p_n does not converge to p_o*. In fact, for v_o singular and v_n regular, let $v_o(x_o) = 0$ at some $x_o \in X$. According to [C] a metric which measures the "distance" between the two preferences v_o and v_n is given by

$$\begin{aligned} d(v_o, v_n) &= \max\{|v_o(x) - v_n(x)|\,;\, x \in X\} \\ &\geq |v_o(x_o) - v_n(x_o)| = |v_n(x_o)| = 1, \end{aligned}$$

and therefore any singular preference *is isolated from* regular preferences in $\Im_c$.

This flaw of the formulation of [C] arises from defining the topology $\Im_c$ on P_c, which heavily relies on the differentiability machinery. However, the defect is not crucial in proving the nonexistence of a continuous rational social choice function in [C]. Her concept that the singular preferences *are separated* from regular preferences is still agreeable, *if restricted to* the changes of individual preferences under social choice mechanism, *and if not limited to* the category of differentiability but released to the given topological structure.

More precisely, when we consider a continuous social welfare function

$$F : P^N \to P$$

where $P^N \equiv P \times \cdots \times P$ (N copies), we adopt $\Im$ as the topology of individual preferences $P \equiv P_{ind}$ in P^N (instead of the social preferences). Consider the whole P_{ind} with topology $\Im$, and let

$$\begin{aligned} P_r &\equiv \{p \in P;\, p \text{ is regular in } \textit{topological} \text{ strucure}\} \\ P_\Sigma &\equiv \{p \in P;\, p \text{ is singular in } \textit{topological} \text{ strucure}\}. \end{aligned}$$

We will show that P_r and P_Σ are the disjoint open sets, which is called separation property of $\Im$.

Separation Property $P = P_r \cup P_\Sigma$ with $P_r \cap P_\Sigma = \phi$ and P_r, P_Σ are both open in P with stratification topology $\Im$.

Proof. The property is evident except that P_r and P_Σ are open. First, $P_r = H(\phi, \phi)$ is open in $(P, \Im)$. Given $p \in P_\Sigma$, $\Lambda^o(p) \neq \phi$. Choose K a nonempty compact subset of $\Lambda^o(p)$. Then $H(K, X) \subset P_\Sigma$, since each preference q in $H(K, X)$ has the interior $\Lambda^o(q)$ at least containing K and therefore q is singular. This proves P_Σ is open in $(P, \Im)$. ■

The separation property says in particular that relative to the topological structure the singular preferences are isolated from the regular preferences in $\Im$. In other words, the statement that a very slight change of a singular individual preference does not become regular is a fact *only in the topological structure, rather than in the differentiable structure.* Example 2.1.2 has shown in the category of differentiability that a singular preference could suddenly shift to regular ones. However, when topological singularities are considered, the individual preference should stay indifferent on an open set of alternatives, which is "large" in a sense. A slight perturbation therefore would not cause an indifferent set to become "thin" at once. Mathematically, this means that P_Σ in open in $\Im$.

In the proof of her nonexistence theorem, the separation property in differentiable sense is unnecessary. Chichilnisky merely used essentially the separation property in topological sense, since the arguments restricted in the linear preferences suffice to work. A linear preference once singular in differentiable structure, is *also* singular in topological structure.

Another defect of the formulation in [C] is the non-integrability of the vector filed v in P_c. If a vector field in P_c is not integrable, *it can hardly be called* a preference in practice. The formulation of [C] defines the preferences with their first order parts, yet leave the zero order parts undetermined. More precisely, according to [C], the preference order between two points x and y in X may be undecided and even *not* well-defined as seen in the following. Let $v \in P_c$ be given by

$$v(x) = (-x_2,\, x_1) \,/\, \sqrt{x_1^2 + x_2^2}. \text{ (see Figure 2.2.3)}$$

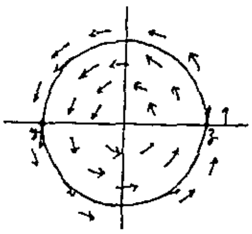

Fig. 2.2.3

Consider $y = (-1, 0)$ and $z = (1, 0)$. If we follow the lower semi circle $\Gamma_- \equiv \{\,(x_1,\, x_2);\ x_1^2 + x_2^2 = 1,\ x_2 \leq 0)\,\}$ from y to z along the preference vector fields $v(x)$ as the tangent vector fields, then z is preferred to y. However when the upper semi circle Γ_+ is followed from z to y, y is preferred to z. Therefore we have $y \prec z$ in v and also $z \prec y$ in v, which leads to a contradiction to the basic definition of a preference order. This simple observation shows that the way defining the set of all preferences on X as in [C] by the set of all preference vector fields *is improper.*

Nevertheless, the points mentioned previously about the formulation of [C] are minor shortcomings. The crucial defect lies in the assumption of the separation property in the social preferences, i.e. Chichilnisky did not differentiate the topologies in P_{ind} and P_{soc} and adopted $\Im_c$ for both the topologies in P_{ind} and in P_{soc}. It is this assumption that makes the nonexistence of a continuous rational social welfare function valid. In the formulation of this paper, we take a different opinion: the social aggregation is more accurate so that some *other topology* should be introduced in social preferences in order to match the empirical world. This viewpoint leads to the existence of a continuous rational welfare function, contrary to Chichilnisky's nonexistence result.

§ 2.3 Stratification approximation theorem

The theorem in this subsection shows the uniform approximation of stratification sets on compact regions for convergent preferences in $\Im$. It justifies figure 2.1.1, which illustrates the stratification convergence, as well as the name "stratification topology" of $\Im$. Let the alternative space X be a locally connected[8], locally compact, T_3-space.

Theorem 1 *(Uniform Approximation Theorem)*

Let $p_n \to p_o$ in $\Im$. Given $\widehat{X}$ compact in X and $x \in \widehat{X}$. If

$$I_x(p_o) \cap \widehat{X} \subset W$$

for some open set W, then $\exists\, N$ and $\exists$ a neighborhood V of x such that $I_v(p_n) \cap \widehat{X} \subset W$, $\forall\, n > N$ and $\forall\, v \in V$. Furthermore, if

$$x \in K \subset Int\, I_x(p_o),$$

for some compact connected set K, then $\exists\, N_1$ and $\exists$ a neighborhood V_1 of x such that $K \subset Int\, I_v(p_n)$, $\forall\, n > N_1$ and $\forall\, v \in V_1$.

Proof. Step 1 Let $W^c \equiv \widehat{X} - W$. Evidently, W^c is a compact set. Let

$$L_1 \equiv W^c \cap Q_x'(p_o),\ L_2 \equiv W^c \cap R_x'(p_o).$$

Since L_1 and L_2 are disjoint and open in W_c, it is easily seen by the finite open covering property of the compact set W^c that both $L_1$and L_2 are compact. Also, $L_1 \prec x \prec L_2$ in p_o. By the argument used in Step 1 of the proof of Prop. 1.2.3, $\exists\, x_1 \in L_1$ and $x_2 \in L_2$ with $L_1 \precsim x_1$ and $x_2 \precsim L_2$ in p_o. By the

---

[8] By X locally connected, we mean $\forall x \in X$, $\forall$ neighborhood U of x the component of U containing x is also a neighborhood of x.

local compactness of X, $\exists$ a neighborhood V of x such that $\overline{V}$ is compact and $\overline{V} \subset R'_{x_1} \cap Q'_{x_2}$. Thus we have

$$L_1 \precsim x_1 \prec \overline{V} \prec x_2 \precsim L_2 \text{ in } p_o.$$

Now $p_o \in G(L_1, \overline{V}) \cap G(\overline{V} \cap L_2)$. By the definition of $p_n \to p_o$ in $\Im$, there exists N such that $p_n \in G(L_1, \overline{V}) \cap G(\overline{V} \cap L_2)$, $\forall n > N$. This means that $\forall n > N$ and $\forall v > V$,

$$L_1 \prec v \prec L_2 \text{ in } p_n.$$

In particular, any point of $\widehat{X}$ indifferent to v in p_n with $n > N$ and $v \in V$ is disjoint from $W^c \equiv L_1 \cup L_2$. Equivalently,

$$I_v(p_n) \cap \widehat{X} \subset W, \; \forall n > N \text{ and } \forall v \in V.$$

Step 2 Given a connected set K with $x \in K \subset Int\, I_x(p_o)$. Since X is a locally connected, locally compact[9], T_3-space, $\exists V$ open in X with its compact closure $\overline{V}$ connected such that

$$x \in K \subset V \subset \overline{V} \subset Int\, I_x(p_o).$$

Denote $\overline{V}$ by $\widetilde{K}$. Evidently, $\forall v \in V$

$$v \in \widetilde{K} \subset Int\, I_v(p_o).$$

We have $p_o \in H(\widetilde{K}, X)$. By Definition 2.1.2, $\exists N_1$ such that $p_n \in H(\widetilde{K}, X)$, $\forall n > N_1$; i.e. $\widetilde{K} \subset \Lambda^o(p_n)$, $\forall n > N_1$. For any fixed $n > N_1$ and *each given* $v \in V$ it is claimed that

$$\widetilde{K} \subset Int\, I_v(p_n).$$

Let $K_o \equiv \widetilde{K} \cap I_v(p_n)$. The indifferent set $I_v(p_n)$ is closed in X. Clearly, $K_o \neq \phi$ since $v \in K_o$ and K_o is closed in $\widetilde{K}$, We then claim that K_o is also open in $\widetilde{K}$. It suffices to show that $K_o \subset Int\, I_v(p_n)$. For any $y \in K_o$, we have $y \in \widetilde{K}$ and $y \sim v$ in p_n due to the definition of K_o. Since $\widetilde{K} \subset \Lambda^o(p_n)$, it holds that $y \in \Lambda^o(p_n)$. By the definition of the singular set $\Lambda^o(p_n)$, $\exists U$ open in $\widehat{X}$ with

$$y \in U \subset I_y(p_n).$$

However, $I_y(p_n) = I_v(p_n)$. Therefore, $y \in Int\, I_v(p_n)$ and hence

$$K_o \subset Int\, I_v(p_n).$$

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[9] Since X is locally compact and of T_3, $\forall x \in K, \exists$ a compact neighborhood u_x of x such that $U_x \subset Int I_x(p_o)$. The component V_x of U_x containing x is a neighborhood, by the local connectedness of X. V_x is connected and closed. A closed set in a compact T_3-space is compact. Therefore V_x is compact. By the finite open covering property of K, it is easy to see that V, being a finite union of V_{x_i} with x_i in K ratiofies the quirements.

In other words, $K_o = \widetilde{K} \cap \mathit{Int}\, I_v(p_n)$. It follows that K_o is open in $\widetilde{K}$. By the connectedness of $\widetilde{K}$, $\widetilde{K}$ contains no proper subset which is both open and closed. We have $\tilde{K} = K_o$, and consequently,

$$K \subset \tilde{K} \subset \mathit{Int}\, I_v(p_n),$$

which completes the proof. ■

The condition that K be connected in Theorem 1 is essential, as seen in the following example.

Example 2.3.1 (see Figure 2.3.1) Let $X = [-2,2] \times [0,1] \subset \mathbb{R}^2$ and let p_n and p_o be defined by the utility functions f_n and f_o respectively, where

$$f_o(x_1,x_2) = \begin{cases} 0; & -2 \leq x_1 \leq -1 \\ x+1; & -1 \leq x_1 \leq 0 \\ -x+1; & 0 \leq x_1 \leq 1 \\ 0; & 1 \leq x_1 \leq 2 \end{cases}$$

$$f_o(x_1,x_2) = \begin{cases} 0; & -2 \leq x_1 \leq -1 \\ x+1; & -1 \leq x_1 \leq 0 \\ (\frac{1}{n}-1)x+1; & 0 \leq x_1 \leq 1 \\ \frac{1}{n}; & 1 \leq x_1 \leq 2 \end{cases}$$

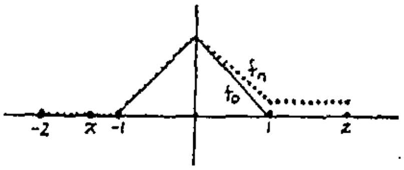

Fig. 2.3.1

Let $x = -\frac{3}{2}$. Clearly, $I_x(p_o) = [-2,-1] \cup [1,2]$, while $I_x(p_n) = [-2,-1]$. We see that at the singular point x, $I_x(p_n)$ dose not tend to $I_x(p_o)$, although $p_n \to p_o$ in $\Im$. However, the singular set $\Lambda(p_n)$ of p_n converges to the singular set $\Lambda(p_o)$ of p_o. In this example, if we let $K = [-\frac{5}{3}, -\frac{4}{3}] \cup [\frac{4}{3}, \frac{5}{3}]$ which is not connected, then the second consequence of Theorem 1 would be false.

Also the following example 2.3.2 shows that the local connectedness of X is essential in Theorem 1.

