Fault-Tolerant Embedding of Longest Paths in Star Graphs with Edge Faults

by

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Abstract

In this paper, we aim to embed a longest path between every two vertices of a star graph with at most *n*-3 random edge faults, where *n* is the dimension of the star graph. Since the star graph is regular of degree *n*-1, *n*-3 (edge faults) is maximum in the worst case. We show that when $n \geq 6$ and there are *n*-3 edge faults, the star graph can embed a longest path between every two vertices, exclusive of two exceptions in which there are at most two vertices missing from the longest path. The probabilities of the two exceptions are analyzed. When $n \geq 6$ and there are *n*-4 edge faults, the star graph can embed a longest path between every two vertices. The situation of $n<6$ is also discussed.

Index Terms: embedding, fault-tolerant embedding, hamiltonian path, longest path, star graph

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1 Introduction

One crucial step on designing a large-scale multiprocessor system is to determine the topology of the interconnection network (network, for short). In recent decades, a lot of networks have been proposed in the literature. The interested readers may refer to [5, 21] for extensive references. Among them, the star graph [1] has been recognized as an attractive alternative to the hypercube. The star graph belongs to the class of Cayley graphs [2], and it possesses many nice topological properties such as recursiveness, symmetry, maximal fault tolerance, sublogarithmic degree and diameter, and strong resilience, which are desirable when we are building an interconnection topology for a parallel and distributed system. Besides, the star graph can embed rings [31], meshes [33], trees [3], and hypercubes [29]. The fault diameter of the star graph was computed in [34]. Efficient communication algorithms for shortest-path routing [1], multiple-path routing [13], multicasting [17], broadcasting [37], gossiping [4], and scattering [17] on the star graph were proposed. A wide range of problems such as sorting [27], merging [31], selection [30], prefix sums [31], ranking [36], Fourier transform [16], and computational geometry [32] were solved on the star graph.

Since faults may happen when a network is put in use, it is practically meaningful and important to consider faulty networks. The problems of diameter [34], routing [15, 40], multicasting [26], broadcasting [39], gossiping [14], embedding [12, 35, 41], and fault-tolerant graphs [7-10, 24] were solved on a variety of faulty networks. This paper is concerned with the problem of embedding on a faulty star graph. Previously related work can be found in [20, 25, 38].

Let F_v denote the set of vertex faults in a faulty star graph. In [38], Tseng *et al.* showed that an n-dimensional star graph with $|F_v|\leq n-3$ can embed a fault-free ring of length at least $n!-4|F_v|$, where $n\geq 4$. In [20], the authors showed that an n-dimensional star graph with $|F_v|\leq n-5$ can embed a fault-free path of length $n!-2|F_v|-2$ ($n!-2|F_v|-1$) between every two vertices of even (odd) distance, where $n\geq 6$. The situations of $|F_v|=n-4$, $n-3$ were also discussed in [20]. In [25], Latifi and Bagherzadeh showed that an n-dimensional star graph with vertex faults can embed a fault-free ring and a fault-free path of each length $n!-m!$, where all vertex faults belong to an m-dimensional star graph and m is minimal.

On the other hand, let F_e denote the set of edge faults in a faulty star graph. In [38], Tseng *et al.* showed that an n-dimensional star graph with $|F_e|\leq n-3$ contains a longest fault-free ring of length $n!$, where $n\geq 4$. In this paper, we show that an n-dimensional star graph with $|F_e|\leq n-4$ can embed a longest fault-free path of length $n!-2$ ($n!-1$) between every two vertices of even (odd) distance, where $n\geq 6$. If $|F_e|=n-3$, there is also a longest fault-free path between every two vertices, exclusive of two exceptions in which at most two vertices miss from the longest fault-free path. Since an n-dimensional star graph is regular of degree n-1, $|F_e|$=n-3 is maximum in the worst case, in order to embed a longest fault-free path between every two vertices.

The rest of this paper is organized as follows. In the next section, necessary definitions and notations are introduced. In Section 3, some fundamental properties are shown which are useful to the embedding of Section 4. In Section 4, a longest fault-free path is embedded between every two vertices of an n-dimensional star graph with $|F_e|\leq n-4$, where $n\geq 6$. If $|F_e|=n-3$, a longest fault-free path can also be embedded between every two vertices, exclusive of two exceptions in which at most two vertices miss from the longest fault-free path. In Section 5, this paper concludes with discussions on boundary situations of n=4, 5 and the probabilities of the two exceptions.

Throughout this paper we use network and graph, processor and vertex, and link and edge, interchangeably.

2 Definitions and notations

We use S_n to denote an n-dimensional star graph. The following is a formal definition of S_n.

Definition 1. *The vertex set of* S_n *is* $\{a_1 a_2 ... a_n \mid a_1 a_2 ... a_n$ *is a permutation of 1, 2, ..., n\}. The edge set of* S_n *is* $\{(a_1 a_2 ... a_n, a_i a_2 ... a_{i-1} a_1 a_{i+1} ... a_n) \mid$ *for all* $2\leq i\leq n\}$.

The vertices of S_n are n! permutations of 1, 2, ..., n, and two of them are adjacent if and only if one can be obtained from the other by swapping the leftmost number with one of the other n-1 numbers. Clearly, S_n is regular of degree n-1. The structures of S_1, S_2, and S_3 are a vertex, an edge, and a cycle of length 6, respectively. The structure of S_4 is shown in Figure 1. Unless specified particularly, $n\geq 4$ for S_n is assumed throughout this paper.

For convenience, we refer to the position of a_i in $a_1 a_2 ... a_n$ as the *ith dimension*, and we say that the edge $(a_1 a_2 ... a_n, a_i a_2 ... a_{i-1} a_1 a_{i+1} ... a_n)$ *belongs to the ith dimension*, where $2\leq i\leq n$. There are S_r's contained in S_n, which are referred to as *embedded* S_r*'s* of S_n, where $1\leq r\leq n$. An embedded S_r is conveniently represented by $\langle s_1 s_2 ... s_n\rangle_r$, where s_1=∗, $s_i \in \{∗, 1, 2, ..., n\}$ for all $2\leq i\leq n$, and exactly r of s_1, s_2, ..., s_n are ∗ (∗ denotes a “don’t care” symbol). For example, $\langle ∗∗∗3\rangle_3$, which represents an embedded S_3 of S_4, contains six nodes 1243, 1423, 2143, 2413, 4123, and 4213. In terms of graph, $\langle ∗∗∗3\rangle_3$ is a subgraph of S_4 induced by {1243, 1423, 2143, 2413, 4123, 4213}. When r=n, $\langle s_1 s_2 ... s_n\rangle_n$ represents S_n. Two basic operations on S_n are defined below.

Definition 2. *An i-partition on* $\langle s_1s_2...s_n\rangle_r$ *partitions* $\langle s_1s_2...s_n\rangle_r$ *into r embedded* S_{r-1}*'s, denoted by* $\langle s_1s_2...s_{i-1}qs_{i+1}...s_n\rangle_{r-1}$ *for all* $q \in\{1, 2, ..., n\}-\{s_1, s_2, \ldots, s_n\}$*, where* $2\leq i\leq n$ *and* $s_i=*$.

For example, executing a 3-partition on $\langle ***15\rangle_3$ produces three embedded S_2's, i.e., $\langle **215\rangle_2$, $\langle **315\rangle_2$, and $\langle **415\rangle_2$.

Definition 3. *An* $(i_1, i_2, ..., i_m)$*-partition on* $\langle s_1s_2...s_n\rangle_r$ *performs an* i_1*-partition, an* i_2*-partition, ..., an* i_m*-partition, sequentially, on* $\langle s_1s_2...s_n\rangle_r$*, where* $i_1i_2\ldots i_m$ *is a permutation of m elements from {2, 3, ..., n}. After executing an* $(i_1, i_2, ..., i_m)$*-partition,* $\langle s_1s_2...s_n\rangle_r$ *is partitioned into* $r(r-1)\cdots(r-m+1)$ *embedded* S_{r-m}*'s.*

For example, when a (3, 2)-partition is applied to $\langle ***15\rangle_3$, a 3-partition on $\langle ***15\rangle_3$ is first executed to produce three embedded S_2's, i.e., $\langle **215\rangle_2$, $\langle **315\rangle_2$, and $\langle **415\rangle_2$. Then, a 2-partition on each embedded S_2 is executed to produce six embedded S_1's, i.e., $\langle *3215\rangle_1$, $\langle *4215\rangle_1$, $\langle *2315\rangle_1$, $\langle *4315\rangle_1$, $\langle *2415\rangle_1$, and $\langle *3415\rangle_1$.

Definition 4. *Two embedded* S_r*'s* $\langle s_1s_2...s_n\rangle_r$ *and* $\langle t_1t_2...t_n\rangle_r$ *are adjacent if* $s_j\neq *$, $t_j\neq *$, *and* $s_j\neq t_j$ *for some* $2\leq j\leq n$, *and* $s_i=t_i$ *for all* $1\leq i\leq n$ *and* $i\neq j$. *The position j is denoted by* $dif(\langle s_1s_2...s_n\rangle_r, \langle t_1t_2...t_n\rangle_r)$.

For example, $\langle **23\rangle_2$ is adjacent to $\langle **13\rangle_2$, and $dif(\langle **23\rangle_2, \langle **13\rangle_2)=3$.

Definition 5. *Suppose* $A_0, A_1, ..., A_{n(n-1)(n-2)\cdots(r+1)-1}$ *are the embedded* S_r*'s that result from executing an* $(i_1, i_2, ..., i_{n-r})$*-partition on* S_n*, where* $1\leq r\leq n-1$. *They form an r-path, denoted by* $P_r=[A_0, A_1, ..., A_{n(n-1)(n-2)\cdots(r+1)-1}]$*, if* A_i *and* A_{i+1} *are adjacent for all* $0\leq i\leq n(n-1)(n-2)\cdots(r+1)-2$. *Each vertex of* P_r*, i.e.,* A_i*, is called an r-vertex. Each edge of* P_r*, i.e.,* (A_i, A_{i+1})*, is called an r-edge.*

For example, Figure 2 shows a P_4 in S_5. The vertices of the P_4 are $\langle ****1\rangle_4$, $\langle ****2\rangle_4$, $\langle ****3\rangle_4$, $\langle ****4\rangle_4$, and $\langle ****5\rangle_4$, which are all 4-vertices. The edges of the P_4 are $(\langle ****5\rangle_4, \langle ****3\rangle_4)$, $(\langle ****3\rangle_4, \langle ****4\rangle_4)$, $(\langle ****4\rangle_4, \langle ****1\rangle_4)$, and $(\langle ****1\rangle_4, \langle ****2\rangle_4)$, which are all 4-edges. We note that an r-vertex is an embedded S_r of S_n and an r-edge comprises $(r-1)!$ edges of S_n. An r-edge $(\langle s_1s_2...s_n\rangle_r, \langle t_1t_2...t_n\rangle_r)$ is considered belonging to the jth dimension, where $j=dif(\langle s_1s_2...s_n\rangle_r, \langle t_1t_2...t_n\rangle_r)$.

Definition 6. *An i-partition on* $P_r=[A_0, A_1, ..., A_{n(n-1)(n-2)\cdots(r+1)-1}]$ *performs an i-partition on* $A_0, A_1, ..., A_{n(n-1)(n-2)\cdots(r+1)-1}$*, respectively, where* $2\leq i\leq n$ *and* $r\geq 2$. *An i-partition on* P_r *is abbreviated to a partition on* P_r *if the position i is "don't care".*

In this paper, we aim to construct a longest path from a vertex (*the beginning vertex*) of S_n to another vertex (*the ending vertex*). As will become clear in Section 4, two classes of P_r's with some good properties, named good P_r's and partially good P_r's, respectively, can help the construction. Below, they are formally defined, where an r-vertex is called the *beginning r-vertex* (the *ending r-vertex*) if it contains the beginning vertex (the ending vertex).

Definition 7. *A $P_r=[A_0, A_1, ..., A_{n(n-1)(n-2)\cdots(r+1)-1}]$ in S_n is good if it satisfies the following three conditions, where $2\le r\le n-1$.*

(1) A_0 and $A_{n(n-1)(n-2)\cdots(r+1)-1}$ are the beginning r-vertex and the ending r-vertex, respectively.

(2) $x_{dif(A_{i-1},A_i)}\neq z_{dif(A_i,A_{i+1})}$ for every three consecutive r-vertices $A_{i-1}=\langle x_1x_2...x_n\rangle_r$, $A_i=\langle y_1y_2...y_n\rangle_r$, and $A_{i+1}=\langle z_1z_2... z_n\rangle_r$ in the P_r, where $1\le i\le n(n-1)(n-2)\cdots(r+1)-2$.

