行政院國家科學委員會專題研究計畫成果報告

圖枝同調之三維流形不變量

計畫類別：☑個別型計畫　□整合型計畫

計畫編號：NSC 89—2115—M002—029

執行期間：89年8月1日至90年7月31日

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註：整合型計畫總報告與子計畫成果報告請分開編印各成一冊，彙整一起繳送國科會。

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中華民國 90 年 10 月 15 日

摘　　要

我們利用量枝的同調類及映射次數的理論希望能取得三維流形的有理數不變量。目前我們只考慮同調三維球的情形。方法如下：首先必須將這個三維流形映射到 $\mathbb{R}$，但大部分是平坦的，在這個映射之下，我們建構一套由 homology theory 取得的不變量。

關鍵詞： 三維同調球，不變量

ABSTRACT

The degree-theoretical approach provides a natural method to construct the rational-valued invariant for the integral homology 3-spheres.

Key words: integral homology 3-sphere, invariant.

Degree Invariant of Integral Homology 3-Spheres

Su-Win Yang

Abstract

The degree-theoretical approach provides a natural method to construct the rational-valued invariants for the integral homology 3-spheres. To do this, we need to embed the 3-manifold in the Euclidean space such that the manifold occupies a flat 3-space except a compact subset.

1 Introduction

Suppose $\overline{M}$ is a 3-dimensional closed smooth manifold whose integral homology groups are the same as that of the 3-sphere S^3. x_0 is a fixed point in $\overline{M}$. Embed $M = \overline{M} - x_0$ in a Euclidean space $\mathbb{R}^n$ such that x_0 is the infinite point of the flat space $\mathbb{R}^3 \times \{0\}$ in $\mathbb{R}^n$ and each open neighborhood of x_0 contains the whole flat space $\mathbb{R}^3 \times \{0\}$ except a compact set.

Let $\Delta(M)$ denote the diagonal subset $\{(x,x) \in M \times M : x \in M\}$ of $M \times M$ and $C_2(M) = M \times M - \Delta(M)$, it is the configuration space of all ordered pairs of distinct points. Compactify $C_2(M)$ suitably such that the compactification has the same homotopy type as $C_2(M)$; also denote the compactification by $C_2(M)$. Thus $C_2(M)$ contains $S(TM)$ as part of the boundary, where $S(TM)$ is the spherical bundle of the tangent bundle TM over M.

The "degree" invariant defined in this article is essentially dependent on a canonical map $f : C_2(M) \longrightarrow S^2$, which is unique up to homotopy. Consider the restriction $h_0 : S(TM) \longrightarrow S^2$ of f to $S(TM)$. The obstruction of h_0 to be homotopic to a fibrewise orthogonal map is an element $Q(h_0)$ in $\pi_5(S^2)$. The value $Q(h_0)$ is an invariant of the integral homology 3-sphere $\overline{M}$. (For the constructions of the map f and the value $Q(h_0)$, please see [14].) For technical reason, we need the following assumption

Assumption (1.1): $Q(h_0) = 0$, for the integral homology 3-sphere $\overline{M}$.

Under the assumption, we shall construct a series of invariants from the cocycles of the graph cohomology defined in Kontsevich [9], more precisely by Bott and Cattaneo [6]. The invariants defined in this article could be understood as a degree theory formulation of the perturbative Chern-Simons theory. The parallel theory of knot invariant is in the author's paper [13], also in Poirier [10].

Remark (1.2): We believe that the assumption holds for any integral homology 3-sphere, but we still can not prove it. If $Q(h_0)$ is not always zero, it could be additive under connected sum and cobordant invariant.

1.1 The canonical map f from $C_2(M)$ to S^2

The construction of the series of invariant shall use the degree theory. To fit everything into the degree theory, the map $f : C_2(M) \longrightarrow S^2$ should be chosen to satisfy the following properties:

(i) On some flat neighborhood N of the infinite point x_0, $N \subset \mathbb{R}^3 \times \{0\}$, and a smaller neighborhood N_1 of x_0, closure(N_1) $\subset$ interior(N),

$$f(x,y) = \frac{\pi(y-x)}{|\pi(y-x)|}$$

for $(x,y) \in C_2(N)$, $(M-N) \times N_1$, or $N_1 \times (M-N)$, where $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^3$ denote the projection

$$\pi(t_1, t_2, \cdots, t_n) = (t_1, t_2, t_3) \quad .$$

(ii) $h_0 = f|_{S(TM)}$ is a fibrewise orthogonal map, that is, for each $x \in M$, the restriction of h_0 to the 2-sphere over x is an orthogonal map.