Example 2.3.2 (See Figure 2.3.2) Let

$$X = \left\{ (t, \sin\frac{1}{t}) \in \mathbb{R}^2; 0 \leq t \leq 1 \right\} \cup \{(0,s); -1 \leq s \leq 1\}.$$

Notice that X is connected but not locally connected. We define continuous functions f and f_n on X as follows. Given $x = (t,s) \in X$,

$$f(x) = \begin{cases} 0 & ; \quad s \in [-\frac{1}{2}, \frac{1}{2}] \\ 2s-1 & ; \quad s \in [\frac{1}{2}, 1] \\ 2s+1 & ; \quad s \in [-1, -\frac{1}{2}] \end{cases}$$

and

$$f_n(x) = \begin{cases} f(x) & ; \quad t \leq t_n \\ g_n(x) & ; \quad t > t_n, \end{cases}$$

Where t_n's are the maximal points of $h(t)$ ordered as $1 > t_1 > t_2 > \cdots > t_n > \cdots \to 0$. and

$$g_n(x) = \begin{cases} 1/n; & s \in [-1/2, 1/2] \\ (2s-1)(1-1/n) + 1/n; & s \in [1/2, 1] \\ (2s+1)(1+1/n) + 1/n; & s \in [-1, -1/2]. \end{cases}$$

Let the preferences p_n and p be defined correspondingly by the utility functions f_n and f. Denote the set $\{(t,s);\, -1/2 \leq s \leq 1/2\}$ by Σ. Then the singular sets

$$\Lambda(p_n) = \Lambda(p_o) = \Sigma.$$

Choose $x = 0$. We have

$$I_x(p_o) = \Sigma \text{ and } I_x(p_n) = \Sigma \cap A_n$$

where $A_n = \{(t,s) \in X;\, t \leq t_n\}$. Evidently, $I_x(p_n)$ does not tend to $I_x(p_o)$ as $n \to \infty$. Let K be the connected set $\{(0,s) \in X;\, s \in [-1/3, 1/3]\}$. Then $x \in K \subset I_x(p_o)$. Given any neighborhood V of x in X, there must exist $v = (\tau, \sigma) \in V \cap \Sigma$ with $\tau > 0$. Choose n_o such that $t_{n_o} < \tau$. Then $\forall\, n > n_o$, $I_v(p_n) = \Sigma \cap A_n^c$ where $A_n^c = \{(t,s) \in X;\, t > t_n\}$. But $K \cap A_n^c = \phi$. Therefore $K \cap I_v(p_n) = \phi$. Then second consequence of Theorem 1 is false in this example, where X is not locally connected.

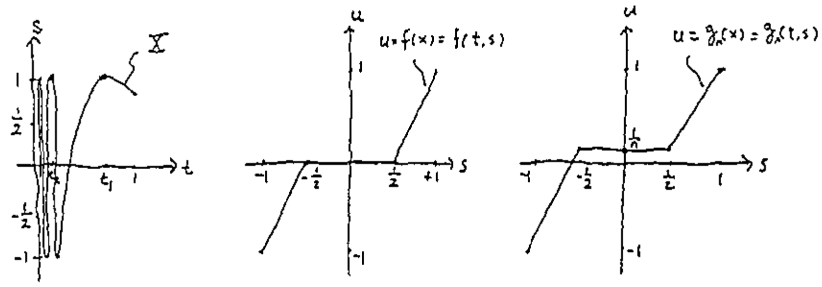

Fig. 2.3.2

§ 3 Continuous rational social welfare functions

§ 3.1 Social aggregation topology $\Im_o$

We assume in § 3 that the alternative space X is a domain with or without boundary in the n-dimensional Euclidean space $\mathbb{R}^n$ or more generally a connected topological manifold[10]. In fact, the results in this section (§ 3) are valid

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[10] By a topological manifold (of dimension n), we mean a topological space in which every point has a neighborhood homeomorphic to $\mathbb{R}^n$. Usually for a topological manifold we assume σ-compactness which means that the whole space is a countable union of compact subsets. With σ-compactness, we can use compact subsets to approximate the given space.

for a general *continuum alternative space*, by which we mean a topological T_3-space which is path-connected, locally connected, σ-compact, locally-compact and has a finite Borel measure μ such that the measure of any nonempty open set is positive. The meanings of the mathematical terms would be explained in the context whenever they are used. However, those who are not familiar with general topology and measure theory may regard the continuum alternative space X as a domain in $\mathbb{R}^n$ on which μ means the n-dimensional volume (or n-Hausdorff measure is technical terminology).

The case of discrete alternative spaces will be treated separately in the next section § 4.

By a preference singular [resp. regular] in § 3, we mean singular [resp. regular] relative to the given topological structure of X, unless otherwise stated.

In § 2, we have introduced for P the stratification topology $\Im$ which separate singular preferences from the regular ones. The following example shows a topology different from $\Im$, but as natural as $\Im$ on the basis that both two topologies respect the topology of X. The new topology does not satisfy the separation property however.

Example 3.1.1 Let the preferences $\overline{p}_n$ and $\overline{p}_o$ be defined on $X = [-1, 1]$ by the utility functions f_n and f_o respectively, where

$$f_n(x) = \begin{cases} -x, & x \in [-1, 0] \\ \frac{x}{n}, & x \in [0, 1] \end{cases}, \quad f_o(x) = \begin{cases} -x, & x \in [-1, 0] \\ 0, & x \in [0, 1] \end{cases}$$

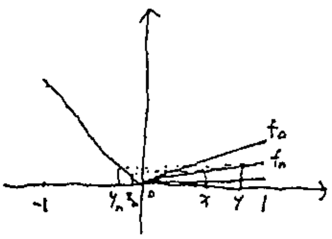


Fig. 3.1.1

We remark that the use of the utility functions f_n and f_o is simply for the expository convenience and is not necessary in the essence. For x and y in $[0, 1]$, $x \sim x_n$ and $y \sim y_n$ where $x_n = -\frac{x}{n}$ and $y_n = -\frac{y}{n}$. As $n \to \infty$, both x_n and y_n tends to 0. Let $\overline{p}_n$ have a limit in P relative to a topology of P respecting the topology of X (see Definition 2.1.3). The two alternatives x and y are forced to be indifferent in the limit preference. In other words, the limit of $\overline{p}_n$ must be $\overline{p}_o$. However, $\overline{p}_n$ diverges in the stratification topology $\Im$. In fact, suppose the contrary with $\overline{p}_n$ converging to a limit q_o in $\Im$. It follows from Corollary 2.1.1 that $x \succsim 0$ in q_o, $\forall x \in [0, 1]$. On the other hand, given $x \in [0, 1]$ we assume that $x \succ 0$ in q_o, then $\exists$ neighborhoods U, V of x, 0 respectively and $\exists N$ such that

$$U \succ V \text{ in } \overline{p}_n, \ \forall n > N.$$

Choosing $w \in (-1,0) \cap V$, we have $x \sim -\frac{x}{n} \prec w$ in $\overline{p}_n$, for $n > -\frac{x}{w}$. This contradicts to $U \succ V$ in $\overline{p}_n$, $\forall n > N$. Hence, $x \sim 0$ in q_o, $\forall x \in [0,1]$, and therefore

$$q_o = \overline{p}_o.$$

But q_o has the singular set $\Lambda(q_o) = \Lambda(\overline{p}_o) = [0,1]$, while each $\overline{p}_n$ is regular. It violates the separation property that $\Im$ satisfies. Therefore, the topology of P with $\lim \overline{p}_n = \overline{p}_o$ should be a new topology different from the stratification topology $\Im$.

Example 3.1.2 Let X be the square $[0,1] \times [0,1]$ and the preferences $\overline{p}_n$ and $\overline{p}_o$ be given by Fig. 3.1.2.

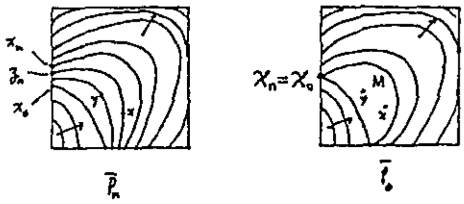


Fig. 3.1.2

Let $x_n \to x_o$ as $n \to \infty$. The curves in the figures indicate indifference sets. For $\overline{p}_o$, the moon-like area M is an indifferent set. Also, $\overline{p}_n$ is regular and $\overline{p}_o$ is singular with $\Lambda(\overline{p}_o) = M$. We see that $\forall x \in M$,

$$x \sim z_n \ \text{ in } \overline{p}_n, \ x_o \precsim z_n \precsim x_n \ \text{ in } \overline{p}_n, \ x_n \to x_o \ \text{ as } n \to \infty.$$

Therefore, if $\overline{p}_n$ tends to a limit preference $\lim_{n\to\infty} \overline{p}_n$ relative to a topology of P which respects the topology of X, we should have

$$x \sim x_o \ \text{ in } \lim_{n\to\infty} \overline{p}_n.$$

Hence,

$$\lim_{n\to\infty} \overline{p}_n = \overline{p}_o.$$

The zero order topology $\Im_o$ defined in Definition 2.1.1 serves as a model of the topologies of P which respects the topology of X and allows the convergence of $\overline{p}_n$ to $\overline{p}_o$ in the above two examples. We note that $\Im_o$ is weaker (or called smaller) than $\Im = \Im_o \cup \Im_1$, i.e. given $p_n, p_o \in P$,

$$p_n \to p_o \ \text{ in } \Im \Rightarrow p_n \to p_o \ \text{ in } \Im_o.$$

In fact, $\Im_o$ is the weakest topology of P that respects the topology of X. This is proved as follows.

Proposition 3.1.2 The zero order topology of P is the weakest (or equivalently, the smallest) topology of P that respects the topology of X.

Proof. It follows from Proposition 2.1.2 that $\Im_o$ respects the topology of X. Given $\Im'$ a topology of P which respects the topology of X. Let $p_n \to p_o$ in $\Im'$, it is claimed that $p_n \to p_o$ in $\Im_o$, i.e. for $p_o \in G(K, L)$ we claim that $\exists\, N$ such that $p_n \in G(K, L)$, $\forall\, n > N$, where K and L are compact sets in X. For any x, y with $x \in K$, $y \in L$, we have $x \prec y$ in p_o. By $\Im'$ respecting the topology of X, $\exists\, U_x, U_y$ neighborhoods of x, y respectively and $\exists\, N_{xy}$ such that $U_x \prec U_y$ in p_n, $\forall\, n > N_{xy}$. Using the finite open covering property of compact sets, we see that $\exists\, N$ with $K \prec L$ in p_n, $\forall\, n > N$. This completes the proof. ■

We search now a topology suitable for P_{soc} of social preferences. Any reasonable topology of P should respects the topology of X. This is certainly a minimum requirement for an acceptable topology of P_{soc}. We claim in the following that the topology of P_{soc} should be $\Im_o$, the smallest topology of P respecting the topology X. The underlying reason is that the social aggregation must be a quantified process, so that the social preferences as the outcomes of the quantified social aggregation should detect the slight difference of a preference over two distinct alternatives even when the difference vanishing into zero. With this observation, a social welfare function F has to be factorized into

$$F : P_{ind}^N \xrightarrow{\widetilde{F}} \mathfrak{F}(X) \xrightarrow{q} P_{soc}$$

where $\mathfrak{F}(X)$ is the space of all real-valued functions on X. For X a topological space, we may restrict the range of $\widetilde{F}$ to be the space $C^o(X)$ of all continuous real-valued functions on X. The factorization means F can be written into the composition $F = q \circ \widetilde{F}$ in

$$F :: P_{ind}^N \xrightarrow{\widetilde{F}} C^o(X) \xrightarrow{q} P_{soc}.$$

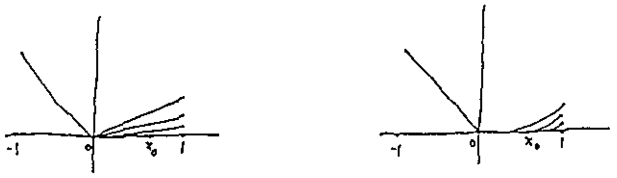
Therefore, the candidate for the topology of P_{soc} of social preferences naturally is the quotient topology inherited from the compact-open topology of $C^o(X)$ through the quotient map q. In the quotient theorem of § 3.2, we prove that the required quotient topology is exactly $\Im_o$, the smallest topology of P respecting the topology of X. This justifies why we adopt $\Im_o$ as the topology of P_{soc} and call it the *social aggregation topology*.

With the social aggregation topology $\Im_o$ of P_{soc}, the preference $\overline{p}_n$ converges to $\overline{p}_o$ in the two examples 3.1.1 and 3.1.2. This just incorporates the reality that for social preferences, an arbitrary small perturbation of $\overline{p}_o$ may result in $\overline{p}_n$ with some large n, or equivalently, the society has the ability sophisticated enough to catch the slight difference of its preference over two distinct alternatives, for instance x and y in the two examples 3.1.1 and 3.1.2, and even detect whether or not the difference vanishing into indifference.