(3) There exists $2\le k\le n$ so that after executing a k-partition on the P_r, the beginning (ending) (r-1)-vertex in A_0 ($A_{n(n-1)(n-2)\cdots(r+1)-1}$) is not connected to A_1 ($A_{n(n-1)(n-2)\cdots(r+1)-2}$).

Definition 8. *A $P_r=[A_0, A_1, ..., A_{n(n-1)(n-2)\cdots(r+1)-1}]$ in S_n is partially good if it satisfies the conditions (1) and (2) of Definition 7 and the condition (3') as follows, where $2\le r\le n-1$.*

(3') There exists $2\le k\le n$ so that after executing a k-partition on the P_r, either the beginning (r-1)-vertex is connected to A_1 and the ending (r-1)-vertex is not connected to $A_{n(n-1)(n-2)\cdots(r+1)-2}$, or the beginning (r-1)-vertex is not connected to A_1 and the ending (r-1)-vertex is connected to $A_{n(n-1)(n-2)\cdots(r+1)-2}$.

Definition 9. *An r-path $P_r=[A_0, A_1, ..., A_{n(n-1)(n-2)\cdots(r+1)-1}]$ forms an r-ring, denoted by $R_r=[A_0, A_1, ..., A_{n(n-1)(n-2)\cdots(r+1)-1}]$ if A_0 and $A_{n(n-1)(n-2)\cdots(r+1)-1}$ are adjacent.*

An i-partition on R_r can be defined similar to Definition 6 (substituting R_r for P_r). After an i-partition on P_r (R_r), each A_j is partitioned into r (r-1)-vertices, where $0\le j\le n(n-1)(n-2)\cdots(r+1)-1$. Since every two of the r (r-1)-vertices are joined with an (r-1)-edge, each A_j can be viewed as a complete graph of r (r-1)-vertices. Throughout this paper, we use K_r^{r-1} to denote the complete graph. We note that each vertex of K_r^{r-1} is an (r-1)-vertex and each edge of K_r^{r-1} is an (r-1)-edge.

Suppose $A_j=\langle s_1...s_{i-1}s_is_{i+1}...s_{j-1}xs_{j+1}...s_n\rangle_r$ and $A_{j+1}=\langle s_1...s_{i-1}s_is_{i+1}...s_{j-1}ys_{j+1}...s_n\rangle_r$ are two neighboring r-vertices in P_r (R_r), where $r\ge 2$, $x\neq y$, and $s_i=*$. After an i-partition, they each are partitioned into r (r-1)-vertices, and the r-edge between them is split into r-1 (r-1)-edges connecting r-1 pairs of (r-1)-vertices that belong to them, respectively. The (r-1)-vertex belonging to A_j (A_{j+1}) that is not connected to A_{j+1} (A_j) is $\langle s_1...s_{i-1}ys_{i+1}...s_{j-1}xs_{j+1}...s_n\rangle_{r-1}$ ($\langle s_1...s_{i-1}xs_{i+1}...s_{j-1}ys_{j+1}...s_n\rangle_{r-1}$).

In this paper we consider S_n with at most n-3 random edges faults, i.e., $|F_e|\leq n$-3. The situation of vertex faults was discussed in [20]. Since S_n is regular of degree n-1, n-3 edges faults are maximum in the worst case, in order to embed a longest fault-free path between every two vertices of S_n. An r-edge in S_n is *fault-free* if it contains no edge fault, and *faulty* else, where $1\leq r\leq n$-1. A P_r (R_r) in S_n is *fault-free* if it contains no edge fault in its r-vertices and r-edges, *healthy* if it contains no faulty r-edge, and *faulty* if it contains one or more faulty r-edges. We note that both a healthy P_r (R_r) and a faulty P_r (R_r) may contain edge faults in its r-vertices.

We use D_i to denote the set of edge faults in S_n that belong to the ith dimension, where $2\leq i\leq n$ and $|D_2|+|D_3|+\cdots+|D_n|\leq n$-3. For example, if F_e={(12345, 32145), (43251, 34251)}, then D_2={(43251, 34251)}, D_3={(12345, 32145)}, and D_4, D_5 empty.

Definition 10. *Suppose $u=u_1u_2...u_n$ and $v=v_1v_2...v_n$ are two vertices of S_n, and let $a_1a_2...a_{n-1}$ be a permutation of 2, 3, ..., n so that $|D_{a_1}|\geq|D_{a_2}|\geq ...\geq|D_{a_{n-1}}|$. Then, $a_1a_2...a_{n-1}$ is crucial with respect to u and v if either of the following two conditions holds.*

(1) $u_{a_1}\neq v_{a_1}$.

(2) There exists $2\leq k\leq n$-1 so that $u_{a_1}=v_{a_1}$, $u_{a_2}=v_{a_2}$, …, $u_{a_{k-1}}=v_{a_{k-1}}$, $u_{a_k}\neq v_{a_k}$, *and* $|D_{a_{k-1}}|>|D_{a_k}|$.

The a_1th (a_kth) dimension is referred to as the break dimension if the condition (1) ((2)) holds.

As shown in Section 4, determining a crucial permutation and finding the break dimension are essential to embedding a longest fault-free path between u and v. A crucial permutation with respect to $u=u_1u_2...u_n$ and $v=v_1v_2...v_n$ can be obtained using the following procedure.

Procdure 1.

(1) Determine a permutation $A=a_1a_2...a_{n-1}$ of 2, 3, ..., n satisfying $|D_{a_1}|\geq|D_{a_2}|\geq ...\geq|D_{a_{n-1}}|$.

(2) Determine the leftmost a_k in A satisfying $u_{a_k}\neq v_{a_k}$.

(3) If k=1, or k>1 and $|D_{a_{k-1}}|>|D_{a_k}|$, then A is a desired one. If k>1 and $|D_{a_1}|=|D_{a_2}|=\cdots=|D_{a_k}|$, then $a_ka_2...a_{k-1}a_1a_{k+1}...a_{n-1}$ (i.e., swap a_1 with a_k) is a desired one. Otherwise, $a_1...a_{l-1}a_ka_{l+1}...a_{k-1}a_la_{k+1}...a_{n-1}$ (i.e., swap a_l with a_k) is a desired one, where $1<l<k$ and $|D_{a_{l-1}}|>|D_{a_l}|=|D_{a_{l+1}}|=\cdots=|D_{a_k}|$.

3 Fundamentals

In this section, some fundamental lemmas are shown which are useful to the embedding of Section 4. A cycle (path) in a graph is a *hamiltonian cycle* (*path*) if it contains every vertex of the graph exactly once. A graph is *hamiltonian* if it contains a hamiltonian cycle. A graph is *hamiltonian-connected* if it contains a hamiltonian path between every two of its vertices [11]. The following three lemmas were proved before.

Lemma 1.*[19] Suppose* $U=\langle u_1u_2...u_n\rangle_r$, $V=\langle v_1v_2...v_n\rangle_r$, *and* $W=\langle w_1w_2...w_n\rangle_r$ *are three consecutive r-vertices of a* P_r *in* S_n, *and let* $p=dif(U, V)$ *and* $q=dif(V, W)$, *where* $2\leq r\leq n-1$. *If* $u_p\neq w_q$, *then after a partition on the* P_r, *each (r-1)-vertex of V is connected to U or W.*

Lemma 2.*[18] Each graph that results from removing d-3 edges* (x_1, y_1), (x_2, y_2), ..., (x_{d-3}, y_{d-3}) *from a complete graph of d vertices is hamiltonian, where* $d\geq 3$. *Further, if for each* (x_i, y_i) *there exists another* (x_j, y_j) *so that they are not incident to the same vertex, then the resulting graph is hamiltonian-connected, where* $1\leq i\leq d-3$, $1\leq j\leq d-3$, *and* $i\neq j$.

Lemma 3.*[18] Each graph that results from removing d-4 edges from a complete graph of d vertices is hamiltonian-connected, where* $d\geq 4$.

The following lemma is clear because $|F_e|\leq n-3$.

Lemma 4. *Suppose* $a_1a_2...a_{n-1}$ *is a permutation of 2, 3, ..., n so that* $|D_{a_1}|\geq|D_{a_2}|\geq ... \geq|D_{a_{n-1}}|$. *Then,* $|D_{a_{n-2}}|=|D_{a_{n-1}}|=0$, *and* $|D_{a_k}|\leq\left\lfloor \frac{n-3}{k} \right\rfloor$ *for* $1\leq k\leq n-3$.

Lemma 5. *Suppose* $u=u_1u_2...u_n$ *and* $v=v_1v_2...v_n$ *are two distinct vertices of* S_n. *If* $|D_k|\leq n-4$ *for some* $2\leq k\leq n$ *and* $u_k\neq v_k$, *then a healthy* P_{n-1} *whose first (last) (n-1)-vertex contains u (v) can be generated after a k-partition on* S_n.

Proof. After a *k*-partition, S_n is partitioned into *n* (*n*-1)-vertices, which form a K_n^{n-1}. Clearly, *u* and *v* belong to two distinct (*n*-1)-vertices of the K_n^{n-1}. Since $|D_k|\leq n-4$, the K_n^{n-1} contains at most *n*-4 faulty (*n*-1)-edges. This lemma then follows as a consequence of Lemma 3. Q.E.D.

***Lemma 6**. Given a healthy R_r which results from executing an $(i_1, i_2, ..., i_{n-r})$-partition on S_n, if $|D_k|\le r-4$ for some $k \in\{2, 3, \ldots, n\}-\{i_1, i_2, ..., i_{n-r}\}$, then a healthy R_{r-1} can be generated after a k-partition on the R_r, where $4\le r\le n-1$ and $n\ge 5$.*

Proof. Suppose $R_r=[A_0, A_1, \ldots, A_{n(n-1)(n-2)\cdots(r+1)-1}]$ whose r-edges are fault-free. After a k-partition on the R_r, each A_i forms a K_r^{r-1}, where $0\le i\le n(n-1)(n-2)\cdots(r+1)-1$. We note that each A_i contains r $(r-1)$-vertices and there are $r-1$ $(r-1)$-edges between every two adjacent r-vertices of the R_r. Since $r\ge 4$, two distinct $(r-1)$-vertices X_i and Y_i can be determined in each A_i so that X_i and Y_i are adjacent to $Y_{(i-1) \bmod n(n-1)(n-2)\cdots(r+1)}$ and $X_{(i+1) \bmod n(n-1)(n-2)\cdots(r+1)}$, respectively. Since the $(r-1)$-edges of each K_r^{r-1} belong to the kth dimension and $|D_k|\le r-4$, a healthy hamiltonian X_i-Y_i path (the abbreviation of a path from X_i to Y_i) can be established in each K_r^{r-1} formed by A_i according to Lemma 3. Hence, a healthy R_{r-1} can be generated by interleaving the obtained hamiltonian paths with $(r-1)$-edges $(Y_i, X_{(i+1) \bmod n(n-1)(n-2)\cdots(r+1)})$. Q.E.D.

The following lemma can be proved similarly.

***Lemma 7.** Given a healthy P_r which results from executing an $(i_1, i_2, ..., i_{n-r})$-partition on S_n and whose first (last) r-vertex is the beginning (ending) r-vertex, if $|D_k|\le r-4$ for some $k \in\{2, 3, \ldots, n\}-\{i_1, i_2, ..., i_{n-r}\}$, then a healthy (or fault-free, see below) P_{r-1} whose first (last) (r-1)-vertex is the beginning (ending) (r-1)-vertex can be generated after a k-partition on the P_r, where $4\le r\le n-1$ and $n\ge 5$.*

Remarks. Since edge faults contained in the P_r may be excluded from the obtained P_{r-1}, the latter will become fault-free if all edge faults are excluded.

***Lemma 8**. Suppose $u=u_1u_2...u_n$ and $v=v_1v_2...v_n$ are the beginning and ending vertices, respectively. Given a healthy R_r which results from executing an $(i_1, i_2, ..., i_{n-r})$-partition on S_n and whose beginning r-vertex and ending r-vertex are the same, if $u_k\ne v_k$ and $|D_k|\le r-4$ for some $k \in\{2, 3, \ldots, n\}-\{i_1, i_2, ..., i_{n-r}\}$, then a healthy P_{r-1} whose first (last) (r-1)-vertex is the beginning (ending) (r-1)-vertex can be generated after a k-partition on the R_r, where $4\le r\le n-1$ and $n\ge 5$.*

Proof. Suppose $R_r=[A_0, A_1, \ldots, A_{n(n-1)(n-2)\cdots(r+1)-1}]$, and without loss of generality, assume u and v belong to A_j, where $0\le j\le n(n-1)(n-2)\cdots(r+1)-1$. After a k-partition on the R_r, each r-vertex of the R_r forms a K_r^{r-1}. Since $|D_k|\le r-4$, by Lemma 3 each K_r^{r-1} with faulty $(r-1)$-edges removed is hamiltonian-connected.