The first property comes from the construction of f in [14]. For the second property, we should use the assumption that $Q(h_0) = 0$ and modify the value of f on $S(TM)$ to an orthogonal map.

Remark (1.3): For the point y in N, the tangent space of M at y is exactly the space $\mathbb{R}^3 \times \{0\}$ and the spherical fibre of $S(TM)$ over y is $S^2 \times \{0\}$. The Property (i) implies that the restriction h_0 to such fibre is nothing but the identity map of S^2.

1.2 Graphs in a 3-manifold

Now, we should go to our main object: the trivalent graphs in the 3-manifold M.

A graph Γ in a space X has two kinds of objects, vertices and edges. A vertex is a point in the space X. An edge is an abstract set of two distinct vertices, the two vertices are said to be the end-points of the edge. A graph is always assumed to have finite vertices and finite edges, but the edges are admitted to have single, double, triple, or more multiplicities. We always denote the set of vertices by $V(\Gamma)$ and the set of edges by $E(\Gamma)$.

A graph with multiple-edges sometimes makes ambiguity. If we assign each edge an integer, everything shall become clear. The refined notion of labelling of a graph, introduced by Poirier[10], is a convenient notation for us.

Assume Γ has k edges (counting the multiplicities). A map $\tau : \{1, 2, \cdots, 2k\} \longrightarrow V(\Gamma)$ is said to be a labelling, if $\{\tau(1), \tau(2)\}$, $\{\tau(3), \tau(4)\}$, $\cdots$, $\{\tau(2k-1), \tau(2k)\}$ are exactly all the edges of Γ, counting all the multiplicities of the edges. (Γ, τ) is called the labelled graph.

Two labellings of the same graph are different by a special type of permutation of $\{1, 2, \cdots, 2k\}$. For preciseness, we describe these permutations in the following.

For a positive integer p, let Σ_p denote the group of all permutations of $\{1, 2, \cdots, p\}$. A permutation $\overline{\sigma}$ in Σ_{2k} is said to cover a permutation σ in Σ_k, if $\overline{\sigma}$ sends the elements in $\{2i-1, 2i\}$ to the elements in $\{2\sigma(i)-1, 2\sigma(i)\}$, for each $i = 1, 2, \cdots, k$. It is easy to see that the number of permutations in Σ_{2k} which cover a fixed permutation σ in Σ_k is exactly equal to 2^k. If $\overline{\sigma}_i$ covers σ_i, $i = 1, 2$, then $\overline{\sigma}_1 \cdot \overline{\sigma}_2$ covers $\sigma_1 \cdot \sigma_2$. Thus the set G_k of all the permutations $\overline{\sigma}$ in Σ_{2k} such that there exists a permutation σ covered by $\overline{\sigma}$ is a subgroup of Σ_{2k}. Furthermore, there is a group homomorphism $G_k \longrightarrow \Sigma_k$, sending a permutation to the permutation covered by it; the kernel of this homomorphism is isomorphic to $\bigoplus_k \mathbf{Z_2}$. Thus G_k has $2^k \times k!$ elements.

For a labelled graph (Γ, τ) with k edges and a permutation σ in G_k, let $\sigma \cdot \tau$ be a new labelling defined by $(\sigma \cdot \tau)(i) = \tau(\sigma^{-1}(i))$, $i = 1, 2, \cdots, 2k$, and $\sigma \cdot (\Gamma, \tau) = (\Gamma, \sigma \cdot \tau)$. This defines an action of G_k on the labellings of k edges, also on the labelled graphs with k edges. Therefore, we can think that a graph Γ is a simplification of the orbit set $G_k \cdot (\Gamma, \tau)$, for some labelled graph (Γ, τ).

Remark (1.4): Using the notation of labelling, we can easily define the trivalency of a graph as follows:

A labelled graph (Γ, τ) is said to be trivalent, if, for each vertex $v \in V(\Gamma)$, the inverse-images of v under τ are exactly three integers in $\{1, 2, \cdots, 2k\}$. Thus if a trivalent graph has m vertices and has k edges then $2k = 3m$.