The situation would be different for individual preferences *under social aggregation apparatus* F. Consider Example 3.1.1 with $X = [0, 1]$. Let an individual have slightly perturbed his preference $\overline{p}_o$, in which all the alternatives in

[0, 1] are totally indifferent, to $\overline{p}_n$. Before the social aggregation apparatus F (or in mathematical terminology, a profile of individual preferences is given as an element in the domain of a social welfare function F), an individual was asked to formulate his new preference explicitly and present it to the society *in an accurate form.* We hardly believe that the individual would then present the new preference in the form $\overline{p}_n$ in which each point in [0, 1] is now totally differently preferred from one another. The change would be considered by him as "a big change", rather than merely a slight perturbation. Moreover, individuals are *not sufficiently sensitive* to distinguish the difference between $\overline{p}_o$ and $\overline{p}_n$ with very large n (See Chichilnisky [C], 1982 and Kreps [Kd], 1996). It corresponds to the mathematical fact that interiors have the "huge inertia", as not to jump to become "thin" in an infinitesimal change. This means in Example 1.3.1 that the interior of the singular set (0, 1] does not shrink with small perturbation into a single point, and in Example 1.3.2 the moon-like area not change into an arc.

One may wonder how an interior point x_o of a singular set of an individual preference changes into a regular point continuously, as it presented to a society. The process is to vary its singular points into regular ones gradually from the boundary of the singular set, rather than from the interior. It reveals the nature of the stratification topology $\Im$ adopted for P_{ind}, comparative to that of the social aggregation topology $\Im_o$ adopted for P_{soc}. We use Example 3.1.3 in Fig.3.1.3 to illustrate the difference.



a continuous change in $\Im_o$ a continuous change in $\Im_1$

Fig. 3.1.3

We conclude this subsection § 3.1 by comparing the essence of our formulation from the one by Chichilnisky [C] on selecting the topology for P_{soc} of social preferences. She adopts $\Im_c$ both for P_{ind} and P_{soc} without any distinction, so that for instance in Example 3.1.1 $\overline{p}_n$ does not converge to $\overline{p}_o$ since singular preferences are separate from the regular ones by the separation property of $\Im_c$. This seems to violate the reality that for the social preferences the convergence is reasonable. The key point of the proof of her nonexistence theorem of continuous rational choice is exactly based on the separation property. By the formulation of the social aggregation topology for P_{soc}, we obtain in § 3.4 the existence theorem on the contrary.

§ 3.2 The quotient theorem

Historically, the viewpoint of utilitarianism on the social choice theory was criticized by Lionel Robbins [R] and others in that the preferences had been naïvely quantified by utility functions without scientific basis. The "new welfare economics," emerged from the eschewal of interpersonal comparisons and cardinality of individual utilities, tried to redefine preferences simply by their preference ordering (see Sen [S2]).

Mathematically, instead of using a (utility) function on the alternative space X to define a preference, one could redefine it by an equivalent class of functions preserving the order of the function value. That is, two real valued functions f and g are in the same class, denoted by $f \sim g$, if and only if there exists a transformation $\eta : S \to T$ which is monotonic (i.e. $\eta(u) < \eta(v)$ whenever $u \prec v$), such that

$$f(x) = \eta \circ g(x), \; \forall x \in X,$$

where $S \equiv f(X)$ and $T \equiv g(X)$ are respectively the images of X under f and g in the real number system $\mathbb{R}^1$. In fact, the equivalence relation $\sim$ leads to the fact that

$$f(x) \leq f(y) \text{ iff } g(x) \leq g(y), \; \forall x, y \in X,$$

and given an equivalent class $[f]$ containing f. A preference p can be correspondingly defined by

$$x \preceq y \text{ in } p \text{ iff } f(x) \leq f(y)$$

without multiplicity. Let $\mathfrak{F}(X)$ be the space of all real valued functions defined on X, the historical improvement of social choice theory shifting from utilitarianism to modern formulation of the welfare economics is to consider in place of $\mathfrak{F}(X)$ the quotient space $\mathfrak{F}(X)/_\sim$ (i.e. the set of all equivalent classes of functions on X) as the space of preferences.

When X is a topological space, it is natural to consider the space $C^o(X)$ of all real valued *continuous* functions on X to replace the function space $\mathfrak{F}(X)$, because the preferences on X should now respect the topology of X. It is evident that P and $C^o(X)/_\sim$ are identical as sets (see also the proof of Theorem 3). In the definition of the equivalence relation $\sim$ between two continuous functions f and g, we have to require the transformation η be a homeomorphism, besides the monotonicity.

Consider the quotient map

$$\pi : C^o(X) \to C^o(X)/_\sim$$

defined by $\pi(f) = [f]$, $\forall f \in C^o(X)$. A natural topology of $C^o(X)$ is the compact open topology which is defined by the subbase

$$\widetilde{B}_o \equiv \{\mathfrak{F}(K, U); \; K \text{ is compact in } X, \; U \text{ is open in } \mathbb{R}^1\}$$

where

$$\mathfrak{F}(k, U) = \{f \in C^o(X); \; f \text{ maps } K \text{ to } U\}.$$

The compact open topology corresponds to the uniform convergence on compact sets for functions in $C^o(X)$. We take for $C^o(X)/_\sim$ the quotient topology of the compact open topology of $C^o(X)$ by the quotient map π. It means that a set G is open is $C^o(X)/_\sim$ if and only if the preimage $\pi^{-1}(G)$ is open in $C^o(X)$.

We now show the quotient theorem that the quotient topology of $C^o(X)/_\sim$ is exactly the zero order topology $\Im_o$ of P.

Theorem 1 (Quotient theorem) $(P, \Im_o)$ is topologically equivalent to $C^o(X)/_\sim$; in other words, these two spaces are homeomorphic.

Proof. Let $\varphi : C^o(X)/_\sim \to P$ be defined by

$$x \preceq y \ \text{ in } p \ \text{ iff } f(x) \leq f(y) \tag{3.2.1}$$

where $[f] \in C^o(X)/_\sim$ and $p = \varphi([f])$. Clearly, φ is well defined.

Step 1 First we claim the φ is an injection. For f, $g \in C^o(X)$ such that $\varphi([f]) = \varphi([g]) = p$. Consider $\eta : f(X) \to g(X)$ defined by

$$\eta(u) = g(x_u),\ \forall u \in f(X).$$

where $x_u \in X$ with $f(x_u) = u$. Note that η is well defined, since for x_u and x_u' in X with $f(x_u) = f(x_u')$, we have $x_u \sim x_u'$ in p and therefore

$$g(x_u) = g(x_u').$$

To claim that $[f] = [g]$, we have to show that the transformation η is a homeomorphism[11]. Let u_o and u_n, $n = 1, 2, \cdots$, be in $f(X)$ such that

$$u_n \to u_o \text{ in } \mathbb{R}^1.$$

We will see that $\eta(u_n) \to \eta(u_o)$ in $\mathbb{R}^1$. It is claimed that given any monotonic subsequence $\{u_{n_k}\}$ of $\{u_n\}$ with

$$u_{n_k} \to u_o \text{ in } \mathbb{R}^1, \text{ as } k \to \infty,$$

there exists $\{x_{n_k}\}$ in X such that

$$x_{n_k} \to x_o \text{ in } X, \text{ as } k \to \infty,$$

and $f(x_{n_k}) = u_{n_k}$, $f(x_o) = u_o$. Assume without loss of generality that u_{n_k} is increasing. Choose x_{n_1} such that $f(x_{n_1}) = u_{n_1}$ and choose $\bar{x}$ such that $f(\bar{x}) = u_o$. Since X is path connected, there exists a continuous map $\gamma : [0,1] \to X$ connecting x_{n_1} to $\bar{x}$ with $\gamma(0) = x_{n_1}$and $\gamma(1) = \bar{x}$. Let x_o be a point on γ such that f first attains the value u_o, i.e. $x_o = \gamma(t_o)$ with

$$t_o = \min\{t;\ f(\gamma(t)) = u_o\}.$$

---

[11] If X is assumed compact, then the proof is trivial, since continuous injection $f_* : X/_\sim \to \mathbb{R}^1$ is an open map. In that case we do not need even connectedness. However, for X non-compact, we use the path-connectedness of X.

Then the value of f on γ approximates continuously to $f(\gamma(t_o))$ from below as t increasingly tends to t_o. Therefore $\exists\, \gamma(t_{n_k})$ convergent to $\gamma(t_o)$ as $k \to \infty$ with $u_{n_k} = f(\gamma(t_{n_k}))$. Let $x_{n_k} = \gamma(t_{n_k})$. Then $x_{n_k} \to x_o$ in X as $k \to \infty$ and $f(x_{n_k}) = u_{n_k}$, as required. Hence, we see that

$$\eta(u_{n_k}) = g(x_{n_k}) \to g(x_o) = \eta(u_o), \text{ as } k \to \infty,$$

by the continuity of g. This implies that $\eta(u_n) \to \eta(u_o)$ as $n \to \infty$, since otherwise $\exists$ a neighborhood U of $\eta(u_o)$ and $\exists$ subsequence $\{u_{n_j}\}$ such that $\eta(u_{n_j}) \in \mathbb{R}^1 - U$. However we can find a monotonic subsequence $\{u'_k\}$ of $\{u_{n_j}\}$, which has $\eta(u'_k) \to \eta(u_o)$ since $u'_k \to u_o$. It contradicts to $\eta(u'_k) \in \mathbb{R}^1 - U$ with $\eta(u_o) \in U$. This show that η is continuous. The exactly same reason yields that η^{-1} is continuous. Therefore, η is a homeomorphism and hence $f \sim g$, $[f] = [g]$.

Step 2 We now claim the *surjectivity* of φ. Given p in P, let

$$f(x) \equiv \mu\left(Q'_x(p) - \Lambda(p)\right). \tag{3.2.2}$$

First we show that $f \in C^o(X)$. To claim f is continuous on X, it suffices to show that for any monotonic sequence $\{x_n\}$ convergent to x in X, we have $f(x_n) \to f(x)$ in $\mathbb{R}^1$. By $\{x_n\}$ monotonic, we mean $\{x_n\}$ decreasing, i.e. $x_n \succeq x_{n+1}, \forall n$, or $\{x_n\}$ increasing, i.e. $x_n \preceq x_{n+1}, \forall n$. For $\{x_n\}$ decreasing, we see that

$$\bigcap_{n=1}^{\infty} Q'_{x_n} = Q_x.$$

In fact, for $y \in \bigcap_{n=1}^{\infty} Q'_{x_n}$, $y \prec x_n$, $\forall n$. In virtue of Proposition 1.2.5 and $x_n \to x$ in X, we have that $y \preceq x$, and therefore $y \in Q_x$. The converse is obvious. Now we have

$$\begin{aligned}
\lim f(x_n) &= \lim \mu\left(Q'_{x_n} - \Lambda\right) = \lim\left(\mu\left(Q'_{x_n}\right) - \mu\left(Q'_{x_n} \cap \Lambda\right)\right) \\
&= \mu\left(\bigcap^{\infty} Q'_{x_n}\right) - \mu\left(\bigcap^{\infty} (Q'_{x_n} \cap \Lambda)\right) \\
&= \mu(Q_x) - \mu\left((\bigcap^{\infty} Q'_{x_n}) \cap \Lambda)\right) \\
&= \mu\left(Q'_x \cup I_x\right) - \mu\left((Q'_{x_n} \cup I_x) \cap \Lambda\right) \\
&= \mu\left(Q'_x\right) + \mu(I_x) - \mu\left((Q'_{x_n} \cap \Lambda) \cup I_x\right) \\
&= \mu\left(Q'_x\right) - \mu\left(Q'_{x_n} \cap \Lambda\right) \\
&= \mu\left(Q'_x - \Lambda\right) = f(x).
\end{aligned}$$

Similarly, we have $f(x_n) \to f(x)$ for increasing $\{x_n\}$, noting that

$$\bigcup_{n=1}^{\infty} Q'_{x_n} = Q'_x.$$

We claim $p = \varphi([f])$. It is clear by (3.2.2) that $x \succsim y$ in p implies $f(x) \geq f(y)$. For the converse, suppose $x \prec y$ in p, we would have $f(y) - f(x) = \mu\left(Q'_y(p) - Q'_x(p) - \Lambda(p)\right)$, in which $Q'_y(p) - Q'_x(p) - \Lambda(p)$ is a non-empty open set. Hence $f(y) - f(x) > 0$, contradicting to the assumption $f(x) \geq f(y)$.