Let U and V denote the beginning and ending $(r-1)$-vertices, respectively, in A_j. Since $u_k\ne v_k$, $U\ne V$ holds. Since there are $r-1$ $(r-1)$-edges between A_j and $A_{(j-1) \bmod n(n-1)(n-2)\cdots(r+1)}$, U or V are connected to $A_{(j-1) \bmod n(n-1)(n-2)\cdots(r+1)}$.

Without loss of generality, we assume that U is connected to $A_{(j\text{-}1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$. It is easy to establish a healthy cycle C=(C_0, C_1, …, $C_{r\text{-}2}$) of length r-1 in the $K_r^{r\text{-}1}$ formed by A_j which contains all the (r-1)-vertices of the $K_r^{r\text{-}1}$ but U. Without loss of generality, suppose V=C_t, where $0 \le t \le r\text{-}2$. Similarly, $C_{(t\text{-}1) \bmod (r\text{-}1)}$ or $C_{(t+1) \bmod (r\text{-}1)}$ is connected to $A_{(j+1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$, and we assume without loss of generality that $C_{(t+1) \bmod (r\text{-}1)}$ is connected to $A_{(j+1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$.

Since $r \ge 4$, for $0 \le i \le n(n\text{-}1)(n\text{-}2)\cdots(r+1)\text{-}1$ and $i \ne j$, two distinct (r-1)-vertices X_i and Y_i can be determined in each A_i so that X_i and Y_i are adjacent to $Y_{(i\text{-}1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$ and $X_{(i+1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$, respectively ($X_{(j+1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$ is adjacent to $C_{(t+1) \bmod (r\text{-}1)}$ and $Y_{(j\text{-}1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$ is adjacent to U). Then, a desired $P_{r\text{-}1}$ can be generated as follows: (1) establish a healthy path ($C_{(t+1) \bmod (r\text{-}1)}$, $C_{(t+2) \bmod (r\text{-}1)}$, …, $C_{r\text{-}2}$, C_0, C_1, …, C_t) in the $K_r^{r\text{-}1}$ formed by A_j, (2) establish a healthy hamiltonian X_i-Y_i path in each $K_r^{r\text{-}1}$ formed by A_i, and (3) interleave the path of (1) and the hamiltonian paths of (2) with (r-1)-edges ($Y_{(j\text{-}1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$, U), ($X_{(j+1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$, $C_{(t+1) \bmod (r\text{-}1)}$), and ($Y_i$, $X_{(i+1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$) for all $i \ne j$. Q.E.D.

Remarks. X_i and Y_i used in the proofs of Lemma 6 and Lemma 8 are referred to as a pair of *entry and exit (r-1)-vertices* of A_i in the rest of this paper.

The following procedure can generate an $R_{r\text{-}1}$ from an R_r=[A_0, A_1, ..., $A_{n(n\text{-}1)(n\text{-}2)\cdots(r+1)\text{-}1}$] that results from executing an (i_1, i_2, ..., $i_{n\text{-}r}$)-partition on S_n, where $4 \le r \le n\text{-}1$ and $n \ge 5$.

Procedure 2.

(1) Execute a k-partition on the R_r, where $k \in \{2, 3, \ldots, n\}\text{-}\{i_1, i_2, ..., i_{n\text{-}r}\}$.

(2) For $0 \le i \le n(n\text{-}1)(n\text{-}2)\cdots(r+1)\text{-}1$, determine a pair of entry and exit (r-1)-vertices, say X_i and Y_i, of each A_i.

(3) Construct a hamiltonian X_i-Y_i path in each $K_r^{r\text{-}1}$ formed by A_i so that its first two (r-1)-vertices are connected to $A_{(i\text{-}1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$ and its last two (r-1)-vertices are connected to $A_{(i+1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$.

(4) Interleave all the hamiltonian paths with (r-1)-edges (Y_i, $X_{(i+1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$).

Lemma 9. *The $R_{r\text{-}1}$ generated by Procedure 2 satisfies the condition (2) of Definition 7.*

Proof. Suppose $B=\langle b_1b_2...b_n\rangle_{r\text{-}1}$, $C=\langle c_1c_2...c_n\rangle_{r\text{-}1}$, and $D=\langle d_1d_2...d_n\rangle_{r\text{-}1}$ are three consecutive (r-1)-vertices in the $R_{r\text{-}1}$, and let p=*dif*(B, C) and q=*dif*(C, D). We show $b_p \ne d_q$ according to three cases. If B=Y_j for some $0 \le j \le n(n\text{-}1)(n\text{-}2)\cdots(r+1)\text{-}1$, then C=$X_{(j+1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$ and D is the second (r-1)-vertex in the hamiltonian path established for the K_r^{r-1} formed by $A_{(j+1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$. Hence, $p \ne q = k$. According to the step (3) of Procedure 2, D is connected to A_j. Recall that the (r-1)-vertex in A_j ($A_{(j+1) \bmod n(n\text{-}1)(n\text{-}2)\cdots(r+1)}$) that is not connected to $A_{(j+1) \bmod n(n-1)(n-2)\cdots(r+1)}$ (A_j) is $\langle b_1...b_{q-1}d_pb_{q+1}...b_n\rangle_{r-1}$ ($\langle d_1...d_{q-1}b_pd_{q+1}...d_n\rangle_{r-1}$), where $b_q=d_p\neq b_p=d_q$ and $b_l=d_l$ for all $1\leq l\leq n$ and $l\notin\{p, q\}$. Hence, if $b_p=d_q$, then D is not connected to A_j, which is a contradiction.

If $D=X_j$ for some $0\leq j\leq n(n-1)(n-2)\cdots(r+1)-1$, then $b_p\neq d_q$ can be shown similarly. Otherwise, if B, C, and D belong to the same r-vertex, then $p=dif(B, C)=dif(B, D)=dif(C, D)=q=k$, which implies $b_p\neq d_q$. Q.E.D.

Remarks. Procedure 2 and Lemma 9 remain valid if P_{r-1} and P_r are substituted for R_{r-1} and R_r, respectively.

We have explained in Section 2 that after a partition on a $P_r=[A_0, A_1, ..., A_{n(n-1)(n-2)\cdots(r+1)-1}]$, there is a unique ($r$-1)-vertex of A_i that is not connected to A_{i+1}, where $0\leq i\leq n(n-1)(n-2)\cdots(r+1)-2$. Hence, the following lemma is immediate.

Lemma 10. *Suppose $u=u_1u_2...u_n$ and $v=v_1v_2...v_n$ are the beginning and ending vertices, respectively. Given a $P_r=[A_0, A_1, ..., A_{n(n-1)(n-2)\cdots(r+1)-1}]$ which results from executing an $(i_1, i_2, ..., i_{n-r})$-partition on S_n and whose first (last) r-vertex is the beginning (ending) r-vertex, if $u_k=x_{dif(A_0,A_1)}$ ($v_k=y_{dif(A_{n(n-1)(n-2)\cdots(r+1)-2}, A_{n(n-1)(n-2)\cdots(r+1)-1})}$) for some $k\in\{2, 3, \ldots, n\}-\{i_1, i_2, ..., i_{n-r}\}$, then the beginning (ending) (r-1)-vertex is not connected to A_1 ($A_{n(n-1)(n-2)\cdots(r+1)-2}$) after a k-partition on the P_r, where $2\leq r\leq n-1$, $A_1=\langle x_1x_2...x_n\rangle_r$, and $A_{n(n-1)(n-2)\cdots(r+1)-2}=\langle y_1y_2...y_n\rangle_r$.*

The following lemma is an immediate consequence of Lemma 3.5 and Lemma 3.6 of [19].

Lemma 11*[19]. In S_n, a good fault-free P_3 can be obtained from a good fault-free P_4, which again can be obtained from a fault-free P_5 whose first (last) 5-vertex is the beginning (ending) 5-vertex, where $n>5$.*

We use $dist(u, v)$ to denote the distance between vertices u and v. The following lemma originally appeared in Lemma 3.10 of [19].

Lemma 12*[19]. Given two distinct vertices u and v of S_n, a longest fault-free u-v path can be obtained from a good fault-free P_3 in S_n. The u-v path has length n!-1 if dist(u, v) is odd, and n!-2 if dist(u, v) is even.*

4 Fault-tolerant embedding of longest paths

In this section, given a crucial permutation $a_1a_2...a_{n-1}$ with respect to u and v, we aim to construct a longest fault-free u-v path in S_n. Since S_n is a bipartite graph with two partite sets of equal size (see [22]), the u-v path has maximum length $n!-1$ if $dist(u, v)$ is odd, and $n!-2$ if $dist(u, v)$ is even. Unless specified particularly, we assume $|F_e|=n-3$ throughout this section. The construction depends on the values of

$|D_{a_1}|\geq|D_{a_2}|\geq ... \geq|D_{a_{n-1}}|$, where $|D_{a_{n-2}}|=|D_{a_{n-1}}|=0$ according to Lemma 4. First, three particular cases: (1) $|D_{a_{n-3}}|=1$, (2) $|D_{a_1}|=n-3$, and (3) $|D_{a_{n-4}}|=1$ and $|D_{a_{n-3}}|=0$, are discussed. The case (1), which implies $|D_{a_1}|=|D_{a_2}|=\cdots=|D_{a_{n-4}}|=1$, is discussed in Section 4.1. The case (2), which implies $|D_{a_2}|=|D_{a_3}|=\cdots=|D_{a_{n-3}}|=0$, is discussed in Section 4.2. The case (3), which implies $|D_{a_1}|=2$ and $|D_{a_2}|=|D_{a_3}|=\cdots=|D_{a_{n-5}}|=1$, is discussed in Section 4.3. A general case of $|D_{a_1}|\neq n-3$ and $|D_{a_{n-4}}|=0$ is discussed in Section 4.4. The main results of this section are shown in Section 4.5. Compared with the other three cases, the discussion for the case (1) is complex and lengthy.

4.1 $|D_{a_{n-3}}|=1$

Since $|F_e|=n-3$, $|D_{a_{n-3}}|=1$ implies $|D_{a_1}|=|D_{a_2}|=\cdots=|D_{a_{n-4}}|=1$. The break dimension may be a_1 or a_{n-2}. The former case is discussed in the following lemma; the latter case is discussed in Lemma 18.

Lemma 13. *Suppose u and v are two vertices of S_n, and let $a_1a_2...a_{n-1}$ be a crucial permutation with respect to u and v, where $n\geq6$. If $|D_{a_{n-3}}|=1$ and a_1 is the break dimension, a longest fault-free u-v path can be constructed in S_n.*

Proof. According to Lemma 12, a longest fault-free u-v path can be obtained from a good fault-free P_3. Hence, we only need to construct a good fault-free P_3. For the purpose, we first construct a healthy P_5 whose first (last) 5-vertex is the beginning (ending) 5-vertex as follows.

Since a_1 is the break dimension, we have $u_{a_1}\neq v_{a_1}$. Moreover, $2\leq a_1\leq n$ and $|D_{a_1}|=1<n-4$. By Lemma 5, a healthy P_{n-1} whose first (last) $(n-1)$-vertex is the beginning (ending) $(n-1)$-vertex can be easily constructed after an a_1-partition. If $n=6$, $P_{n-1}=P_5$. If $n>6$, we suppose that a healthy P_j whose first (last) j-vertex is the beginning (ending) j-vertex can be constructed after an $(a_1, a_2, \ldots, a_{n-j})$-partition, where $6\leq j\leq n-1$. Since $|D_{a_{n-j+1}}|=1\leq j-4$, by Lemma 7 a healthy P_{j-1} whose first (last) $(j-1)$-vertex is the beginning (ending) $(j-1)$-vertex can be constructed from the P_j after an a_{n-j+1}-partition. Let $j=n-1$ initially. By induction on j, a desired $P_5=[A_{5,0}, A_{5,1}, \ldots, A_{5,n(n-1)(n-2)\cdots6-1}]$ can be obtained as $j=6$.

Next we show that a good fault-free P_3 can be constructed from the P_5 after an (a_{n-4}, a_{n-3})-partition. First an a_{n-4}-partition is executed on the P_5, and so its each 5-vertex forms a K_5^4. Let U_4 (V_4) be the beginning (ending) 4-vertex, and $W=\langle w_1w_2...w_n\rangle_4$ ($Z=\langle z_1z_2...z\rangle_4$) be the 4-vertex in $A_{5,0}$ ($A_{5,n(n-1)(n-2)\cdots 6-1}$) with $w_{a_{n-4}} = u_{a_{n-3}}$ ($z_{a_{n-4}} = v_{a_{n-3}}$), where $u=u_1u_2...u_n$ and $v=v_1v_2...v_n$ are assumed. A desired P_3 can be constructed according to the following three cases.