Step 3 We claim that φ is a homeomorphism. We show that φ is continuous, i.e. $\varphi^{-1}(G(K,L))$ for any $G(K,L)$ in $\mathcal{B}_o$ is open in $C^o(X)/_\sim$. Let the preimage $(\varphi \circ \pi)^{-1}(G(K,L))$ be denoted by A. Then

$$A = \{f \in C^o(X);\ f(x) < f(y),\ \forall x \in K,\ y \in L\}$$

and

$$A = \bigcup_{r \in \mathbb{R}^1} \left(\mathfrak{F}(K, R_r^-) \cap \mathfrak{F}(L, R_r^+)\right),$$

where $R_r^- \equiv \left\{s \in \mathbb{R}^1;\ s < r\right\}$ and $R_r^+ \equiv \left\{s \in \mathbb{R}^1;\ s > r\right\}$. Thus, A is open in $C^o(X)$. By the definition of the quotient topology on $C^o(X)/_\sim$, $\varphi^{-1}(G(K,L))$ is open in $C^o(X)/_\sim$.

Step 4 Finally, we claim that φ^{-1} is also continuous, which is the nontrivial part of the proof. This is equivalent to show that

$$p_n \to p \text{ in } \Im_o \Rightarrow p_n \to p \text{ in } C^o(X)/_\sim .$$

It suffices to prove that given f with $p = \varphi([f])$ and given any finite number of compact set $K_1, \ldots, K_h$ in X and open sets $U_1, \cdots, U_h$ in $\mathbb{R}^1$ with $f \in \bigcap_{i=1}^{h} \mathfrak{F}(K_i, U_i)$, there exists N and f_n with $p = \varphi([f_n])$ such that $f_n \in \bigcap_{i=1}^{h} \mathfrak{F}(K_i, U_i)$, $\forall\, n > N$. Let $\widehat{X}$ be a compact subset of X which contains $\bigcup_{i=1}^{h} K_i$ and let W be a closed finite interval of $\mathbb{R}^1$ which contains $\bigcup_{i=1}^{h} f(K_i)$. Denote

$$\varepsilon \equiv \min_{i=1,\ldots,h} dist\left(f(K_i),\ \mathbb{R}^1 - U_i\right) \tag{3.2.3}$$

where $dist\,(A,B)$ means the distance between two sets A and B in $\mathbb{R}^1$. We subdivide W into $2m+1$ intervals $[a_0, a_1]$, $[a_1, a_2]$, $\cdots$, $[a_{2m}, a_{2m+1}]$, each having length less than $\varepsilon/3$. Let I_j and I'_j denote the intervals

$$\begin{aligned} I_j &\equiv [a_{2j}, a_{2j+1}],\ j = 0, 1, \cdots, m, \\ \text{and } I'_j &\equiv [a_{2j+1}, a_{2j+2}],\ j = 0, 1, \cdots, m-1, \end{aligned}$$

and let g_n be an arbitrarily chosen function in $C^o(X)$ such that $\varphi([g_n]) = p_n$, by the way given in Step 2, for example. We adjust the value of g_n by defining $\zeta_n : \mathbb{R}^1 \to \mathbb{R}^1$ to construct the required $f_n \equiv \zeta_n \circ g_o$ with $f_n \sim g_n$. Let

$$X_j \equiv f^{-1}(I_j) \cap \widehat{X},\ X'_j \equiv f^{-1}(I'_j) \cap \widehat{X}.$$

As $X_o, \cdots, X_m$ are disjoint compact sets in $\widehat{X}$ and $X_j \prec X_l$ in p, $\forall\, j, l$ with $j < l$, there exists N_1 such that $X_j \prec X_l$ in p_n for $n > N_1$ and $j < l$. We have that for $n > N_1$,

$$g_n(x_j) < g_n(x_l),$$

$\forall\, x_j \in X_j,\ x_l \in X_l$ and $j < l$. Hence, we can freely define a monotonic homeomorphism $\zeta_n : \mathbb{R}^1 \to \mathbb{R}^1$ such that ζ_n maps $g_n(X_j)$ homeomorphically onto I_j with $n > N_1$. This yields that

$$f_n(X_j) \subset f(X_j) = [a_{2j}, a_{2j+1}],\ \forall\, n > N_1 \text{ and } \forall\, j = 0, 1, \cdots, m. \tag{3.2.4}$$

As for f_n on X_j', we let

$$b_j = \frac{(a_{2j} + a_{2j+1})}{2}$$

and let $Z_j \equiv f^{-1}(\{b_j\}) \cap \widehat{X}$. It follows that

$$b_j < f(x_j') < b_{j+1},\ x_j' \in X_l'.$$

Again $Z_o, \cdots, Z_m$ and $X_o', \cdots, X_{m-1}'$ are disjoint compact sets in X. By the similar reasoning, there exists N_2 such that for $n > N_2$,

$$g_n(z_j) < g_n(x_j') < g_n(z_{j+1}),\ \forall\, z_j \in Z_j \text{ and } x_j' \in X_j'.$$

Therefore, for $z_j \in Z_j$ and $x_j' \in X_j'$, we have

$$a_{2j} \leq f_n(z_j) = \zeta_n \circ g_n(z_j) < \zeta_n \circ g_n(x_j') < \zeta_n \circ g_n(z_{j+1}) = f_n(z_{j+1}) \leq a_{2j+3},$$

i.e.

$$a_{2j} < f_n(x_j') < a_{2j+3},$$

where $n > N \equiv \max\{N_1, N_2\}$ and the formula (3.2.4) is used to claim the first and the last inequalities. Hence,

$$f_n(X_j') \subset (a_{2j}, a_{2j+3}),\ \forall\, n > N. \tag{3.2.5}$$

Combining (3.2.4) and (3.2.5), it holds that

$$|f_n(x) - f(x)| < \varepsilon,$$

for $x \in \widehat{X}$, $n > N$. But $\widehat{X} \supset \bigcup\limits_{i=1}^{h} K_i$. By (3.2.3), we have proved that

$$f_n \in \mathsf{F}(K_i, U_i),\ \forall\, n > N,$$

as required. ■

§ 3.3 Graph topology $\Im_G$

The topologies $\Im_o$ and $\Im_1$ are defined to describe the features of preference convergence when the zero order and first order behaviors of preferences are considered respectively. We have called $\Im_o$ the social aggregation topology and defined the stratification topology $\Im$ by $\Im_o \cup \Im_1$. Evidently, $\Im \underset{\neq}{\supset} \Im_o$, hence $\Im$ is stronger than $\Im_o$; in other words,

$$p_n \rightarrow p_o \text{ in } \Im \Rightarrow p_n \rightarrow p_o \text{ in } \Im_o.$$

The converse is not true in general, as Example 2.1.5 and Example 3.1.1 illustrated. However, we have

Proposition 3.3.1 If p_o and p_n with large n are regular, Then

$$p_n \rightarrow p_o \text{ in } \Im \Leftarrow p_n \rightarrow p_o \text{ in } \Im_o.$$

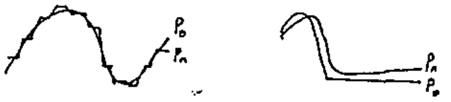
Proof. Given $p_o \in H(K, W)$. Since p_o is regular, $K \subset \Lambda_o(p_o) = \phi$; therefore $K = \phi$. Hence, $H(K, W) = H(\phi, W)$. Also by p_n regular for large n, $p_n \in H(\phi, W)$, for $n >$some N. In fact, $\phi \in W$ is a trivial condition. Hence $H(\phi, W) = P_r$ where P_r denotes the set of all regular preferences. We have $p_n \rightarrow p_o$ in $\Im_1$. But it is given that $p_n \rightarrow p_o$ in $\Im_o$. Therefore, $p_n \rightarrow p_o$ in $\Im = \Im_o \cup \Im_1$. ∎

We consider two prototypes of preferences (p_n, p_o) for which $p_n \rightarrow p_o$in $\Im_o$, but not in $\Im$:

Type A p_o is regular, but $\exists$subsequence $\{p_n\}$ with each p_{n_i} singular;

Type B p_n is regular for large n, but p_o is singular.

They are illustrated in Figure 3.3.1



Type.A Type B

Figure 3.3.1

Example 2.1.5 and Example 3.1.1 represent the two types A and B correspondingly.

Given $p_n \rightarrow p_o$ in $\Im_o$, Proposition 2.1.1 describes a necessary condition of $x \succ y$ in p_o by

(J): $\exists$neighborhoods U, V of x, y respectively and $\exists\, N$ such that

$$U \succ V \text{ in } p_n, \forall n > N.$$

In general, the sufficiency is not true, as shown in the following examples.

Example 3.3.1 Let p_n and p_o be defined by the utility functions illustrated in Figure 3.3.2, where $x \sim x_n$ and $y \sim y_n$ in p_n and $x_n \to c$, $y_n \to c$ as $n \to \infty$. Evidently, $p_n \to p_o$ both in $\Im_o$ and $\Im$. By Corollary 2.1.1, $x \sim c \sim y$ in p_o. But (x, y) satisfies (J).

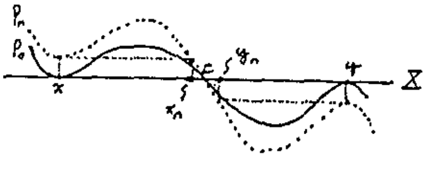
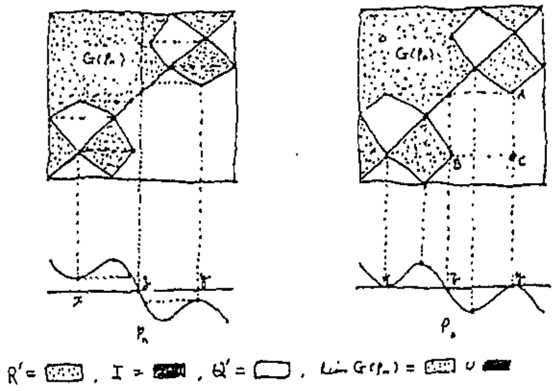


Fig. 3.3.2

The condition (J) is closely related to a third topology of P, which we call the *graph topology* $\Im_G$ defined by the convergence of graphs of preferences is $X \times X$. More precisely, letting the graph $G(p)$ of a preference p be the set $\{(x,y) \in X \times X;\ x \succsim y \text{ in } p\}$, *we define* $p_n \to p_o$ in $\Im_G$ if the graph $G(p_o)$ of p_o is identical with $\lim G(p_n)$ where $\lim G(p_n) \equiv R' \cup I$ with $R' \equiv \{(x,y) \text{ in } X \times X;\ (x,y) \text{ satisties (J)}\}$, $Q' \equiv \{(x,y) \text{ in } X \times X;\ (y,x) \in R'\}$ and $I \equiv X \times X - R' \cup Q'$. The convergence $p_n \to p_o$ in $\Im_G$ is also denoted by "$G(p_n) \to G(p_o)$ as subsets of $X \times X$". We see in the following that even for Example 3.3.1 where $p_n \to p_o$ both in $\Im_o$ and $\Im$, p_n diverges in $\Im_G$, since $\lim G(p_n)$ does not correspond to a preference order satisfying even the transitivity. In fact, Figure 3.3.3 shows that $A = (y, z)$ and $B = (z, x)$ belong to $\lim G(p_n)$, yet $C = (y, x)$ does not. In this sense, we do not consider the graph topology $\Im_G$ a candidate for the topology of P in our approach.



$$G(p_o) = \lim G(p_n) \cup \{C\} \neq \lim G(p_n)$$
Fig 3.3.3

Instead of considering $G(p_n) \to G(p_o)$ pointwisely as in the above treatment, one may choose $G(p_n) \to G(p_o)$ *in measure*, so that an indifference set of measure zero is negligible. More precisely, we say "$G(p_n) \to G(p_o)$ in measure" if $\mu^2(G(p_n)\Delta G(p_o)) \to 0$ as $n \to \infty$, where μ^2 is the product measure $\mu \times \mu$ of the measure μ defined in X, and $A\Delta B$ denotes the different set $(A-B)\cup(B-A)$.

It is observed that for the preferences (p_n, p_o) of Type A, we have $G(p_n) \to G(p_o)$ in measure. However, this is not the case for $(p_n, p_o)$of Type B. The Figures 3.3.4 and 3.3.5, corresponding to Example 2.1.5 and Example 3.1.1 would illustrate these observations.

Type A: (Example 2.1.5, $n = 3$)

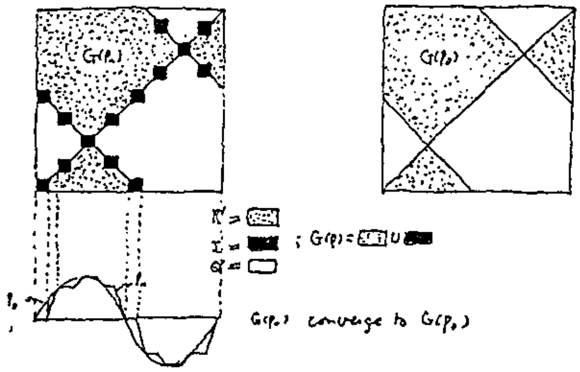


Fig. 3.3.4

Type B: (Example 3.1.1)

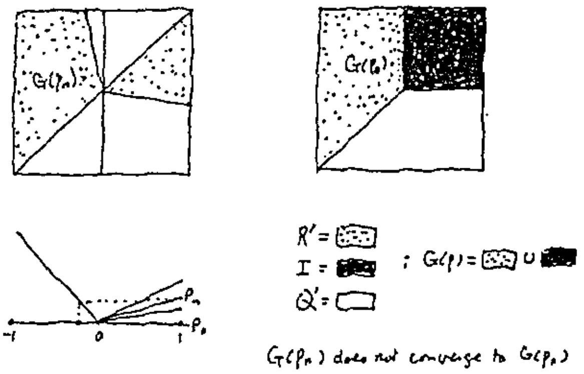


Fig.3.3.5

With this observation, the graph topology, either in the sense of measure or in pointwise convergence, is not proper to be used for the preference space considered in the social choice theory.

§ 3.4 Lifting Theorem

Let X be a continuum alternative space. We show in the following lifting theorem that there exists a continuous map ξ which assigns to each preference $p \in (P, \Im)$ a (utility) function $f = \xi(p) \in C^o(X)$, where $(P, \Im)$ means the space

P of preferences equipped with the stratification topology and $C^o(X)$ equipped with compact-open topology.. Here the topological singular set $\Lambda(p)$ plays a crucial role in making ξ continuous.. Recall that μ is the Borel measure of the continuum alternative space. In particular when X is a finite domain in $\mathbb{R}^n$, $\mu(S)$ is the n dimensional volume of S, for an open set S in X.

Theorem 2 (Lifting Theorem) Let X be a continuum alternative space and let

$$\xi : (P, \Im) \to C^o(X),$$

where $\xi(p) = f$, $\forall p \in P$, is defined by $f(x) = \mu(Q_x^{'}(p) - \Lambda(p))$. Then ξ is a continuous map.

Proof. As the compact-open topology of $C^o(X)$ is concerned, we have to show $f_n(x)$ tends to $f_o(x)$ uniformly on any given compact subset $\widehat{X}$ of X, if $p_n \to p_o$ in $\Im$ where

$$\begin{aligned} f_n(x) &\equiv \mu\left(Q_x'(p_n) - \Lambda(p_n)\right) \\ f_o(x) &\equiv \mu\left(Q_x'(p_o) - \Lambda(p_o)\right). \end{aligned}$$

To avoid the complication of notation, we denote $\widehat{X}$ by X, assuming X compact. Denote $D_v(p) \equiv Q_v^{'}(p) - \Lambda(p)$. Given $\varepsilon > 0$, we first prove that $\forall x \in X$, $\exists N$ and $\exists$ a neighborhood V of x such that

$$|\mu(D_v(p_n)) - \mu(D_v(p_o))| < \varepsilon, \forall n > N \text{ and } \forall v \in V$$

in the following steps.

Step 1 Given $x \in X$, choose a compact set $K^{'} \subset D_x(p_o)$, $K \subset Int\, I_x(p_o)$ and $K^{''} \subset R_x^{'}(p_o)$ such that

$$\mu(D_x(p_o) - K^{'}) < \varepsilon/4, \ \mu(Int\, I_x(p_o) - K) < \varepsilon/4, \text{ and } \mu(R^{'}(p_o) - K^{''}) < \varepsilon/4.$$

Let

$$\Lambda_x^o(p_o) = \{z \in \Lambda^o(p_o);\ z \prec x \text{ in } p_o\}$$

Since X is σ-compact, $\Lambda_x^o(p_o)$ has at most countable components $I_i \equiv Int\, I_{z_i}(p_o)$, $z_i \in \Lambda_x^o(p_o)$. As $\Lambda_x(p_o) = \bigcup_{i=1}^{\infty} I_{z_i}(p_o)$ is assumed closed in $Q_x'(p_o)$, $\partial(\Lambda_x(p_o)) = \bigcup_{i=1}^{\infty} \partial I_{z_i}(p_o)$. Hence

$$\begin{aligned} \mu(\Lambda_x(p_o)) &= \mu(\partial\Lambda_x(p_o) \cup \Lambda_x^o(p_o)) = \mu(\partial\Lambda_x(p_o)) + \mu(\Lambda_x^o(p_o)) \\ &= \mu\left(\bigcup_{i=1}^{\infty} \partial I_{z_i}(p_o)\right) + \mu\left(\bigcup_{i=1}^{\infty} I_i\right) \\ &= \bigcup_{i=1}^{\infty} \mu(\partial I_{z_i}(p_o)) + \mu\left(\bigcup_{i=1}^{\infty} I_i\right) = \mu\left(\bigcup_{i=1}^{\infty} I_i\right) \end{aligned}$$

where the last equality follows from the assumption that $\mu(\partial(I_x(p)) = 0, \forall x \in X, \forall p \in P$ given in § 1.3. Choose an integer M with

$$\mu(\Lambda_x(p_o)) = \mu(\bigcup_{i=1}^{\infty} I_i) = \mu(\bigcup_{i=1}^{M} I_i) + \mu(\bigcup_{i=M+1}^{\infty} I_i),$$

such that the last term $< \frac{\varepsilon}{8}$. Also choose $K_i \subset I_i$ such that

$$\sum_{i=1}^{M} \mu(I_i - K_i) < \varepsilon/8,$$

We see that

$$\mu(\Lambda_x(p_o) - \bigcup_{i=1}^{M} K_i) < \varepsilon/8 + \varepsilon/8 = \varepsilon/4.$$

Let $U \equiv X - (\bigcup_{i=1}^{M} K_i) \cup K \cup K''$. Then we have

$$K' \subset D_x(p_o) \subset U$$

with

$$\begin{aligned} \mu(U - K') &< \mu(\Lambda_x \cup D_x \cup I_x \cup R_x' - (\bigcup_{i=1}^{M} K_i) \cup K' \cup K \cup K'') \\ &= \mu(\Lambda_x - \bigcup_{i=1}^{M} K_i) + \mu(D_x - K') + \mu(I_x - K) + \mu(R_x' - K'') \\ &< \varepsilon. \end{aligned}$$

Step 2 Let K' be divided into:

$$\begin{aligned} K_o' &\equiv \{y \in K';\, y \prec K_1 \text{ in } p_o\} \\ K_i' &\equiv \{y \in K';\, K_i \prec y \prec K_{i+1} \text{ in } p_o\}, \forall i = 1, 2, ..., M-1 \\ K_M' &\equiv \{y \in K';\, K_M \prec y \prec x \text{ in } p_o\}. \end{aligned}$$

In general, for a compact set $K = K_a \cup K_b \subset X$ with K_a and K_b separates form each other; i.e. $\exists$ disjoint open sets U and V of X such that $K_a \subset U$ and $K_b \subset V$, we have K_a and K_b compact, by applying the finite open covering property. Consequently, each K_i', with $i = 0, 1, ..., M$, is compact, since each pair K_i' and $K' - K_i'$ separates form each other by their definition. Choose a compact neighborhood $\overline{V}$ of x such that

$$K' \prec \overline{V} \prec K'' \text{ in } p_o \text{ and } K_i \prec \overline{V} \text{ in } p_o, \forall i = 0, 1, ..., M.$$

Then

$$p_o \in G(K_o', K_1, K_1', ..., K_M, K_M', K \cup \overline{V}, K'').$$ [12]

By $p_n \to p_o$ in $\Im_o$, $\exists\, N_1$ such that

$$p_n \in G(K_o^{'}, K_1, K_1^{'}, \cdots, K_M, K_M^{'}, K \cup \overline{V}, K^{''}),\ \forall\, n > N_1. \qquad (3.4.1)$$

On the other hand, letting $\overset{M}{\cup} K_i \equiv K_o \cup K_1 \cup \cdots \cup K_M$, we have

$$(\overset{M}{\cup} K_i) \cup K \subset \Lambda^o(p_o) \subset \Lambda(p_o) \subset X - K^{'} \cup K^{''}.$$

i.e. $p_o \in H((\overset{M}{\cup} K_i) \cup K, X - K^{'} \cup K^{''})$. Also by $p_n \to p_o$ in $\Im_1$, $\exists\, N_2$ such that

$$p_n \in H((\overset{M}{\cup} K_i) \cup K,\ X - K^{'} \cup K^{''}). \qquad (3.4.2)$$

Step 3 It is claimed that $\exists\, N$ such that $\forall\, n > N$ and $\forall\, v \in \overline{V}$,

$$K^{'} \subset Q_v^{'}(p_n) - \Lambda(p_n) \subset U.$$

where U is defined in Step 1. Let $N \equiv \max\{N_1, N_2\}$. By the formula (3.4.2)

$$(\overset{M}{\cup} K_i) \cup K \subset \Lambda(p_n) \subset X - K^{'} \cup K^{''}, \forall\, n > N. \qquad (3.4.3)$$

Hence, $Q_v^{'}(p_n) - \Lambda(p_n) \subset X - (\overset{M}{\cup} K_i) \cup K,\ \forall\, n > N$. By the formula (3.4.1), $v \prec K^{''}$ in p_n, $\forall\, n > N$. Thus, $Q_v^{'}(p_n) \cap K^{''}$ is empty. Therefore, $\forall\, n > N$, we have

$$Q_v^{'}(p_n) - \Lambda(p_n) \subset X - (\overset{M}{\cup} K_i) \cup K \cup K^{''} \equiv U.$$

On the other hand, the second inclusion in the formula (3.4.3) implies $K^{'} \cap \Lambda(p_n) = \phi$. But again by the formula (3.4.1), $K^{'} \prec v$ in p_n, $\forall\, n > N$. Hence, $K^{'} \subset Q_v^{'}(p_n)$ and therefore,

$$K^{'} \subset Q_v^{'}(p_n) - \Lambda(p_n) \subset U,\ \forall\, n > N.$$

Step 4 By the previous steps, we conclude that given $x \in X$, $\exists\, N_x$ and a neighborhood $\overline{V}_x$ of x such that $K^{'} \subset D_v(p_n) \subset U, \forall\, n > N_x$. In particular, $K^{'} \subset D_v(p_o) \subset U$, since by choosing $p_1 = p_2 = \cdots = p_n = \cdots = p_o$, we certainly have $p_n \to p_o$ in $\Im$. Therefore,

$$D_v(p_n) \Delta D_v(p_o) \subset U - K^{'}$$

and

$$\begin{aligned} |f_n(v) - f(v)| &= |\mu(D_v(p_n)) - \mu(D_v(p_o))| \\ &\leq |\mu(D_v(p_n) \Delta D_v(p_o))| \leq \mu(U - K^{'}) < \varepsilon. \end{aligned}$$

Finally, covering X by a finite number of such V_x's, i.e. $X \subset \bigcup_{j=1}^{h} V_{x_j}$. Choose $N_o \equiv \max\{N_{x_j};\ j = 1, 2, ..., h\}$, we have

$$|f_n(x) - f(x)| < \varepsilon,$$

$\forall\, x \in X$ and $\forall\, n > N_o$, where N_o is independent of x. This completes the proof. ∎

§ 3.5 Existence of continuous rational social welfare functions

We are now ready to prove the existence theorem of rational social welfare functions in our framework.

Theorem 3 Let X be a domain of $\mathbb{R}^n$ or a general continuum alternative space, and let P be the totality of preferences on X. Then there exist social welfare functions

$$F_\nu : P^N \to P, \quad \nu = 1, 2, \cdots, N,$$

which are continuous, anonymous, and satisfy the strong Pareto condition. On the domain of F_ν, P of individual preferences on X is equipped with the first order topology (also called the stratification topology) $\Im$, and on the range of F_ν, P of social preference is equipped with the zero order topology (also called the social aggregation topology) $\Im_o$. . The precise definitions of the anonymity (AN) and the strong Pareto condition (PA) are given in the proof.

Proof. *Step 1* Let $P_{ind} \equiv (P, \Im)$ denote the set P with the topology $\Im$ and $P_{soc} \equiv (P, \Im_o)$ denote the set P with the topology $\Im_o$. Consider the following sequence of maps:

$$P_{ind}^N \xrightarrow{\xi_*^N} (C^o(X))^N \xrightarrow{G_\nu} C^o(X) \xrightarrow{\pi} C^o(X)/_\sim \xrightarrow{\varphi} P_{soc}, \tag{M}$$

where P_{ind}^N is the topology product space $P_{ind} \times P_{ind} \times \cdots \times P_{ind}$ with N copies of P_{ind}. The fact that the composition $F_\nu \equiv \varphi \circ \pi \circ G_\nu \circ \xi_*^N$ is exactly the desired rational social welfare function for each $\nu = 1, 2, \cdots, N$ will be proved step by step as follows..