Case 1. (U_4, W) is faulty. Suppose $D_{a_{n-3}}$ contains the edge fault ($x_1x_2...x_n$, $y_1y_2...y_n$), where $y_1= x_{a_{n-3}}$, $y_{a_{n-3}}=x_1$, and $x_i=y_i$ for all $1\le i\le n$ and $i \notin \{1, a_{n-3}\}$. The edge fault ($x_1x_2...x_n$, $y_1y_2...y_n$) is located in some 4-vertex. If ($x_1x_2...x_n$, $y_1y_2...y_n$) belongs to U_4, then $x_i=y_i=u_i$ for $i \in \{a_1, a_2,\ldots, a_{n-4}\}$, and $\{y_1, y_{a_{n-3}}\} \subset \{u_1, u_{a_{n-3}}, u_{a_{n-2}}, u_{a_{n-1}}\}$.

When $(y_1, y_{a_{n-3}})=(u_{a_{n-3}}, u_{a_{n-2}})$, a partially good healthy P_4 is first constructed by establishing a hamiltonian path for each K_5^4 as follows. Let $T=\langle t_1t_2...t_n\rangle_4$ be the 4-vertex in $A_{5,0}$ with $t_{a_{n-4}} = u_{a_{n-1}}$. The path ($U_4$, T, B_3, B_4, B_5) is established as a healthy hamiltonian path for the K_5^4 formed by $A_{5,0}$, where B_3, B_4, and B_5 are the other three 4-vertices in $A_{5,0}$ than U_4 and T, and both B_4 and B_5 are connected to $A_{5,1}$. The path (D_1, D_2, D_3, Z, V_4) is established as a healthy hamiltonian path for the K_5^4 formed by $A_{5,n(n-1)(n-2)\cdots 6-1}$, where D_1, D_2, and D_3 are the other three 4-vertices in $A_{5,n(n-1)(n-2)\cdots 6-1}$ than V_4 and Z, and both D_1 and D_2 are connected to $A_{5,n(n-1)(n-2)\cdots 6-2}$. For $1\le i\le n(n-1)(n-2)\cdots 6-2$, a healthy hamiltonian $X_{5,i}$-$Y_{5,i}$ path for each K_5^4 formed by $A_{5,i}$ can be established so that the first two 4-vertices are connected to $A_{5,i-1}$ and the last two 4-vertices are connected to $A_{5,i+1}$, where $X_{5,i}$ and $Y_{5,i}$ are a pair of entry and exit 4-vertices of $A_{5,i}$. The obtained healthy hamiltonian paths interleaved with 4-edges form a healthy $P_4=[A_{4,0}, A_{4,1}, \ldots, A_{4,n(n-1)(n-2)\cdots 5-1}]$.

The P_4 is partially good for the reasons as follows. The condition (1) of Definition 7 holds because $A_{4,0}$ is the beginning 4-vertex and $A_{4,n(n-1)(n-2)\cdots 5-1}$ is the ending 4-vertex. Since the construction of the P_4 coincides with Procedure 2, the condition (2) of Definition 7 is assured by Lemma 9. The condition (3’) of Definition 8 is assured by Lemma 10 (an a_{n-3}-partition is necessary in order to satisfy the condition (3’)).

An a_{n-3}-partition is then executed on the P_4, and so its each 4-vertex forms a K_4^3. Let U_3 be the beginning 3-vertex, $E=\langle e_1e_2...e_n\rangle_3$ be the 3-vertex with $e_{a_{n-3}} = u_{a_{n-1}}$, $F=\langle f_1f_2...f_n\rangle_3$ be the 3-vertex with $f_{a_{n-3}} = u_{a_{n-2}}$, and G be the other 3-vertex, all in $A_{4,0}(=U_4)$. We note that (U_3, F) is faulty because $x_1x_2...x_n$ belongs to U_3 and $y_1y_2...y_n$ belongs to F, and E is not connected to $A_{4,1}(=T)$ because $e_{a_{n-3}} = u_{a_{n-1}} = t_{a_{n-4}} = t_{dif(A_{4,0},A_{4,1})}$. A healthy hamiltonian path for the K_4^3 formed by $A_{4,0}$ can be established as (U_3, E, F, G).

On the other hand, let V_3 be the ending 3-vertex, $J=\langle j_1j_2...j_n\rangle_3$ be the 3-vertex with $j_{a_{n-3}} = v_{a_{n-1}}$, and K_1

and K_2 be the other two 3-vertices, all in $A_{4,n(n-1)(n-2)\cdots5-1}$ (=V_4). We note that V_3 is not connected to $A_{4,n(n-1)(n-2)\cdots5-2}$ as a consequence of condition (3') of Definition 8. A healthy hamiltonian path for the K_4^3 formed by $A_{4,n(n-1)(n-2)\cdots5-1}$ can be established as (K_1, K_2, J, V_3).

For $1 \le i \le n(n-1)(n-2)\cdots5-2$, a healthy hamiltonian $X_{4,i}$-$Y_{4,i}$ path for each K_4^3 formed by $A_{4,i}$ can be established so that the first two 3-vertices are connected to $A_{4,i-1}$ and the last two 4-vertices are connected to $A_{4,i+1}$, where $X_{4,i}$ and $Y_{4,i}$ is a pair of entry and exit 4-vertices of $A_{4,i}$. The obtained healthy hamiltonian paths interleaved with 4-edges form a fault-free P_3. Since the construction of the P_3 coincides with Procedure 2, the P_3 is good according to Lemma 9 and Lemma 10 (an a_{n-1}-partition is necessary in order to satisfy the condition (3) of Definition 7).

The discussion for (y_1, $y_{a_{n-3}}$)=($u_{a_{n-2}}$, $u_{a_{n-3}}$) or (u_1, $u_{a_{n-3}}$) or ($u_{a_{n-3}}$, u_1) or (u_1, $u_{a_{n-1}}$) or ($u_{a_{n-1}}$, u_1) is all the same. On the other hand, the discussion for (y_1, $y_{a_{n-3}}$)=($u_{a_{n-1}}$, $u_{a_{n-2}}$) or ($u_{a_{n-2}}$, $u_{a_{n-1}}$) or ($u_{a_{n-2}}$, u_1) or (u_1, $u_{a_{n-2}}$) or ($u_{a_{n-1}}$, $u_{a_{n-3}}$) or ($u_{a_{n-3}}$, $u_{a_{n-1}}$) is very similar, and we only show the difference. For (y_1, $y_{a_{n-3}}$)=($u_{a_{n-1}}$, $u_{a_{n-2}}$) or ($u_{a_{n-2}}$, $u_{a_{n-1}}$), (U_3, E, G, F) is established as a healthy hamiltonian path for the K_4^3 formed by $A_{4,0}$. For (y_1, $y_{a_{n-3}}$)=($u_{a_{n-2}}$, u_1) or (u_1, $u_{a_{n-2}}$) or ($u_{a_{n-1}}$, $u_{a_{n-3}}$) or ($u_{a_{n-3}}$, $u_{a_{n-1}}$), $T=\langle t_1t_2\ldots t_n\rangle_4$ is changed to the 4-vertex in $A_{5,0}$ with $t_{a_{n-4}} = u_{a_{n-2}}$ and (U_3, F, E, G) is established as a healthy hamiltonian path for the K_4^3 formed by $A_{4,0}$. An a_{n-2}-partition is necessary in order to satisfy the condition (3) of Definition 7.

If ($x_1x_2\ldots x_n$, $y_1y_2\ldots y_n$) belongs to V_4, then $x_i=y_i=v_i$ for $i \in \{a_1, a_2, \ldots, a_{n-4}\}$ and $\{y_1, y_{a_{n-3}}\} \subset \{v_1, v_{a_{n-3}}, v_{a_{n-2}}, v_{a_{n-1}}\}$. A partially good healthy P_4 can be obtained all the same (and so the ending 3-vertex in the P_4 is not connected to $A_{4,n(n-1)(n-2)\cdots5-2}$). An a_{n-3}-partition is then executed on the P_4. Let V_3 be the ending 3-vertex, $H=\langle h_1h_2\ldots h_n\rangle_3$ be the 3-vertex with $h_{a_{n-3}} = v_{a_{n-1}}$, $L=\langle l_1l_2\ldots l_n\rangle_3$ be the 3-vertex with $l_{a_{n-3}} = v_{a_{n-2}}$, and M be the other 3-vertex, all in $A_{4,n(n-1)(n-2)\cdots5-1}$(=$V_4$).

When (y_1, $y_{a_{n-3}}$)=($v_{a_{n-3}}$, $v_{a_{n-2}}$), (V_3, L) is faulty because $x_1x_2\ldots x_n$ belongs to V_3 and $y_1y_2\ldots y_n$ belongs to L. Then, (U_3, E, F, G) and (V_3, H, L, M) are established as healthy hamiltonian paths for the K_4^3's formed by $A_{4,0}$ and $A_{4,n(n-1)(n-2)\cdots5-1}$, respectively. A good fault-free P_3 can be constructed similarly (an a_{n-1}-partition is necessary in order to satisfy the condition (3) of Definition 7). When (y_1, $y_{a_{n-3}}$)=($v_{a_{n-2}}$, $v_{a_{n-3}}$) or ($v_{a_{n-3}}$, v_1) or (v_1, $v_{a_{n-3}}$) or ($v_{a_{n-1}}$, v_1) or (v_1, $v_{a_{n-1}}$), a good fault-free P_3 can be constructed all the same.

For the other values of y_1 and $y_{a_{n-3}}$, some modifications are necessary. If (y_1, $y_{a_{n-3}}$)=($v_{a_{n-1}}$, $v_{a_{n-2}}$) or ($v_{a_{n-2}}$, $v_{a_{n-1}}$), then (V_3, H, M, L) is established as a healthy hamiltonian path for the K_4^3 formed by $A_{4,n(n-1)(n-2)\cdots 5\text{-}1}$. If ($y_1$, $y_{a_{n-3}}$)=($v_{a_{n-2}}$, v_1) or (v_1, $v_{a_{n-2}}$) or ($v_{a_{n-1}}$, $v_{a_{n-3}}$) or ($v_{a_{n-3}}$, $v_{a_{n-1}}$), then $T=\langle t_1t_2...t_n\rangle_4$ is changed to the 4-vertex in $A_{5,0}$ with $t_{a_{n-4}}=u_{a_{n-2}}$, and (U_3, F, E, G) and (V_3, L, H, M) are established as healthy hamiltonian paths for the K_4^3's formed by $A_{4,0}$ and $A_{4,n(n-1)(n-2)\cdots 5\text{-}1}$, respectively.

If ($x_1x_2...x_n$, $y_1y_2...y_n$) belongs to some other 4-vertex than U_4 and V_4, then a partially good healthy P_4 can be obtained all the same. After an $a_{n\text{-}3}$-partition on the P_4, the faulty 3-edge is located in some $A_{4,k}$, where $1\leq k\leq n(n\text{-}1)(n\text{-}2)\cdots 5\text{-}2$. Since the P_4 is partially good, by Lemma 1 each 3-vertex of $A_{4,k}$ is connected to $A_{4,k\text{-}1}$ or $A_{4,k+1}$. It is not difficult to establish a healthy hamiltonian path for the K_4^3 formed by $A_{4,k}$ so that the first (last) two vertices in the hamiltonian path are connected to $A_{4,k\text{-}1}$ ($A_{4,k+1}$). Similarly a good fault-free P_3 can be constructed (an $a_{n\text{-}1}$-partition is necessary in order to satisfy the condition (3) of Definition 7).

Case 2. (V_4, Z) is faulty. The discussion is similar to Case 1.

Case 3. Both (U_4, W) and (V_4, Z) are fault-free. It is not difficult to construct a good P_4=[$A_{4,0}$, $A_{4,1}$, …, $A_{4,n(n\text{-}1)(n\text{-}2)\cdots 5\text{-}1}$] from the P_5 after an $a_{n\text{-}4}$-partition so that $A_{4,1}=W$ and $A_{4,n(n\text{-}1)(n\text{-}2)\cdots 5\text{-}2}=Z$ (an $a_{n\text{-}3}$-partition is necessary in order to satisfy the condition (3) of Definition 7). Since $|D_{a_{n-4}}|=1$, there is at most one faulty 4-edge in the P_4. Without loss of generality, we assume that the P_4 contains one faulty 4-edge ($A_{4,r}$, $A_{4,r+1}$), where $1\leq r\leq n(n\text{-}1)(n\text{-}2)\cdots 5\text{-}3$.