Step 2 Let $C_+^o(X)$ be the subspace $\{g \in C^o(X);\ g\,(x) > 0,\ \forall\, x \in X\} \subset C^o(X)$. Define $\xi_* : P_{ind} \to C_+^o(X)$ by

$$f = \xi_*(p), \ \ \forall\, p \in P_s$$

with[13] $f(x) = \mu\left(Q_x'(p) - \Lambda(p)\right) + 1 > 0$, $\forall\, x \in X$. By Step 2 of the proof of the quotient theorem (Theorem 1), we have shown that $f \in C_+^o(X)$. Using the lifting theorem (Theorem 2), we see that ξ_* is a continuous map on P_{ind}. Therefore, ξ_*^N is continuous on P_{ind}^N.

Step 3 Define $G_\nu : \left(C_+^o(X)\right)^N \to C^o(X)$ by

$$G_\nu(f_1, f_2, \cdots, f_N) = \left(\binom{N}{\nu}^{-1} \sum f_{i_1} f_{i_2} \cdots f_{i_\nu}\right)^{1/\nu}$$

for each $\nu \in \{1, 2, \cdots, N\}$ and $(f_1, f_2, \cdots, f_N) \in \left(C_+^o(X)\right)^N$, where the summation $\sum$ ranges over all the possible combinations $I_\nu = \{i_1, \cdots, i_\nu\}$ of ν

integers contained in $\{1, 2, \cdots, N\}$. The map G_ν is clearly well-defined and continuous when restricted to $\left(C_+^o(X)\right)^N$. Consider the quotient map $\pi : C^o(X) \to C^o(X) / \sim$ as defined in the proof of the quotient theorem in §3.2. It is continuous by the definition of quotient topology on $C^o(X)/_\sim$ that a set A is open in $C^o(X)/_\sim$ iff its preimage $\pi^{-1}(A)$ is open in $C^o(X)$. Finally, $\varphi : C^o(X)/_\sim \to P_{soc} \equiv (P, \Im_o)$ also defined in § 3.2 is continuous, by the quotient theorem. Therefore the composition map

$$F_\nu = \varphi \circ \pi \circ G_\nu \circ \xi_*^N$$

of continuous maps is continuous.

Step 4 It is clear by the symmetry of G_ν in its entities that F_ν *is anonymous*, i.e. F_ν satisfies

AN $F_\nu(p_1, p_2, \cdots, p_N) = F_\nu(p_{\sigma_1}, p_{\sigma_2}, \cdots, p_{\sigma_N}), \forall (p_1, p_2, \cdots, p_N) \in P^N$ and $\forall$ permutation σ of indices $1, 2, \cdots, N$.

Furthermore, F_ν satisfies the strong Pareto condition, i.e.

PA $x \succeq y$ in $p_i, \forall i = 1, 2, \cdots, N \Rightarrow x \succeq y$ in $F_\nu(p_1, p_2, \cdots, p_N)$ and if furthermore $\exists i$ in $\{1, 2, ..., N\}$ such that $x \succ y$ in p_i then $x \succ y$ in $F_\nu(p_1, p_2, \cdots, p_N)$,

since each $f_i = \xi_*(p_i) > 0$ and the symmetric function G_ν is strictly increasing in $(f_1, f_2, \cdots, f_N)$. The proof is completed. ■

Theorem 3 states the existence of rational continuous social welfare functions when the topologies of P in the domain and the range are given by $\Im$ and $\Im_o$ differently to fit the real world. The following theorem 4 will show the same existence theorem for the situation in which the domain is restricted to the set of all n-tuples of regular preferences, while the two topologies are identical.

Theorem 4 (Restricted domain) Let the alternative space X and the preference space P on X are as in Theorem 3. Then there exist social welfare functions

$$F_\nu : P_r^N \to P,\ \nu = 1, 2, \cdots, N,$$

which are continuous, anonymous, and satisfy the strong Pareto condition. Here P_r denotes the totality of all regular preferences, and both topologies of P_r in the domain and P in the range of F_ν share the same topology $\Im_o$.[14]

Proof. By Proposition 3.3.1, the two topologies $\Im_o$ and $\Im$ are identical as restricted to the regular preferences, i.e.

$$(P_r, \Im_o) = (P_r, \Im).$$

Therefore, the map ξ_* defined in Theorem 3 is still continuous if it is restricted to $(P_r, \Im_o)$. The rest part of this proof follows exactly as in Theorem 3. ■

Remark 3.5.1 It is clear that the strong Pareto condition implies the *unanimity* which is defined by

(UN) $F(p, p, \cdots, p) = p, \forall p \in P$, where F is the social welfare function.

Chichilnisky proved in [C] that there exists no social welfare function $F : P_c^N \to P_c$ which is continuous, anonymous and unanimous. In her framework, she adopted $\Im_c$ for P_c without distinguishing the topology for the individual preferences from that for the social preferences.

Finally, we define *the quantified social welfare function* by

$$\widetilde{F} : P^N \to C^o(X).$$

We say $\widetilde{F}$ *satisfies the strong Pareto condition* if

$$x \succsim y \text{ in } p_i, \ \forall i = 1, 2, \cdots, N \Longrightarrow u(x) \geq u(y)$$

where $u = \widetilde{F}(p_1, p_2, ..., p_N) \in C^o(X)$, and furthermore,

$$x \succsim y \text{ in } p_i, \ \forall i = 1, 2, \cdots, N, \text{ with } x \succ y \text{ in } p_j \text{ for some } j \Longrightarrow u(x) > u(y).$$

By the proof of Theorem 3, we also obtain the existence theorem of quantified social welfare functions; which in fact is a strong existence theorem.

Theorem 5 Let the alternative space X and the preference space P on X are as in the theorem 3. Then there exist quantified social welfare functions

$$\widetilde{F}_\nu : P^N \longrightarrow C^o(X), \ \forall \nu = 1, 2, \cdots, N.$$

which are continuous, anonymous and satisfies the strong Pareto condition. The preference space P is equipped with the stratification topology and the function space $C^o(X)$ with the compact-open topology.

§ 4 Analysis of Arrow's framework

§ 4.1 A degree theorem

Consider in this section that the alternative space X is simply a finite discrete set. Since the topology of X is now trivial, a preference order on X respecting the topology of X is a void condition. Using the terminology introduced in § 1, a preference order on X is automatically a preference on X. That X is a finite discrete set was the case considered by Kenneth Arrow in 1951. Arrow first proved the impossibility of rational social choices. His impossibility theorem in our language is the following.

Arrow's Impossibility Theorem

Let X be a finite set of at least three alternatives, denoted by $\#X = n \geq 3$, P be the totality of all preferences on X and N be a positive integer denoting the number of the individuals (or the voters) with the set of individuals $V = \{v_1, \cdots, v_N\}$. There exists no (social welfare) function $F : P^N \to P$ with $F(\mathrm{p}) = \overline{p}$, $\mathrm{p} = (p_1, p_2, \cdots, p_N)$[15] satisfying the following conditions:

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[15] p, the N-tuple of preferences, is usually called a *profile*.

(CS) (Citizen sovereignty) F is surjective.

(PA) (Strong Pareto Condition)

$$x \succsim y \text{ in } p_i, \ \forall i^{16} \Rightarrow x \succsim y \text{ in } \overline{p},$$

and if "$\exists\, j$ with $x \succ y$ in p_j" is added to the antecedent, then $x \succ y$ in $\overline{p}$.

(I) (Independent of Irrelevant Alternatives)

Given p, q$\in P^N$ and $x, y \in X$ such that for any individual v_i,

$$x \succsim y \text{ in } p_i \Longleftrightarrow x \succsim y \text{ in } q_i, \text{ and}$$

$$x \precsim y \text{ in } p_i \Longleftrightarrow x \precsim y \text{ in } q_i,$$

which is written as $p_i(x, y) = q_i(x, y)$. Then

$$\overline{p}(x, y) = \overline{q}(x, y)$$

where $\overline{p} = F(\mathrm{p})$ and $\overline{q} = F(\mathrm{q})$.

(ND) (Nondictatorship) There exists no *dictator* of F. By a dictator of F, we mean an individual v_k such that

$$F(\mathrm{p}) = p_k$$

for any $\mathrm{p} \in P^N$.

Instead of interpreting the result as a deductive impossibility theorem, we establish on the contrary *a constructive degree theorem*, which detects a more essential intrinsic connection between the degree of Arrow's independency and that of dominance. The argument we use to obtain the degree theorem is basically the same argument that Arrow invented to prove the impossibility theorem. However, Arrow's impossibility theorem is negative for possible social choice, while our degree theorem is positive, emphasizing constructively a line to study the *possible* rational social choices in practice.

By a *social welfare function* we mean a function $F : \Omega \to P$, where $\Omega \subset P^N$.

Definition 4.1.1 *(a)* Given $F : \Omega \to P$, $Z \subset X$ is called a *set of Arrow's independence* for any pair x, y in Z, if the social choice preference on the (unordered) pair $\{x, y\}$[17] is independent of the given individual preferences on the other pairs, i.e. $\forall\, x, y \in Z$ and $\forall$ p,q $\in \Omega$,

$$p_i(x, y) = q_i(x, y), \ \forall\, i \Rightarrow \ \overline{p}(x, y) = \overline{q}(x, y). \qquad (I_a)$$

---

[17] By $\{x, y\}$ we mean the *unordered* pair x, y regarded as a set of two elements x and y, while (x, y) means the ordered pair in the product space $X \times X$.

(b) The *degree of Arrow's independency of* F is defined by the maximal number of sets of Arrow's independence; i.e.

$$\delta_I(F) \equiv \max\{\#Z;\ Z \text{ is a set of Arrow's independence}\}^{18}.$$

Definition 4.1.2 (a) Given $F : \Omega \to P$, $Z \subset X$ is called *a dominated set* if there exists v_k such that $\forall\, \mathrm{p} \in \Omega$,

$$F(\mathrm{p})|_Z = p_k|_Z,$$

which means that the social preference restricted to any pairs in Z is the same as the k-th individual preference restricted to the pairs in Z. The individual v_k is called a dominator over Z.
(b) The *degree of dominance of* F is defined by the maximal number of dominated sets, i.e. by

$$\delta_D(F) \equiv \max\{\#Z;\ Z \text{ is a set of dominance}\}.$$

Theorem 6 Given $F : P^N \to P$, a social welfare function with the Pareto condition[19](**PA′**), we have

$$1 \leq \delta_D(F) \leq \delta_I(F) \leq n,$$

where $n = \#X$ being the cardinal number of alternative set X. Furthermore, if $\delta_I(F) \geq 3$, then

$$\delta_D(F) \geqq \delta_I(F).$$

and therefore $\delta_D(F) = \delta_I(F)$ by the first statement.

We give some examples with the assumption of $\#X \geq 3$ to illustrate these two kinds of degree before proving the theorem. All the social welfare function F in the following five examples satisfy (**CS**) and (**PA**).

Example 4.1.1 (Simple Majority) Let F be defined on a restricted domain P_m^N where

$$P_m \equiv \{p \in P; \exists x \in X \text{ such that } x \succ y \text{ in } p, \forall y \in X \text{ with } y \neq x\}$$

Given $x \in X$ and $\mathrm{p} = (p_1, \ldots, p_N) \in P_m^N$, let

$$f(x) = \#\{v_i \in V;\ x \succ y \text{ in } p_i, \forall\, y \in X \text{ with } y \neq x\}$$

Define

$$x \succsim y \text{ in } F(\mathrm{p}) \text{ iff } f(x) \geq f(y).$$

---

[19] (**PA′**) $x \succ y$ in p_i, $\forall i \Rightarrow x \succ y$ in $\overline{p}$, where $\overline{p} = F(\mathrm{p})$.

In this example, $\delta_I(F) = 1$ and $\delta_D(F) = 1$.

Example 4.1.2 (Arithmetic Mean of Inferiority) Let $F : P^N \to P$ be defined as follows. For the i-th individual preference $p_i \in P$, let the utility function f_i be defined by

$$f_i(x) = \#Q_x(p_i),\ \forall x \in X.$$

Also let $f = (f_1 + f_2 + \ldots + f_N)/N$. Then define $F(\mathrm{p})$ for $\mathrm{p} = (p_1, p_2, \ldots, p_N)$ by

$$x \succsim y \text{ in } F(\mathrm{p}) \text{ iff } f(x) \geq f(y).$$

Clearly, $\delta_I(F) = 1 = \delta_D(F)$.

Example 4.1.3 (Voting of Amendment Bill) Given $X = \left\{L, L', C\right\}$ three bills formulated in a meeting, where C is the original bill and L a bill newly proposed with L' its amendment.. The social choice system F requires to cast the first vote between L and L', and then the second between C and the winning bill of the first vote to decide the final choice. It is clear that $Z = \left\{L, L'\right\}$ is a set of Arrow's independence, but $\left\{L, L', C\right\}$ is not. Therefore, $\delta_I(F) = 2$. On the other hand, $\delta_D(F) = 1 < \delta_I(F)$ since there is no dominator over any two distinct bills.

Example 4.1.4 Define a social welfare function $F : P^N \to P$ by

$$F(p_1, \ldots, p_N) = p_1.$$

Then $\delta_D(F) = n = \delta_I(F)$, where $n = \#X$.

Example 4.1.5 For each m with $3 \leq m \leq n$, we construct an example with $\delta_D(F) = m = \delta_I(F)$. Given a subset Z of X with $\#Z = m$, define for each profile p the utility function f of $\overline{p} = F(\mathrm{p})$ by

$$f(x) \equiv \#\{z \in Z;\, z \precsim x \text{ in } p_1\},\ \text{for } x \in Z$$

and

$$f(x) \equiv \tfrac{1}{N}(f_1 + f_2 + \cdots + f_N),\ \text{for } x \in X - Z$$

where $f_i(x) \equiv \#\{y \in X;\, y \precsim x \text{ in } p_i\}$. We see that for $x \in Z$, $f(x) \in [1, m]$, and for $x \in X - Z$, $f(x) \in [1, n]$. Let

$$x \succsim y \text{ in } \overline{p} \text{ iff } f(x) \geq f(y).$$

Clearly, Z is a set of Arrow independence with $\#Z$ attaining $\delta_I(F)$. On the other hand,

$$F(p_1, \ldots, p_N)|_Z = p_1|_Z.$$

We have $\delta_D(F) = m = \delta_I(F)$.

Proof. It is straightforward to see the first consequence. According to the definition, any singleton set $\{x\}$ is a dominated set, since the only relation $x \succsim x$ is trivial to in any preference. Therefore, $\delta_D(F) \geq 1$. Now given Z a dominated set of F with $\#Z = \delta_D(F)$, then $F(\mathrm{p})|_Z = p_k|_Z$ for some individual v_k. Clearly, it follows that Z is also a set of Arrow's independence of F. Hence $\delta_D(F) \leq \delta_I(F)$. The proof of $\delta_D(F) \geq \delta_I(F)$ when $\delta_I(F) \geq 3$ is basically the Arrow's argument. For easier reference, we present again the argument, while taking a more concise version of Sen [S2]. The only difference of the following proof from the known argument is that we have to be careful about the restriction of the Arrow's independency. It is denoted by $I\{x, y, ..., w\}$ that $\{x, y, ..., w\}$ is a set of Arrow's independence. We also follow Sen in defining that a subset G of V *decisive* over the order pair (x, y), denoted by $\overline{D}_G(x, y)$, if

$$x \succ y \ \text{ in } p_i, \forall v_i \in G \Rightarrow x \succ y \ \text{ in } \overline{p}.$$

When "$x \prec y$ in p_j, $\forall v_j \in V - G$ "is added to the antecedent of the last implication, we say G *almost decisive* over (x, y), denoted $D_G(x, y)$. Note that Z is a dominated set by v_k iff $\overline{D}_{v_k}(x, y)$ for any ordered pair (x, y) with x, y in Z. We now proceed the proof by steps:

Step 1 (Field Expansion Lemma) We claim first that

$$I\{x, y, z\} + D_G(x, y) \Rightarrow \overline{D}_G\{x, y, z\}$$

where $\overline{D}_G\{x, y, z\}$ means that G is decisive over any order pair in $\{x, y, z\}$ with x, y, z distinct. Let the profile p be given such that $z \succ x \succ y$ in p_i, $\forall v_i \in G$, and $y \succsim z \succ x$ or $z \succsim y \succ x$ in p_j, $\forall \underset{j}{v_i} \in V - G$. This is illustrated by the diagram:

$$\mathrm{p}: \begin{array}{c|c} G & V - G \\ \hline x & [y, z] \\ y & x \\ z & \end{array}$$

where $[y, z]$ means undecideness of the preference order between y and z. Denote $\bar{p} = F(\mathbb{P})$. Since $z \succ x$ in $\overline{p}$ by (PA$'$) and $x \succ y$ in $\overline{p}$ by $D_G(x, y)$, we have $z \succ y$ in $\overline{p}$ when ~~$\overline{p} = F(\mathrm{p})$~~. By $I\{z, y\}$, we then have $\overline{D}_G(z, y)$. Thus,

$$I\{z, y\} + D_G(x, y) \Rightarrow \overline{D}_G(z, y). \tag{4.1.1}$$

Similarly, considering the following diagram

$$\mathbb{P}': \begin{array}{c|c} G & V - G \\ \hline x & y \\ y & [z, x] \\ z & \end{array}$$

we have

$$I\{x,z\} + D_G(x,y) \Rightarrow \overline{D}_G(x,z). \tag{4.1.2}$$

Repeating the formulas (4.1.1) and (4.1.2) by changing order pairs of alternatives among $\{x,y,z\}$, we easily obtain

$$I\{x,y,z\} + D_G(x,y) \Rightarrow \overline{D}_G\{x,y,z\}. \tag{4.1.3}$$

Again by i~~terating~~ (4.1.3), ~~for example~~, we have

$$I\{x,y,a,b\} + D_G(x,y) \overset{\cancel{(4.1.3)}}{\Longrightarrow} \overline{D}_G\{x,y,a\}$$

and therefore $I\{y,a,b\} + D_G(y,a)$ is valid. Again by (4.1.3), we have

~~$I\{x,y,a,b\} + D_G(x,y) \Longrightarrow$~~ $I\{y,a,b\} + D_G(y,a) \overset{(4.1.3)}{\Longrightarrow} \overline{D}_G\{y,a,b\}$,

Iterating the process, ~~we have~~ it is clear that ~~we have~~

$$I(Z) + D_G(x,y) \Rightarrow \overline{D}_G(Z), \tag{4.1.4}$$

for any subset Z of X.

Step 2 (Group Contraction) Let $G = G_1 \cup G_2$ with $G_1 \cap G_2 = \phi$, we will show

$$I(Z) + D_G(x,y) \Rightarrow \overline{D}_{G_1}(Z) \text{ or } \overline{D}_{G_2}(Z), \tag{4.1.5}$$

$\forall\, Z \subset X$. Let x, y, z be three distinct alternatives in X. Consider

$$\mathbb{P}'' : \begin{array}{c|c|c} G_1 & G_2 & V-G \\ \hline x & z & y \\ y & x & z \\ z & y & x \end{array}$$

If $y \succsim z$ in $\overline{p}''$ where $\overline{p}'' \equiv F(\mathrm{p}'')$, then by $D_G(x,y)$ and $x \succ y$ in p_i, $\forall\, v_i \in G$, we have $x \succ y \succsim z$ in $\overline{p}''$. Since $x \succ z$ in $\overline{p}''$ and $I\{x,z\}$,we have $D_{G_1}(x,z)$. By the formula (4.1.4) we have $\overline{D}_{G_1}(Z)$. On the other hand, if $y \prec z$ in $\overline{p}''$, we have $D_{G_2}(z,y)$ based on $I\{z,y\}$. Again by (4.1.4) we have $\overline{D}_{G_2}(Z)$. Hence (4.1.5) is proved.

Step 3 (Iterated Contraction) Let $Z \subset X$ be given with $I(Z)$. Let $G = V$ at the beginning. By $(\mathbf{PA}')$, we have $D_G(x,y)$ for a given ordered pair (x,y). Iterating (4.1.5) for finite times (no more than N times), we finally have $\overline{D}_{v_k}(Z)$ for some $v_k \in V$. This shows that

$$\delta_D(F) \geq \delta_I(F)$$

as required. ☒

Theorem 6 generalizes Arrow's impossibility theorem. In fact, Arrow's independence (I) means $\delta_I(F) = n$, where $n = \#X$, being the number of the alternative set X. On the other hand, the existence of a dictator means $\delta_D(F) = n$. With the assumption of Pareto condition $(\mathbf{PA}')$ we have $\delta_D(F) = \delta_I(F) = n$, which yields the existence of a dictator. Therefore, (PA′), (I) and (ND) are inconsistent.

However the significance of Theorem 6 is neither limited to generalize Arrow's theorem, nor to strengthen the impossibility of a rational social choice. On the contrary, it says that Arrow's independence has an innate bond with the possible dominance of some individuals over alternatives. Once the degree $\delta_I(F)$ of Arrow's independency is no less than 3, the degree $\delta_D(F)$ of dominance satisfies $\delta_D(F) \geq \delta_I(F)$, where Arrow's independent condition (I) is not as rational as it ever appears. To investigate rational social choices, we have to limit our consideration to the cases of $\delta_I(F) = 1$ or 2 (as in the Examples 4.1.1, 4.1.2 and 4.1.3), which ironically have long been regarded as irrational simply because they do not satisfy the irrational principle (I).

The historical background of Arrow's impossibility theorem is also interesting to draw attention. Given a Condorcet triple[20], there contains no paradox in itself unless we exclude singularities of the social preference[21]. It leads to the Condorcet paradox when pairwise majority voting is introduced. Arrow's impossibility theorem is historically a generalization of the Condorcet paradox to a general setting of profiles in which the pairwise independence (I) is satisfied. A reasonable interpretation of Arrow's theorem is the nonexistence of a rational sotaal aggregation which *satisfies the pairwise independence* (I). As whether the condition (I) could therefore be regarded rational remains debatable, it is too facile to conclude from Arrow's theorem the nonexistence of a rational social aggregation.

Mas-Colell, Whinston and Green stated in [MC-W-G], "What Arrow's theorem does tell us, in essence, is that the institutional detail and procedures of the political process cannot be neglected." However, there evidently exist no logical connection between Arrow's theorem and the relevance of procedures and rules to social aggregation. Instead, the latter is the consequence of the quantified degree theorem [Theorem 6] which says that $\delta_I(F) < 3$ are the cases worthwhile to be studied in the practical world. The studies about the procedures of political process such as the examples 4.1.1 to 4.1.3 and the ones in [AS-B] and [S-B], which do not involve dynamical process as mentioned in the end of the introduction of this paper, are all included in the realm with $\delta_I(F) < 3$, while Arrow's theorem *only* governs the complementary case that $\delta_I(F) = n \geq 3$.

In § 4.2 we will define a revised rational independence condition (I_*) which respects the given stratification structure of individual preferences. With (I_*)

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[20] See [MC-W-G]. A Condorcet triple is a profile (p_1, p_2, p_3) with $N = n = 3$ and $x \succ y \succ z$ in p_1, $y \succ z \succ x$ in p_2, $z \succ x \succ y$ in p_3. By pairwise majority voting we mean $a \succsim b$ in the social preference $F(\mathrm{p})$ iff

$$\#\{v_i;\ a \succ b \text{ in } p_i\} \geq \#\{v_i;\ a \prec b \text{ in } p_i\}.$$

For a Condorcet triple, the pairwise majority voting causes non-transitive social preference. This is called Condorcet Paradox.

[21] See the following section § 4.2 for details.

an existence theorem of rational social choices will be established.

§ 4.2 Singularities and existence

We will show that Arrow's independence condition (I) precludes the cancellation in social aggregation, although any social aggregation by nature is a process of averaging individual preferences that make the cancellation inevitable. As a corollary, the preclusion of cancellation then excludes singularities from consideration.

Theorem 7 (Preclusion of cancellation ~~in social~~ aggregation ~~by (I)~~) Let $F : P^N \to P$ be a social welfare function satisfying (PA) and (I) with $n = \#X \geq 3$. Given $x, y \in X$ and $\mathbf{p} = (p_1, ..., p_N) \in P^N$, x is not indifferent to y in $F(\mathbf{p})$,unless x is indifferent to y in p_i, $\forall i$.

Proof. Suppose $x \sim y$ in $F(\mathrm{p})$. Let $z \in X$, distinct from x and from y.
It suffices by (PA) to consider the case that $\exists\, G_1, G_2$ two proper subsets of V such that

$$p : \begin{array}{ccc} G_1 & G_2 & V-(G_1 \cup G_2) \\ \hline x & y & \\ y & x & x \sim y \end{array}$$

(In fact, if $G_1 \neq \phi$ and $G_2 = \phi$, then $x \succ y$ in $F(\mathrm{p})$ by (PA)). Denote $G = G_1 \cup G_2$. ~~Suppose $x \sim y$ in $F(\mathrm{p})$~~. Consider

$$\mathrm{p}' : \begin{array}{ccc} G_1 & G_2 & V-G \\ \hline x & y & \\ y & z & x \sim y \sim z \\ z & x & \end{array}, \text{ and } \mathrm{p}'' : \begin{array}{ccc} G_1 & G_2 & V-G \\ \hline x & z & \\ z & y & x \sim y \sim z \\ y & x & \end{array},$$

leaving open the preference order on other pairs, we have by (PA) that $y \succ z$ in $F(\mathrm{p}')$. But $I(x,y)$ implies

$$x \succsim y \text{ in } F(\mathrm{p}) \text{ iff } x \precsim y \text{ in } F(\mathrm{p}')$$

and

$$x \precsim y \text{ in } F(\mathrm{p}) \text{ iff } x \precsim y \text{ in } F(\mathrm{p}')$$

Hence, $x \sim y$ in $F(\mathrm{p}')$. The transitivity of $F(\mathrm{p}')$ yields that

$$x \succ z \text{ in } F(\mathrm{p}'). \tag{4.2.1}$$

On the other hand, $z \succ y$ in $F(\mathrm{p}'')$, still by (PA). Similarly $x \sim y$ in $F(\mathrm{p}'')$ by $I(x,y)$ and

$$z \succ x \text{ in } F(\mathrm{p}'').$$

by the transitivity of $F(\mathrm{p}'')$. Again by $I(z, x)$, we have

$$z \succ x \text{ in } F(\mathrm{p}'),$$

contradicting to (4.2.1). The proof is thus completed. ■

Extending the idea about the regularities and the singularities developed in § 2.2, we define them for the preferences on the discrete finite space as follows.