An $a_{n\text{-}3}$-partition is then executed on the P_4, and so its each 4-vertex forms a K_4^3. Since $|D_{a_{n-3}}|=1$, there is one faulty 3-edge in some K_4^3 (equivalently, one faulty 3-edge in some 4-vertex of the P_4). If the faulty 3-edge is located in $A_{4,0}$, then a healthy hamiltonian path for the K_4^3 formed by $A_{4,0}$ is established as (S, M, O, Q), where S is the beginning 3-vertex, $M=\langle m_1m_2...m_n\rangle_3$ satisfies $m_{a_{n-3}}\neq u_1$ (hence, $m_{a_{n-3}}\in\{u_{a_{n-1}}, u_{a_{n-2}}\}$), and O and Q are the other two 3-vertices. Without loss of generality, we assume $m_{a_{n-3}}=u_{a_{n-2}}$. A hamiltonian path for the K_4^3 formed by $A_{4,n(n\text{-}1)(n\text{-}2)\cdots 5\text{-}1}$ is established as (S', M', O', Q'), where S' is the ending 3-vertex, $M'=\langle m'_1\, m'_2 \ldots m'_n\rangle_3$ satisfies $m'_{a_{n-3}}=v_{a_{n-2}}$, and O' and Q' are the other two 3-vertices. Since ($A_{4,r}$, $A_{4,r+1}$) contains two fault-free 3-edges, a good fault-free P_3 can be constructed similarly (an $a_{n\text{-}2}$-partition is necessary in order to satisfy the condition (3) of Definition 7).

If the faulty 3-edge is located in $A_{4,n(n\text{-}1)(n\text{-}2)\cdots 5\text{-}1}$, the discussion is similar. Otherwise, if the faulty 3-edge is located in some $A_{4,k}$, a healthy hamiltonian path for the K_4^3 formed by $A_{4,k}$ can be established so that its first (last) two vertices are connected to $A_{4,k-1}$ ($A_{4,k+1}$), where $1 \le k \le n(n-1)(n-2)\cdots 5-2$. A good fault-free P_3 can be constructed similarly (an a_{n-2}-partition is necessary in order to satisfy the condition (3) of Definition 7).

Q.E.D.

The following two Lemmas are prerequisite to Lemma 18.

Lemma 14. *Suppose u and v are two vertices of S_n, and $a_1a_2...a_{n-1}$ is a crucial permutation with respect to u and v. If a_{n-2} is the break dimension, dist(u, v)=1 or 2 or 3.*

Proof. Suppose $u=u_1u_2...u_n$ and $v=v_1v_2...v_n$. Since a_{n-2} is the break dimension, $u_{a_{n-2}} \neq v_{a_{n-2}}$ and $u_{a_i} = v_{a_i}$ for all $1 \le i \le n-3$. Thus, $\{u_1, u_{a_{n-2}}, u_{a_{n-1}}\}=\{v_1, v_{a_{n-2}}, v_{a_{n-1}}\}$. If $u_1=v_1$, then $u_{a_{n-2}} = v_{a_{n-1}}$ and $u_{a_{n-1}} = v_{a_{n-2}}$, which implies $dist(u, v)=3$. If $u_1 \neq v_1$ and $u_{a_{n-1}} = v_{a_{n-1}}$, then $u_1 = v_{a_{n-2}}$ and $u_{a_{n-2}} = v_1$, which implies $dist(u, v)=1$. If $u_1 \neq v_1$ and $u_{a_{n-1}} \neq v_{a_{n-1}}$, then $dist(u, v)=2$. Q.E.D.

Lemma 15. *Suppose u and v are two vertices of S_n, and $a_1a_2...a_{n-1}$ is a crucial permutation with respect to u and v. If $|D_{a_1}|=|D_{a_2}|= \cdots =|D_{a_{n-3}}|=1$ and a_{n-2} is the break dimension, then after executing an (a_1, a_2, ..., a_{n-3})-partition on S_n, a fault-free R_3 can be generated so that $x_{dif(X,Y)} \neq z_{dif(Y,Z)}$ for every three consecutive 3-vertices $X=\langle x_1x_2...x_n\rangle_3$, $Y=\langle y_1y_2...y_n\rangle_3$, and $Z=\langle z_1z_2...z_n\rangle_3$ in the R_3.*

Proof. When $n=4$, a desired R_3 can be easily obtained after an a_1-partition because $|D_{a_1}|=1$. When $n>4$, a healthy R_4 is first constructed so that $x_{dif(X,Y)} \neq z_{dif(Y,Z)}$ for every three consecutive 4-vertices $X=\langle x_1x_2...x_n\rangle_4$, $Y=\langle y_1y_2...y_n\rangle_4$, and $Z=\langle z_1z_2...z_n\rangle_4$ in the R_4. When $n=5$, a desired R_4 can be easily obtained after an a_1-partition on S_n. When $n>5$, a healthy R_{n-1} is first constructed after an a_1-partition on S_n. A healthy $R_5=[A_{5,0}, A_{5,1}, ..., A_{5,n(n-1)(n-2)\cdots 6-1}]$ can result directly as $n=6$, or after an (a_2, a_3, …, a_{n-5})-partition according to Lemma 6 as $n>6$.

An a_{n-4}-partition is then executed on the R_5, and so its each 5-vertex forms a K_5^4. Since $|D_{a_{n-4}}|=1$, we assume without loss of generality that the faulty 4-edge is located in the K_5^4 formed by some $A_{5,k}$, where $0 \le k \le n(n-1)(n-2)\cdots 6-1$. Let X_k and Y_k be the two 4-vertices of $A_{5,k}$ that are not connected to $A_{5,(k+1) \bmod n(n-1)(n-2)\cdots 6}$ and $A_{5,(k-1) \bmod n(n-1)(n-2)\cdots 6}$, respectively. If $X_k \neq Y_k$, by Lemma 3 a healthy hamiltonian X_k-Y_k path can be established for the K_5^4 formed by $A_{5,k}$. If $X_k=Y_k$, a healthy hamiltonian path (B_1, B_2, X_k, B_4, B_5) for the K_5^4 formed by $A_{5,k}$ can be easily established, where B_1, B_2, B_4, and B_5 are all 4-vertices of $A_{5,k}$. For $0 \le i \le n(n-1)(n-2)\cdots 6-1$ and $i \neq k$, a healthy hamiltonian $X_{5,i}$-$Y_{5,i}$ path for each K_5^4 formed by $A_{5,i}$ can be established so that its first two 4-vertices are connected to $A_{5,(i-1) \bmod n(n-1)(n-2)\cdots 6}$ and its last two 4-vertices are connected to $A_{5,(i+1) \bmod n(n-1)(n-2)\cdots 6}$, where $X_{5,i}$ and $Y_{5,i}$ are a pair of entry and exit 4-vertices of $A_{5,i}$. A healthy R_4 can thus result. According to Lemma 9, the R_4 satisfies the condition (2) of Definition 7.

Next, we construct a desired R_3 from the R_4 as follows. Suppose the R_4=[$A_{4,0}$, $A_{4,1}$, ..., $A_{4,n(n-1)(n-2)\cdots 5-1}$]. First, an a_{n-3}-partition is executed on the R_4, and so its each 4-vertex forms a K_4^3. Since $|D_{a_{n-3}}|$=1, we assume without loss of generality that the faulty 3-edge is located in the K_4^3 formed by some $A_{4,k}$, where $0 \le k \le n(n-1)(n-2)\cdots 5-1$. Let B and D be the two 3-vertices of $A_{4,k}$ that are not connected to $A_{(k+1) \bmod n(n-1)(n-2)\cdots 5}$ and $A_{(k-1) \bmod n(n-1)(n-2)\cdots 5}$, respectively, and E and F be the other two 3-vertices of $A_{4,k}$. Since the R_4 satisfies the condition (2) of Definition 7, $B \neq D$ by Lemma 1. Hence, E and F are connected to both $A_{(k-1) \bmod n(n-1)(n-2)\cdots 5}$ and $A_{(k+1) \bmod n(n-1)(n-2)\cdots 5}$. A healthy hamiltonian path for the K_4^3 formed by $A_{4,k}$ is established as (B, E, D, F) if (E, F) is faulty, (B, F, E, D) if (E, B) or (F, D) is faulty, and (B, E, F, D) if (B, D) or (F, B) or (E, D) is faulty. Similarly, an R_3 can be constructed by establishing healthy hamiltonian paths for the other K_4^3's. Since the construction of the R_3 coincides with Procedure 2, it has the desired property according to Lemma 9. Q.E.D.

Recall that the structure of S_3 is a cycle of length six. The following two lemmas were shown in [38].

Lemma 16*[38]. Suppose X and Y are two adjacent 3-vertices in a P_3, and let (c_0, c_1, ..., c_5) be the cycle formed by X. The vertices of X that are connected to Y are c_j and $c_{(j+3) \bmod 6}$ for some $0 \le j \le 5$.*

Lemma 17 *[38]. Suppose $X=\langle x_1x_2...x_n\rangle_3$, $Y=\langle y_1y_2...y_n\rangle_3$, and $Z=\langle z_1z_2...z_n\rangle_3$ are three consecutive 3-vertices in a P_3. If $x_{dif(X,Y)} \neq z_{dif(Y,Z)}$, the two vertices of Y that are connected to X are disjoint from the two of Y that are connected to Z.*

Lemma 18*. Suppose u and v are two vertices of S_n, and $a_1a_2...a_{n-1}$ is a crucial permutation with respect to u and v. If $|D_{a_{n-3}}|$=1 and a_{n-2} is the break dimension, a longest fault-free u-v path in S_n can be constructed as dist(u, v)≠3, and a fault-free u-v path of length n!-3 in S_n can be constructed as dist(u, v)= 3.*

Proof. By Lemma 14, $dist(u, v)$=1 or 2 or 3. Since S_n is a bipartite graph, it suffices to show a fault-free u-v path of length n!-1 as $dist(u, v)$=1, n!-2 as $dist(u, v)$=2, and n!-3 as $dist(u, v)$=3. According to Lemma 15, after executing an (a_1, a_2, ..., a_{n-3})-partition on S_n, a fault-free R_3=[A_0, A_1, ..., $A_{n(n-1)(n-2)\cdots 4-1}$] can be generated so that $x_{dif(X,Y)} \neq z_{dif(Y,Z)}$ for every three consecutive 3-vertices $X=\langle x_1x_2...x_n\rangle_3$, $Y=\langle y_1y_2...y_n\rangle_3$, and $Z=\langle z_1z_2...z_n\rangle_3$ in the R_3. Since a_{n-2} is the break dimension, $u_{a_i} = v_{a_i}$ for all $1 \le i \le n$-3. Hence, u and v are located in the same A_k

for some $0 \le k \le n(n-1)(n-2)\cdots 4-1$. In subsequent discussion, we represent A_k by a cycle $(c_0, c_1, c_2, c_3, c_4, c_5)$.

First we consider $dist(u, v)=1$. Without loss of generality, suppose $c_0=u$ and $c_1=v$. According to Lemma 16 and Lemma 17, two cases need to be discussed below.

Case 1. One of u and v is connected to $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ or $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$, and the other is not connected to either $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ or $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$. Without loss of generality, assume that u is connected to $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$. Hence, c_2 and c_5 are connected to $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ according to Lemma 16 and Lemma 17. Let $u=y_k$, and for $i=(k+1) \bmod n(n-1)(n-2)\cdots 4$, $(k+2) \bmod n(n-1)(n-2)\cdots 4$, …, $n(n-1)(n-2)\cdots 4-1$, 0, 1, ..., $(k-1) \bmod n(n-1)(n-2)\cdots 4$, sequentially determine two adjacent vertices, denoted by x_i and y_i, from each A_i so that $y_{(i-1) \bmod n(n-1)(n-2)\cdots 4}$ is adjacent to x_i and $y_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ is adjacent to c_2 or c_5. Lemma 16 and Lemma 17 assure the existence of x_i and y_i. Thus a fault-free u-v path can be generated whose length is $n!-1$ if $y_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ is adjacent to c_5, and $n!-4$ if $y_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ is adjacent to c_2. The latter is impossible because each u-v path with odd $dist(u, v)$ has odd length.

Case 2. One of u and v is connected to $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ and the other is connected to $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$. Without loss of generality, assume that u is connected to $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$ and v is connected to $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$. Then, c_3 and c_4 are connected to $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$ and $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$, respectively, according to Lemma 16. Let $c_3=y_k$, and determine x_i and y_i from each A_i like Case 1 ($y_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ is adjacent to c_4). A fault-free u-v path of length $n!-1$ can be generated.

Then we consider $dist(u, v)=2$. Without loss of generality, suppose $u=c_0$ and $v=c_2$. According to Lemma 16 and Lemma 17, two cases need to be discussed below.

Case 1. One of u and v is connected to $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ or $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$, and the other is not connected to either $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ or $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$. Without loss of generality, assume that u is connected to $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$. Let $c_3=y_k$, and determine x_i and y_i from each A_i like the situation of $dist(u, v)=1$ ($y_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ is adjacent to c_4). A fault-free u-v path of length $n!-2$ can be generated.

Case 2. One of u and v is connected to $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$ and the other is connected to $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$. Without loss of generality, assume that u is connected to $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$ and v is connected to $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$. Then, c_3 and c_5 are connected to $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$ and $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$, respectively, according to Lemma 16. Let $u=y_k$, and determine x_i and y_i from each A_i like Case 1 ($y_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ is adjacent to c_5). A fault-free u-v path of length $n!-2$ can be generated.

Finally we consider $dist(u, v)=3$. Without loss of generality, suppose $u=c_0$ and $v=c_3$. According to Lemma 16 and Lemma 17, two cases need to be discussed below.

Case 1. Both u and v are connected to $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ or $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$. Without loss of generality, assume that u and v are connected to $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$, and c_1 and c_4 are connected to $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$. Let $u=y_k$, and determine x_i and y_i from each A_i like the situation of $dist(u, v)=1$ ($y_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ is adjacent to c_1). A fault-free u-v path of length $n!-3$ can be generated.

Case 2. Both u and v are not connected to either $A_{(k-1) \bmod n(n-1)(n-2)\cdots 4}$ or $A_{(k+1) \bmod n(n-1)(n-2)\cdots 4}$. A fault-free u-v path of length $n!-3$ can be generated similarly. Q.E.D.

4.2 $|D_{a_1}|=n-3$

Since $|F_e|=n-3$, $|D_{a_1}|=n-3$ implies $|D_{a_2}|=|D_{a_3}|=\cdots=|D_{a_{n-3}}|=0$. The break dimension may be a_1 or a_2. The former is discussed in Lemma 19; the latter is discussed in Lemma 20.

Lemma 19. *Suppose u and v are two vertices of S_n, and $a_1a_2...a_{n-1}$ is a crucial permutation with respect to u and v, where $n\geq 6$. If $|D_{a_1}|=n-3$ and a_1 is the break dimension, a longest fault-free u-v path can be constructed in S_n.*

Proof. First an a_1-partition is executed on S_n, and so a K_n^{n-1} is obtained. Let f denote the number of faulty $(n-1)$-edges in the K_n^{n-1}. If $f<n-3$, or $f=n-3$ and for each faulty $(n-1)$-edge there exists another faulty $(n-1)$-edge so that they have no $(n-1)$-vertex in common, then according to Lemma 2 and Lemma 3, removing all faulty $(n-1)$-edges from the K_n^{n-1} results in a hamiltonian-connected graph. For both cases, there is a fault-free $P_{n-1}=[A_0, A_1, ..., A_{n-1}]$ in S_n, where A_0 and A_{n-1} are the beginning and ending $(n-1)$-vertices, respectively. If $n=6$, by Lemma 11 a good fault-free P_3 can be obtained from the $P_{n-1}=P_5$. According to Lemma 12, a longest fault-free u-v path can be obtained from the P_3. If $n>6$, a fault-free P_5 whose first (last) 5-vertex is the beginning (ending) 5-vertex can be obtained by Lemma 7. Similarly, a longest fault-free u-v path can be obtained by the aid of Lemma 11 and Lemma 12.

On the other hand, if $f=n-3$ and there exists a faulty $(n-1)$-edge which has a common $(n-1)$-vertex with each of the other $n-4$ faulty $(n-1)$-edges, then according to Lemma 3 removing $n-4$ faulty $(n-1)$-edges from the K_n^{n-1} results in a hamiltonian-connected graph. Hence, a P_{n-1} containing at most one faulty $(n-1)$-edge whose first (last) $(n-1)$-vertex is the beginning (ending) $(n-1)$-vertex can be obtained. If there is no faulty $(n-1)$-edge in the P_{n-1}, a longest fault-free u-v path can be obtained all the same as the above. If the P_{n-1} contains one faulty $(n-1)$-edge, a longest fault-free u-v path can be constructed according to the following two cases.

Case 1. n>6. An a_2-partition is executed on the P_{n-1}, and so its each (n-1)-vertex forms a K_{n-1}^{n-2}. Since the faulty (n-1)-edge contains exactly one edge fault, it contains n-3≥4 fault-free (n-2)-edges. The other (n-1)-edges of the P_{n-1} are fault-free, and each contains n-2≥5 fault-free (n-2)-edges. It is not difficult to construct a fault-free P_{n-2} whose first (last) (n-2)-vertex is the beginning (ending) (n-2)-vertex. A longest fault-free u-v path can be constructed, similarly (depending on n=7 or n>7), by the aid of Lemma 7, Lemma 11, and Lemma 12.

Case 2. n=6. Suppose the $P_{n-1}(=P_5)=[A_0, A_1, ..., A_5]$, and assume (A_k, A_{k+1}) is the faulty 5-edge, where $0 \le k < 5$. Since $|D_{a_2}|=0$ and (A_k, A_{k+1}) contains three fault-free 4-edges, a good fault-free P_4 can be obtained after an a_2-partition. A longest fault-free u-v path can be obtained by Lemma 11 and Lemma 12. Q.E.D.

Lemma 20. *Suppose u and v are two vertices of* S_n, *and* $a_1a_2...a_{n-1}$ *is a crucial permutation with respect to u and v, where* $n \ge 6$. *If* $|D_{a_1}|=n-3$ *and* a_2 *is the break dimension, a longest fault-free u-v path can be constructed in* S_n.

Proof. Suppose $u=u_1u_2...u_n$ and $v=v_1v_2...v_n$. A longest fault-free u-v path can be constructed according to the following two cases.

Case 1. n>6. First, an a_1-partition is executed on S_n, and so a K_n^{n-1} results. Since $|D_{a_1}|=n-3$, there are at most n-3 faulty (n-1)-edges in the K_n^{n-1}. By Lemma 2 a fault-free $R_{n-1}=[A_0, A_1, ..., A_{n-1}]$ can be obtained. Since $u_{a_1}=v_{a_1}$, u and v are located in the same A_k, where $0 \le k \le n-1$. An a_2-partition is then executed on the R_{n-1}, and so its each (n-1)-vertex forms a K_{n-1}^{n-2}. Since $u_{a_2} \ne v_{a_2}$, u and v are located in different (n-2)-vertices. By Lemma 8, a fault-free P_{n-2} whose first (last) (n-2)-vertex is the beginning (ending) (n-2)-vertex can be obtained from the R_{n-1} because $|D_{a_2}|=0$. If n=7, a longest fault-free u-v path can be obtained according to Lemma 11 and Lemma 12. If n>7, a longest fault-free u-v path can be obtained by the aid of Lemma 7, Lemma 11, and Lemma 12.

Case 2. n=6. We first consider $u_{a_i}=v_{a_i}$ for i=3, 4, 5. Hence, $u_1 \ne v_1$, $u_{a_2} \ne v_{a_2}$, and $dist(u, v)$=1. A fault-free R_3 can be obtained from $S_n=S_6$ after an (a_1, a_3, a_4)-partition so that $x_{dif(X,Y)} \ne z_{dif(Y,Z)}$ for every three consecutive 3-vertices $X=\langle x_1x_2...x_6\rangle_3$, $Y=\langle y_1y_2...y_6\rangle_3$, and $Z=\langle z_1z_2...z_6\rangle_3$ in the R_3. The construction is very similar to the construction of a fault-free R_3 in Lemma 15. Then a longest fault-free u-v path can be constructed from the R_3. The construction is similar to the construction of a longest fault-free u-v path for the situation of $dist(u, v)$=1 in Lemma 18.

We then consider $u_{a_k} \neq v_{a_k}$ for some 3≤k≤5. Without loss of generality, suppose $u_{a_5} \neq v_{a_5}$. An a_1-partition is applied to S_6, and so a K_6^5 results. Since $u_{a_1} = v_{a_1}$, u and v are located in the same 5-vertex, denoted by S. Suppose $Q=\langle q_1q_2...q_6\rangle_5$ with $q_{a_1} = u_{a_5}$ and $Q'=\langle q'_1\, q'_2 \ldots q'_6\rangle_5$ with $q'_{a_1} = u_{a_2}$ are another two 5-vertices. Since $|D_{a_1}|=n$-3=3, (S, Q) or (S, Q') contains at most one edge fault. Without loss of generality, suppose (S, Q) contains one edge fault. By the aid of Lemma 3, an R_5=[S, T, W, Z, O, Q] containing at most one faulty 5-edge can be obtained from the K_6^5, where T, W, Z, and O are the other 5-vertices ($Q' \in \{T, W, Z, O\}$). The faulty 5-edge, if it exists, contains exactly one edge fault.

Next, a good or partially good P_4 is constructed from the R_5 as follows. An a_5-partition is applied to the R_5, and so its each 5-vertex forms a K_5^4. Let U_4 and V_4 ($U_4 \neq V_4$) denote the beginning and ending 4-vertices, respectively. Since $u_{a_5} = q_{a_1} = q_{dif\,(Q,S)}$, U_4 is not connected to Q. A healthy hamiltonian path for the K_5^4 formed by S is established as (F, J, U_4, V_4, L), where F, $J=\langle j_1j_2j_3j_4j_5j_6\rangle_4$ with $j_{a_5} \neq u_1$, and $L=\langle l_1l_2l_3l_4l_5l_6\rangle_4$ with $l_{a_5} \neq v_1$ are the other three 4-vertices in S. We note that both F and J are connected to Q and both V_4 and L are connected to T, or both F and J are connected to T and both V_4 and L are connected to Q (in order to satisfy the condition (2) of Definition 7). Following Procedure 2, an R_4 can be constructed from the R_5. If the 4-edge (U_4, V_4) of the R_4 is broken, a P_4=[A_0, A_1, ..., A_{29}] will result, where $A_0=U_4$, $A_1=J$, $A_{28}=L$, and $A_{29}=V_4$. The P_4 contains at most one faulty 4-edge, and (A_0, A_1), (A_1, A_2), and (A_{28}, A_{29}) are fault-free.

Clearly, the P_4 satisfies the condition (1) of Definition 7. The condition (2) of Definition 7 also holds as a consequence of Lemma 9. We note that $j_{a_5} \in \{u_{a_2},\ u_{a_3},\ u_{a_4}\}$ and $l_{a_5} \in \{v_{a_2},\ v_{a_3},\ v_{a_4}\}$. Without loss of generality, suppose $j_{a_5} = u_{a_3}$. If an a_3-partition is executed on the P_4, the beginning 3-vertex is not connected to A_1 because $u_{a_3} = j_{a_5} = j_{dif\,(A_0,A_1)}$. The ending 3-vertex is connected to A_{28} if $l_{a_5} \neq v_{a_3}$, and is not connected to A_{28} if $l_{a_5} = v_{a_3}$ because $l_{a_5} = l_{dif\,(A_{28},A_{29})}$. Hence, the P_4 is good if $l_{a_5} = v_{a_3}$, and partially good if $l_{a_5} \neq v_{a_3}$.

An a_3-partition is then executed on the P_4. If the P_4 is good, a good fault-free P_3 can be obtained from the P_4, even if the P_4 contains one faulty 4-edge, exclusive of (A_0, A_1), (A_1, A_2), and (A_{28}, A_{29}). If the P_4 is partially good, a good fault-free P_3 can be constructed as follows. A hamiltonian path for the K_4^3 formed by A_{29} is established as (V_3, G, H, Y), where V_3 is the ending 3-vertex, $G=\langle g_1g_2g_3g_4g_5g_6\rangle_3$ is the 3-vertex that is not connected to $A_{28}=L$, and H and Y are the other two 3-vertices in A_{29}. We note that $g_{a_3}=l_{dif(A_{28},A_{29})}=l_{a_5}\neq v_1$. Without loss of generality, we assume $g_{a_3}=v_k$, where $k\in\{2, 3, \ldots, 6\}-\{a_1, a_3, a_5\}$. A hamiltonian path for the K_4^3 formed by A_0 is established as (U_3, G', H', Y'), where U_3 is the beginning 3-vertex, $G'=\langle g'_1\, g'_2\, g'_3\, g'_4\, g'_5\, g'_6\rangle_3$ has $g'_{a_3}=u_k$, and H' and Y' are the other two 3-vertices in A_0.