Definition 4.2.1 A preference p on a finite set X is called *singular* if $\exists$ distinct x, y in X such that $x \sim y$ in p. Otherwise, it is called *regular.*

Literally, Theorem 7 can be restated by

Theorem 7′ (exclusion of singularities by (I)) Under the assumption of the strong Pareto condition (PA), the Arrow's independence condition (I) excludes any singularity appeared in the social preference, except the same singularity is given in *each* of the individual preferences.

Since to deal with the singularities is inevitable for any discipline of science, the theorem $7'$ gives a strong reason to question the rationality of Arrow's independence condition (I).

We agree that the independence of irrelevant alternatives in some modified sense would still be regarded as a principle of rationality. As the stratification structure of individual preference is, *a priori*, built in the preferences presented to the society, a rational independence condition *must respect* the stratification structure.. The following (I*) is such a condition.

(I*) (Independence respecting stratification structure) Given x, y in X and p, q in P^N, if the subpreference sets are invariant, i.e.

$$Q_x(p_i) = Q_x(q_i),\ Q_y(p_i) = Q_y(q_i)$$

$\forall\, v_i \in V$, then

$$x \succsim y \text{ in } F(\mathrm{p}) \text{ iff } x \succsim y \text{ in } F(\mathrm{q}).$$

The modified in dependence condition (I*) means that the social preference on a pair (x, y) remains unchanged, as the individual preferences on (x, y) alters yet preserving the stratification structure given at x and y for each individual preference.

Theorem 8 (Existence Theorem of Rational SWF) Let X be a finite set. P be the totality of preferences on X. There exists $F : P^N \to P$, called social welfare function, such that (CS), (PA), (I*) and (AN)[22] are satisfied.

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[22] (AN) means the anonymity. For the definition, see § 3.5.

Remark 4.2.1 It is evident that the anonymity (**AN**) implies the nondictatorship (**ND**). In fact, suppose v_1 is a dictator, then $p_1 = F(p_1, p_2, ..., p_N) = F(p_2, p_1, ..., p_N) = p_2$, leading to a contradiction, if p_1 and p_2 are chosen distinct.

Proof of Theorem 8 Consider again the sequence of maps

$$P^N \xrightarrow{\xi^N} (\mathfrak{F}(X))^N \xrightarrow{G_\nu} \mathfrak{F}(X) \xrightarrow{\pi} \mathfrak{F}(X)/_\sim \xrightarrow{\varphi} P,$$

defined in § 3.5, except here we take $\mathfrak{F}(X)$, the totality of functions on X, to replace $C^o(X)$ and redefine $\xi : P \to F(x)$ by $\xi(p) = f$, $\forall p \in P$, with

$$f(x) = \#Q_x(p), \forall\, x \in X,$$

i.e. $f(x)$ is the number of the alternatives y's with $y \precsim x$ in p. Notice that G_ν in the ν-th symmetric function[23] where $\nu = 1, 2, \cdots, n$, π is the quotient map[24] and φ is bijective[25].It is clear that all the rational principles are satisfied. ■

§ 4.3 Rationality of proximity property

Besides Arrow and Chichilnisky, Baigent [B] also proved an impossibility theorem which said that the proximity preservation (**PX**), the unanimity (**UN**) and the anonymity (**AN**) are inconsistent for social choice. As the three properties are considered rational, Baigent hence claimed the impossibility of a rational social choice.

According to Baigent, the *proximity preservation* (**PX**) of a social welfare function

$$F : P^N \to P$$

is defined by

$$(\mathbf{PX}) \qquad d_\delta(\mathrm{p},\mathrm{q}) \leq d_\delta(\mathrm{p},\mathrm{r}) \Rightarrow \delta(F(\mathrm{p}), F(\mathrm{q})) \leq \delta(F(\mathrm{p}), F(\mathrm{r}))$$

for any p, q, $\mathrm{r} \in P^N$, where δ is any given metric in P and d_δ is given by

$$d_\delta(\mathrm{p},\mathrm{q}) = \sum_{i=1}^{N} \delta(p_i, q_i) \tag{4.3.1}$$

It was regarded as a rational principle since it means that the slighter a profile of individual preference changes, the smaller the social preference varies. The proximity preservation (**PX**) appears to be the extended notion of continuity to the discrete case.

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[23] See Step 3 of the proof of theorem 3 in §3.5.

[24] See the begining of §3.2.

[25] See the formula (3.2.1) and Step 1 of Theorem 1 in §3.2

However, we question the rationality of **(PX)** by showing that it violates the spacial arrangement. In other words, **(PX)** itself is *geometrically impossible.* Therefore, it should not be accepted as a rational principle as it appears.

Theorem 9 Let V be a metric space[26] with a metric δ, V^N be the N-product space with the metric d_δ defined by (4.3.1) and $F : V^N \to V$ be an arbitrarily given map, where $N > 1$. Assume F maps the diagonal points of V^N injectively into V, i.e. $\forall$ distinct $v, w \in V$, $F(v, v, ..., v) \neq F(w, w, ..., w)$. (The assumption is called the *diagonal injectivity* **(DI)**). Then the proximity preservation **(PX)** is false.

Example 4.3.1 We show in this example that the diagonal injectivity assumption **(DI)** is essential for the theorem 9 to be valid. Let $V \equiv \{a, b, c_1, c_2, ..., c_h\}$ with $h \geq 9$, and let

$$\delta(a,b) = 100,\ \delta(a,c_i) = 90,\ \delta(b,c_i) = 10, \tag{4.3.2}$$
$$\delta(c_i, c_j) = \begin{cases} 0; & i = j, \\ 1; & i \neq j. \end{cases}$$

$\forall\ i, j \in \{1, 2, ..., h\}$. Define $F : V^N \to V,\ N = 2$ by mapping each of the nine sets:

$$\{(a,a)\},\ \{(b,b)\},\ \{(a,b)\},\ \{(b,a)\},$$
$$\{(a,c_i);\ i = 1, ..., h\},\ \{(b,c_i);\ i = 1, ..., h\},$$
$$\{(c_i,a);\ i = 1, ..., h\}, \{(c_i,b);\ i = 1, ..., h\}$$
$$\{(c_i,c_j);\ i, j = 1, ..., h\},$$

into $\{(c_k, c_k);\ k = 1, ..., h\}$ injectively. Then **(PX)** is satisfied.

Proof of Theorem5

Step 1 Let $a, b \in V$ attain the diameter of V, i.e.

$$\delta(a,b) = \max_{v,w \in V} \delta(v,w) \equiv diam\, V.$$

Denote $A = (a, a, a, ..., a), B = (b, b, b, ..., b) \in V^N$. Also let

$$D = (a, b, b, ..., b) \in V^N.$$

By the injectivity assumption, the image of all diagonal points covers V, since it have the same cardinality with that of V. There exists $C = (u, u, ..., u)$ such that $F(D) = F(C)$. However,

$$d_\delta(D, C) = \delta(a, u) + \delta(b, u) + \delta(b, u) + \cdots + \delta(b, u) \geq \delta(a, b) = d_\delta(D, B).$$

Suppose the proximity property **(PX)** were true, then $\delta(F(D), F(C)) \geq \delta(F(D), F(B)) \geq 0$. But $\delta(F(D), F(C)) = 0$. Hence, $\delta(F(D), F(B)) = 0$, i.e. $F(B) = F(D) = F(C)$. This yields that $B = C$.

---

[26] By V a metric space, we mean a set V with a map $\delta : V \times V \to \mathbb{R}$ such that $\forall x, y, z \in V$, (i) $\delta(x,y) \geq 0$ and the equality holds iff $x = y$, (ii) $\delta(x,y) = \delta(y,x)$, and (iii) $\delta(x,y) + \delta(y,z) \geq \delta(x,z)$.

Step 2 For any $v \in V$, consider the point $E = (v, b, b, ..., b)$, we have

$$d_\delta(E, D) = \delta(v, a) \leq \delta(b, a) = d_\delta(B, D).$$

By **(PX)**, it follows that

$$\delta(F(E), F(D)) \leq \delta(F(B), F(D)) = 0.$$

Hence $F(E) = F(D) = F(B)$, i.e. the whole column α_1 defined by

$$\alpha_1 \equiv \{(v, b, ..., b) \in V^N, v \in V\}$$

was mapped by F to a same point in V.

Step 3 We claim that the entire domain V^N would now be mapped under F to the same point. Consider

$$D_2 = (a, a, b, b, ..., b),\ B_2 = (b, a, b, b, ..., b).$$

Since

$$d_\delta(D_2, D) = \delta(a, b) = d_\delta(B, D),$$

we have by **(PX)** that $\delta(F(D_2), F(D)) = \delta(F(B), F(D)) = 0$; hence, $F(D_2) = F(D)$. Similarly $F(B_2) = F(D)$. It is then clear that the set

$$\alpha_2 \equiv \{(v, w, b, b, ..., b);\ v, w \in V\}$$

are mapped by F to a same point in V. Continuing the procedure inductively, we would have V^N mapped by F to a same point, which would violate **(DI)**.
■

Theorem 9 says that the proximity preservation **(PX)** is geometrically impossible, if assumed the diagonal injectivity **(DI)** which is a mild condition much weaker than the unanimity **(UN)**. One therefore can hardly accepts **(PX)** as a rational principle and concludes from the inconsistency of **(PX)**, **(UN)** and **(AN)** that a rational social choice is impossible as claimed by Baigent.

Furthermore, the proximity preservation **(PX)** is far beyond the extension notion of the continuity of social welfare functions considered by Chichilnisky. The continuity means the "smaller the change" in individual preferences, the "smaller the change" in social preferences. However, it does not claim that the "larger the change" in individual preferences, the "larger the change" in social preferences. Unfortunately, the proximity preservation **(PX)** required the latter, which practically is false even in the geometric content. The following figure in continuum case would illustrate this better. Consider Figure 4.3.1, where $\overline{p} = F(p)$, $\overline{q} = F(q)$, $\overline{r} = F(r)$, with $\mathbf{p}, r$ mapped to the same $\overline{p}$ while having the distance from $\mathbf{p}$ to $\mathbf{r}$ larger than that from p to q. The proof of Theorem 9 is essentially based on the last observation in the figure.

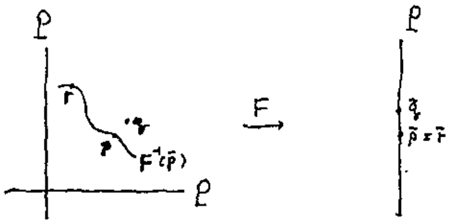

Fig. 4.3.1

If one agrees that the proximity preservation (PX) is irrational, then Baigent's theorem could not be regarded as an impossibility theorem of rational social choice.

Remark 4.3.1 Finally, we point out that the proof of Baigent's theorem [B] is not valid for odd N. We follow his notations to illustrate the mistake. In the proof of his lemma 1, the last inequality

$$d_\delta(\underline{s}, \underline{p}) = d_\delta(\underline{t}, \underline{p}) + \delta(p, q) \leq \frac{n}{2}\delta(p, q)$$

dose not necessarily hold for odd n. Although

$$d_\delta\ (\underline{t}, \underline{p}) < \frac{n}{2}\delta(p, q),$$

the fact $d_\delta(\underline{s}, \underline{p}) < (\frac{n}{2} + 1)\delta(p, q)$ implies the desired inequality only when n is even. The counter example of the lemma 1 is illustrated as follows. Let $n = 3$, $p \neq q$. Let F map the four points (p, p, p), (q, p, p), (p, p, q) and (p, q, p) to p, and map (q, q, q), (q, q, p), (q, p, q) and (p, q, q) to q. Taking $\underline{t} = (q, p, p)$, $\underline{s} = (q, q, p)$, we have $d_\delta(\underline{t}, \underline{p}) = \delta(p, q)$, $d_\delta(\underline{s}, \underline{p}) = 2\delta(p, q) > \frac{3}{2}\delta(p, q)$, *not* $\leq \frac{3}{2}\delta(p, q)$ as claimed.

Theorem 9 is logically a strong version of Baigent's theorem, although it challenges the rationality of the proximity property (PX) on the contrary. It also affords a correct proof with much weaker requirement.

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