Following Procedure 2, a fault-free P_3 can be obtained from the P_4 because there is at most one faulty 4-edge in the P_4 and the faulty 4-edge, if it exists, contains one edge fault. The P_3 is good for the following reasons. The condition (1) of Definition 7 clearly holds. The conditions (2) and (3) of Definition 7 are assured by Lemma 9 and Lemma 10, respectively (a k-partition is necessary in order to satisfy the condition (3)). By Lemma 12 a longest fault-free u-v path can be obtained. Q.E.D.

4.3 $|D_{a_{n-4}}|=1$ and $|D_{a_{n-3}}|=0$

Since $|F_e|=n-3$, $|D_{a_{n-4}}|=1$ and $|D_{a_{n-3}}|=0$ imply $|D_{a_1}|=2$ and $|D_{a_2}|=|D_{a_3}|=\cdots=|D_{a_{n-5}}|=1$. The break dimension may be a_1 or a_2 or a_{n-3}, which are discussed in Lemma 21, Lemma 22, and Lemma 23, respectively.

Lemma 21. *Suppose u and v are two vertices of S_n, and $a_1a_2...a_{n-1}$ is a crucial permutation with respect to u and v, where $n\geq 6$. If $|D_{a_{n-4}}|=1$, $|D_{a_{n-3}}|=0$, and a_1 is the break dimension, a longest fault-free u-v path can be constructed in S_n.*

Proof. First, an a_1-partition is applied to S_n, and so a K_n^{n-1} can result. Since $|D_{a_1}|=2\leq n-4$, by Lemma 3 a healthy P_{n-1} whose first (last) (n-1)-vertex is the beginning (ending) (n-1)-vertex can be obtained from the K_n^{n-1}. Hence, a longest fault-free u-v path can be constructed from the P_{n-1} depending to n=6 or n>6, which is similar to the construction of a longest fault-free u-v path in Lemma 13. Q.E.D.

Lemma 22. *Suppose u and v are two vertices of S_n, and $a_1a_2...a_{n-1}$ is a crucial permutation with respect to u and v, where $n\geq 6$. If $|D_{a_{n-4}}|=1$, $|D_{a_{n-3}}|=0$, and a_2 is the break dimension, a longest fault-free u-v path can be constructed in S_n.*

Proof. Suppose $u=u_1u_2...u_n$ and $v=v_1v_2...v_n$. A longest fault-free u-v path can be constructed according to the following two cases.

Case 1. n>6. First, an a_1-partition is applied to S_n, and so a K_n^{n-1} can result. Since $|D_{a_1}|$=2≤n-3, by Lemma 2 a healthy R_{n-1} can be obtained from the K_n^{n-1}. Since $u_{a_1} = v_{a_1}$, u and v are located in the same (n-1)-vertex of the R_{n-1}. Since $|D_{a_2}|$=1≤(n-1)-4, by Lemma 8 a fault-free P_5 (as n=7) or a healthy P_{n-2} (as n>7) whose first and last (n-2)-vertices are the beginning and ending (n-2)-vertices, respectively, can be obtained from the R_{n-1} after an a_2-partition. If n=7, a longest fault-free u-v path can be obtained according to Lemma 11 and Lemma 12. If n>7, a longest fault-free u-v path can be obtained by the aid of Lemma 7, Lemma 11, and Lemma 12.

Case 2. n=6. An a_1-partition is first applied to S_n=S_6. Since $u_{a_1} = v_{a_1}$, u and v are located in the same 5-vertex, denoted by S. Suppose Q=$\langle q_1 q_2 ... q_6\rangle_5$ with $q_{a_1} = u_{a_2}$ and Q'=$\langle q'_1\, q'_2 \ldots q'_6\rangle_5$ with $q'_{a_1} = v_{a_2}$ are another two 5-vertices. We note that Q≠Q' because $u_{a_2} \neq v_{a_2}$. Since $|D_{a_1}|$=2, 5-edge (S, Q) or (S, Q') contains at most one edge fault. Without loss of generality, suppose (S, Q) contains one edge fault. By Lemma 3 an R_5=[S, T, W, Z, O, Q] with one faulty 5-edge can be generated, where T, W, Z, and O are the other 5-vertices ($Q' \in \{T, W, Z, O\}$). A longest fault-free u-v path can be constructed from the R_5, similar to the construction of a longest fault-free u-v path for the case of n=6 in Lemma 20. Q.E.D.

Lemma 23. *Suppose u and v are two vertices of* S_n*, and* $a_1 a_2 ... a_{n-1}$ *is a crucial permutation with respect to u and v, where n≥6. If* $|D_{a_{n-4}}|$*=1,* $|D_{a_{n-3}}|$*=0, and* a_{n-3} *is the break dimension, then* S_n *contains a longest fault-free u-v path if n>6, and a fault-free u-v path of length at least 6!-3 (6!-4) if n=6 and dist(u, v) is odd (even).*

Proof. Suppose u=$u_1 u_2 ... u_n$ and v=$v_1 v_2 ... v_n$. A healthy R_{n-1} in which u and v belong to the same (n-1)-vertex can be obtained after an a_1-partition. An a_{n-3}-partition is then executed on the R_{n-1}. Since $u_{a_{n-3}} \neq v_{a_{n-3}}$, u and v belong to two different (n-2)-vertices. If n>6, by Lemma 8 a fault-free P_5 (as n=7) or a healthy P_{n-2} (as n>7) whose first and last (n-2)-vertices are the beginning and ending (n-2)-vertices, respectively, can be obtained from the R_{n-1}. Then a longest fault-free u-v path can be constructed from the P_5 or P_{n-2}, similar to the construction of a longest fault-free u-v path in Lemma 13.

If n=6, a good P_3=[A_0, A_1, ..., A_{119}] can be obtained after executing an (a_1, a_3, a_2)-partition on S_n=S_6, which is similar to the generation of a good fault-free P_3 for the case of n=6 in Lemma 20. Since $|D_{a_2}|$=1, the P_3 contains at most one faulty 3-edge. The faulty 3-edge, if it exists, contains one edge fault. Let ($c_{i,0}$, $c_{i,1}$, $c_{i,2}$, $c_{i,3}$, $c_{i,4}$, $c_{i,5}$) be the cycle representing A_i, where 0≤i≤119. Suppose $dist$(u, v) is odd. A u-v path of length at least 6!-3 can be constructed as follows. Since the P_3 is good, after a k-partition for some $k \in \{a_4, a_5\}$ the beginning 2-vertex is not connected to A_1. We note that a 2-vertex is an edge. Without loss of generality, we assume $u=c_{0,0}$ and the edge $(c_{0,0}, c_{0,5})$ is the beginning 2-vertex that is not connected to A_1. So, $c_{0,1}$ and $c_{0,4}$ are connected to A_1. If the 2-edge between A_0 and A_1 that is incident on $c_{0,1}$ is faulty, then the path $(c_{0,0}, c_{0,1}, c_{0,2}, c_{0,3}, c_{0,4})$ in A_0 is included. Otherwise, the path $(c_{0,0}, c_{0,5}, c_{0,4}, c_{0,3}, c_{0,2}, c_{0,1})$ is included.

Similarly, a path of length 5 in each A_j can be included if the two 2-edges between A_j and A_{j+1} are fault-free, and a path of length at least 4 can be included otherwise, where $1 \le j \le 118$. Since there is exactly one faulty 2-edge, a path of length 6!-7 or 6!-8 that starts from u and ends at a vertex, say $c_{118,r}$ $(0 \le r \le 5)$, of A_{118} can result. It is not difficult to include a $c_{118,r}$-v path of length 6 (5) if the u-$c_{118,r}$ path has length 6!-7 (6!-8) (recall that a u-v path has odd length if $dist(u, v)$ is odd). Hence, there exists a u-v path of length 6!-1 or 6!-3. The discussion for even $dist(u, v)$ is very similar. Q.E.D.

4.4 $|D_{a_1}| \ne n$-3 and $|D_{a_{n-4}}|=0$

When $|D_{a_1}| \ne n$-3 and $|D_{a_{n-4}}|=0$, $n>6$ holds for otherwise ($n=6$) $|D_{a_{n-4}}|=|D_{a_2}|=0$ and $|D_{a_{n-3}}|=|D_{a_3}|=0$ imply $|D_{a_1}|=n$-3, which is a contradiction. The break dimension may be a_1 or a_2 or … or a_{n-4}.

Lemma 24. *Suppose u and v are two vertices of S_n, and $a_1a_2...a_{n-1}$ is a crucial permutation with respect to u and v, where n>6. If $|D_{a_1}| \ne n$-3, $|D_{a_{n-4}}|=0$, and a_k is the break dimension, a longest fault-free u-v path can be constructed in S_n, where $1 \le k \le n$-5.*

Proof. We first consider that the break dimension is a_1. Since $|D_{a_1}| \le n$-4, by Lemma 3 a healthy P_{n-1} whose first (last) (n-1)-vertex is the beginning (ending) (n-1)-vertex can be obtained after an a_1-partition on S_n. A fault-free P_5 whose first (last) 5-vertex is the beginning (ending) 5-vertex can be obtained from the P_{n-1} as follows. Suppose a healthy P_{n-j+1} whose first (last) (n-j+1)-vertex is the beginning (ending) (n-j+1)-vertex is obtained after an $(a_1, a_2, ..., a_{j-1})$-partition on S_n, where $2 \le j \le n$-5. By Lemma 4, $|D_{a_j}| \le \left\lfloor \frac{n-3}{j} \right\rfloor \le n$-$j$-3, because n-j-3-$\left\lfloor \frac{n-3}{j} \right\rfloor \ge n - j - 3 - \frac{n-3}{j} = \frac{nj - j^2 - 3j - n + 3}{j} = \frac{n(j-1) - j^2 - 3j + 3}{j} \ge \frac{(j+5)(j-1) - j^2 - 3j + 3}{j} = \frac{j-2}{j} \ge 0$. By Lemma 7, a P_{n-j} whose first (last) (n-j)-vertex is the beginning (ending) (n-j)-vertex can be obtained from the P_{n-j+1} after an a_j-partition. Let j=2 initially. By induction on j, a desired P_5 can result as j=n-5. By Lemma 11 and Lemma 12, a longest fault-free u-v path can be obtained from the P_5.

Then we consider that the break dimension is a_r, where $2 \le r \le n$-5. Suppose $u=u_1u_2...u_n$ and $v=v_1v_2...v_n$, where $u_{a_1}=v_{a_1}$, $u_{a_2}=v_{a_2}$, …, $u_{a_{r-1}}=v_{a_{r-1}}$, and $u_{a_r}\neq v_{a_r}$. Since $|D_{a_1}|\leq n$-4, by Lemma 3 a healthy R_{n-1} in which u and v are located in the same (n-1)-vertex can be obtained after an a_1-partition on S_n. A healthy R_{n-r+1} in which u and v belong to the same (n-r+1)-vertex can be obtained from the R_{n-1} as follows. Suppose an R_{n-i+1} in which u and v belong to the same (n-i+1)-vertex is obtained after an (a_1, a_2, ..., a_{i-1})-partition on S_n, where 2≤i≤r-1. Since $|D_{a_i}|\leq\left\lfloor\frac{n-3}{i}\right\rfloor\leq n$-$i$-3, by Lemma 6 a healthy R_{n-i} in which u and v belong to the same (n-i)-vertex can be obtained from the R_{n-i+1} after an a_i-partition. Let i=2 initially. By induction on i, a desired R_{n-r+1} can result as i=r-1. Since $|D_{a_r}|\leq n$-r-3, by Lemma 8 a healthy P_{n-r} whose first (last) (n-r)-vertex is the beginning (ending) (n-r)-vertex can be obtained from the R_{n-r+1} after an a_r-partition. By the aid of Lemma 7, Lemma 11, and Lemma 12, a longest fault-free u-v path can be generated from the P_{n-r}.

Q.E.D.

Lemma 25. *Suppose u and v are two vertices of S_n, and $a_1a_2...a_{n-1}$ is a crucial permutation with respect to u and v, where n>6. If $|D_{a_1}|\neq n$-3, $|D_{a_{n-4}}|=0$, and a_{n-4} is the break dimension, a longest fault-free u-v path can be constructed in S_n.*

Proof. Since $|F_e|$=n-3, either $|D_{a_1}|=3$ and $|D_{a_2}|=|D_{a_3}|=\cdots=|D_{a_{n-5}}|=1$, or $|D_{a_1}|=|D_{a_2}|=2$ and $|D_{a_3}|=\cdots=|D_{a_{n-5}}|=1$. A fault-free R_5 in which u and v belong to the same 5-vertex can be constructed after an (a_1, a_2, ..., a_{n-5})-partition on S_n, similar to the construction of a healthy R_{n-r+1} in Lemma 24 (n-4 is substituted for r because a_{n-4} is the break dimension). Since $|D_{a_{n-4}}|=|D_{a_{n-3}}|=0$, it is not difficult to construct a good fault-free P_3 from the R_5 after an (a_{n-4}, a_{n-3})-partition. By Lemma 12 a longest fault-free u-v path can be obtained from the P_3.

Q.E.D.

4.5 Main results

The results of Lemma 13, Lemma 18, Lemma 19, Lemma 20, Lemma 21, Lemma 22, Lemma 23, Lemma 24, and Lemma 25 are summarized as follows. When $|F_e|$=n-3 and n≥6, a longest fault-free u-v path can be obtained in S_n, exclusive of two exceptions: (1) $|D_{a_1}|=|D_{a_2}|=\cdots=|D_{a_{n-3}}|=1$, a_{n-2} is the break dimension, and $dist(u, v)$=3, and (2) n=6, $|D_{a_1}|=2$, $|D_{a_2}|=1$, $|D_{a_3}|=0$, and a_3 is the break dimension. The longest fault-free u-v path has length n!-1 if $dist(u, v)$ is odd, and n!-2 if $dist(u, v)$ is even. The fault-free u-v path obtained for (1) has length n!-3. The fault-free u-v path obtained for (2) has length at least 6!-3 if $dist(u, v)$ is odd, and at least 6!-4 if $dist(u, v)$ is even.

Since S_n is a bipartite graph and contains n! vertices, the following theorem is an immediate consequence of our discussion.

Theorem 1. *Suppose u and v are two vertices of S_n, where n≥6. If $|F_e|$=n-3, S_n can embed a longest fault-free u-v path, exclusive of two exceptions: (1) $|D_{a_1}|=|D_{a_2}|= \cdots =|D_{a_{n-3}}|=1$, a_{n-2} is the break dimension, and dist(u, v)=3, and (2) n=6, $|D_{a_1}|=2$, $|D_{a_2}|=1$, $|D_{a_3}|=0$, and a_3 is the break dimension. The longest fault-free u-v path has length n!-1 (n!-2) if dist(u, v) is odd (even). For the two exceptions, at most two vertices are excluded from the longest paths.*

When $|F_e|=n$-4, S_n can embed a longest fault-free u-v path for the two exceptional cases. For the exceptional case (1), $|D_{a_1}|=|D_{a_2}|= \cdots =|D_{a_{n-4}}|=1$ and a_{n-3} is the break dimension. A P_{n-1} whose first (last) (n-1)-vertex is the beginning (ending) (n-1)-vertex can be obtained after an a_{n-3}-partition on S_n. Since $|D_{a_{n-3}}|=0$, the P_{n-1} is healthy. A longest fault-free u-v path can be constructed from the P_{n-1}, similar to the construction of a longest fault-free u-v path from a healthy P_{n-1} in Lemma 13. For the exceptional case (2), $|D_{a_1}|=2$, $|D_{a_2}|=0$, and a_2 is the break dimension. By Lemma 3, a fault-free R_5 can be constructed after an a_1-partition on $S_n=S_6$. Since $|D_{a_2}|=|D_{a_3}|=0$, a good fault-free P_3 can be constructed from the R_5 after an (a_2, a_3)-partition. By Lemma 12, a longest fault-free u-v path can be constructed from the P_3. Therefore, we have the following corollary.

Corollary 1. *Suppose u and v are two vertices of S_n, where n≥6. If $|F_e|$=n-4, S_n can embed a longest fault-can embed a longest fault-free u-v path for arbitrary two vertices u and v of S_n, where n≥6.*

5 Concluding remarks

We considered S_n with n≥6 in Theorem 1 and Corollary 1. The situations of S_4 and S_5 are now discussed. Suppose u and v are two vertices of S_4. If $|F_e|=n$-3=1, then $|D_{a_1}|=1$ and $|D_{a_2}|=|D_{a_3}|=0$. The break dimension is a_1 or a_2. If a_2 is the break dimension, $dist$(u, v)=1 or 2 or 3 according to Lemma 14. The following theorem can be obtained by exhaustively examining S_4.

Theorem 2. *Suppose u and v are two vertices of S_4 with one edge fault, and $a_1a_2a_3$ is a crucial permutation with respect to u and v. If a_1 is the break dimension, S_4 can embed a fault-free u-v path of length at least 4!-7 (4!-6) if dist(u, v) is odd (even). If a_2 is the break dimension, S_4 can embed a longest fault-free u-v* *path if dist(u, v)=1 or 2, and a fault-free u-v path of length 4!-3 if dist(u, v)=3. The longest fault-free u-v path has length 4!-1 (4!-2) if dist(u, v)=1 (2).*

If $|F_e|=n-4=0$, it is not difficult to embed a longest fault-free *u*-*v* path in S_4. The discussion for S_5 is similar, and the following two theorems can be obtained by exhaustively examining S_5.

Theorem 3. *Suppose u and v are two vertices of S_5 with two edge faults, and $a_1a_2a_3a_4$ is a crucial permutation with respect to u and v. If $|D_{a_1}|=2$ and a_1 is the break dimension, S_5 can embed a fault-free u-v path of length at least 5!-5 (5!-8) if dist(u, v) is odd (even). If $|D_{a_1}|=2$ and a_2 is the break dimension, or $|D_{a_1}|=|D_{a_2}|=1$ and a_1 is the break dimension, S_5 can embed a fault-free u-v path of length at least 5!-11 (5!-8) if dist(u, v) is odd (even). If $|D_{a_1}|=|D_{a_2}|=1$ and a_3 is the break dimension, S_5 can embed a longest fault-free u-v path if dist(u, v)=1 or 2, and a fault-free u-v path of length 5!-3 if dist(u, v)=3. The longest fault-free u-v path has length 5!-1 (5!-2) if dist(u, v)=1 (2).*

***Theorem 4**. Suppose u and v are two vertices of S_5 with one edge fault, and $a_1a_2a_3a_4$ is a crucial permutation with respect to u and v. If a_1 is the break dimension, S_5 can embed a longest fault-free u-v path whose length is 5!-1 (5!-2) if dist(u, v) is odd (even). If a_2 is the break dimension, S_5 can embed a fault-free u-v path of length at least 5!-3 (5!-4) if dist(u, v) is odd (even).*

It is also noted that longest paths for fewer pairs of *u* and *v* can be determined in both S_4 and S_5 than S_n with $n\geq6$, because good fault-free P_3's for fewer pairs of *u* and *v* can be constructed in the former than the latter. In our method, a good fault-free P_3 is prerequisite to a longest fault-free path.

The probabilities of the two exceptions in Theorem 1 can be analyzed as follows. Under the assumption that an edge fault has equal probability of falling into each dimension, we need to solve the "partition of *m*" problem first, in order to compute the probabilities. The "partition of *m*" problem (see page 12 of [23]) asks the number of different ways to write a positive integer *m* as a sum of positive integers, disregarding their order. For example, *m*=4 can be written as 1+1+1+1 or 2+1+1 or 2+2 or 3+1 or 4. When *m*=3, 4, 5, 6, 7, 8, 9, and 10, the answers are 3, 5, 7, 11, 14, 20, 27, and 35, respectively. The "partition of *m*" problem has been recognized as a quite difficult problem.

In order to estimate the probability of the exception (1), we assume that each dimension has equal probability of being the break dimension and *dist*(*u*, *v*) has equal probability of being any possible value. Since *dist*(*u*, *v*)=1 or 2 or 3 by Lemma 14 and the break dimension may be a_1 or a_{n-2}, the probability of the exception (1) for *n*=6 is computed as $\frac{1}{3}\times\frac{1}{2}\times\frac{1}{3}=\frac{1}{18}$. Similarly, the probabilities for *n*=7, 8, 9, 10, 11, 12, and 13 are $\frac{1}{30}$, $\frac{1}{42}$, $\frac{1}{66}$, $\frac{1}{84}$, $\frac{1}{120}$, $\frac{1}{162}$, and $\frac{1}{210}$, respectively. The probabilities tend to drop as *n*'s increase. The probability of the exception (2) can be estimated similarly. Since the break dimension may be a_1 or a_2 or a_3, the probability of the exception (2) is computed as $\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$ as n=6, and 0 as $n\neq 6$.

It is still unknown whether the fault-free *u*-*v* paths we constructed for the two exceptions in Theorem 1 are the longest or not.

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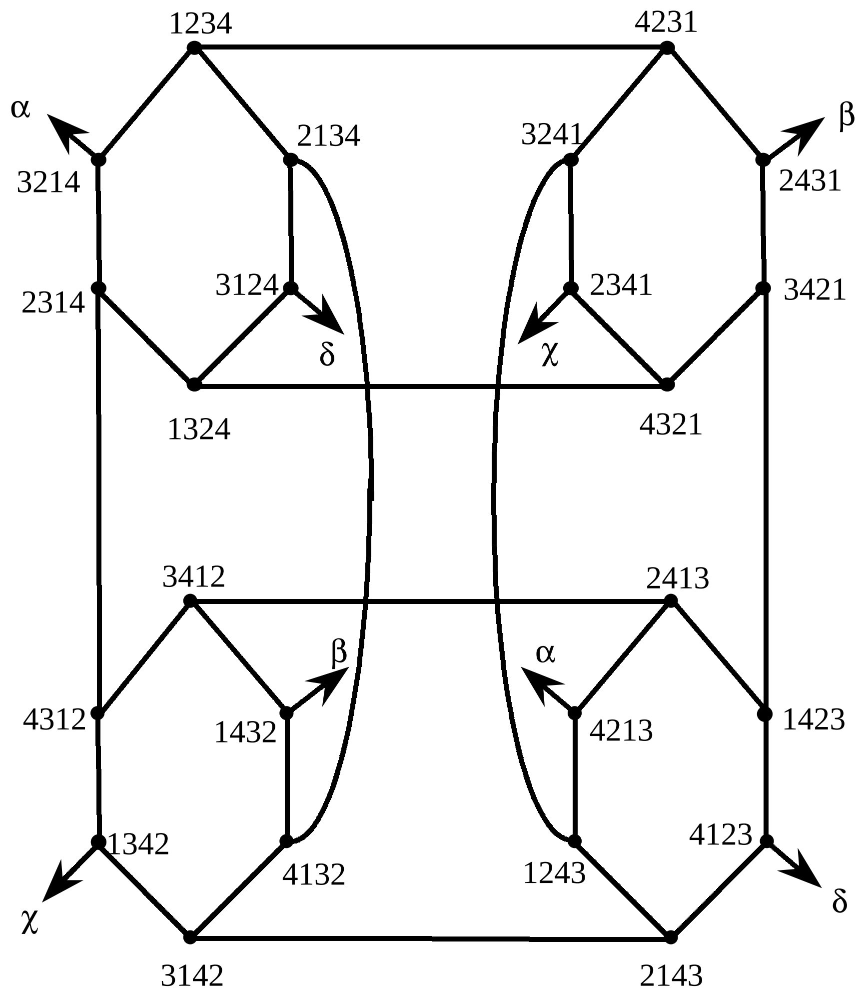

Figure 1. The structure of S_4.

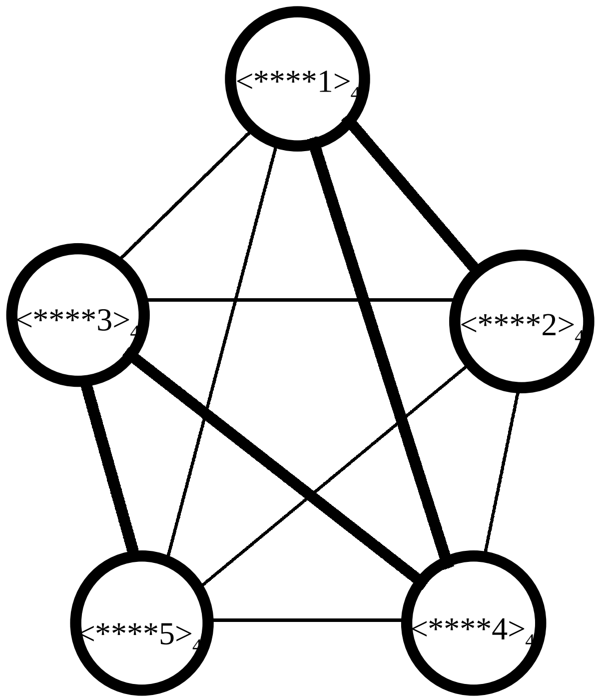

Figure 2. A P_4 (shown in bolded lines) contained in S_5.