1.3 Equivalence of graphs

Two labelled graphs (Γ_1, τ_1) and (Γ_2, τ_2) are said to be equivalent, if there are a bijection $g : V(\Gamma_1) \longrightarrow V(\Gamma_2)$ and a permutation σ in G_k such that $\tau_2 \circ \sigma = g \circ \tau_1$. And (g, σ) is called an **equivalence** from (Γ_1, τ_1) to (Γ_2, τ_2). When $(\Gamma_1, \tau_1) = (\Gamma_2, \tau_2)$, (g, σ) is called an **automorphism** of (Γ_1, τ_1).

If $(g', \sigma') : (\Gamma_2, \tau_2) \longrightarrow (\Gamma_3, \tau_3)$ is also an equivalence, then $(g', \sigma') \circ (g, \sigma)$ $= (g' \circ g, \sigma' \circ \sigma)$ is an equivalence from (Γ_1, τ_1) to (Γ_3, τ_3).

It is easy to see that if Γ has k edges, all are single edges, then there are $2^k \times k!$ different labellings for Γ and the graph Γ with these $2^k \times k!$ different labellings are all equivalent labelled graphs.

We are interested in the graphs in the 3-manifold M.

Definition (1.5) Suppose (Γ, τ) is a graph in M with labelling τ. Let $C(\Gamma, M)$ denote the space of all labelled graphs in M which are equivalent to (Γ, τ). The space $C(\Gamma, M)$ is independent of the labelling τ.

There is a canonical smooth map $\Psi : C(\Gamma, M) \longrightarrow \prod_k C_2(M)$ defined by: For any (Γ', τ') in $C(\Gamma, M)$,

$$\Psi(\Gamma', \tau') = ((\tau'(1), \tau'(2)), (\tau'(3), \tau'(4)), \cdots, (\tau'(2k-1), \tau'(2k))) \ .$$

Note: $(\tau'(2i-1), \tau'(2i))$ is an element in $C_2(M)$, for each i, $1 \le i \le k$.

We also need the map $f : C_2(M) \longrightarrow S^2$ introduced above. Consider the product map $\prod_k f : \prod_k C_2(M) \longrightarrow \prod_k S^2$ and let $\Phi = (\prod_k f) \circ \Psi : C(\Gamma, M) \longrightarrow \prod_k S^2$, it is the map of our main concern in the degree-theoretical approach of 3-manifold invariants.

Main purpose and arrangements:

The main purpose of this article is to show that if some finite trivalent graphs Γ_1, Γ_2, $\cdots, \Gamma_l$ form a cocycle in the graph cohomology defined in [6] then the degree of the map Φ from the union of $C(\Gamma_i, M)$, $i = 1, 2, \cdots, l$, into $\prod_k S^2$ is a well-defined integer. For the purpose of "cohomology theory", we need an orientation for the graphs. There are some well-known methods for the orientation of trivalent graphs. But we need a "real" orientation for the spaces $C(\Gamma_i, M)$ such that their codimension 1 boundaries shall cancel effectively. Thus we arrange the paper as follows: In Section 2, we define the orientability of the graphs and the coherent G_k-orientation for the spaces $C(\Gamma, M)$. In Section 3, describe the connected components of $C(\Gamma, M)$ and their codimension 1 boundaries associated with subsets of vertices of Γ. In Section 4, we describe a compactification of the configuration space $EQ(\Gamma)$ to introduce the "infinite part" boundaries and give the precise staements for the degeneracy of such boundaries. In Section 5, we extend the compactification in Section 4 to the one needed for the all kinds of boundaries and discuss the degeneracies of some kinds of codimension 1 boundaries. In Section 6, we give the proofs of the propositions stated in Section 4.

2 Orientation of the space $C(\Gamma, M)$

The orientation of $C(\Gamma, M)$ is important for the degree theory and the orientation of Γ is also needed in the graph cohomology defined in [9, 6]. The orientation defined in the following are related naturally to the orientation of the graphs defined in [6] and [9].

At first, we discuss the orientability of a graph and we can delete the non-orientable graphs.

G_k also denotes the subgroup of Σ_{2k} consisting of the permutations covering permutations in Σ_k.

For any permutation $\sigma \in \Sigma_p$, let $\delta(\sigma)$ denote the sign of the permutation σ, that is, δ is the group homomorphism from Σ_p to $\{1, -1\}$, sending every transposition to -1.

Suppose (Γ, τ) is a labelled graph and (g, σ) is an automorphism of (Γ, τ), that is, an equivalence from (Γ, τ) to itself. Consider g as a permutation of the set $V(\Gamma)$, we have the sign $\delta(g)$ of g.

Definition (2.1): A labelled graph (Γ, τ) is said to be orientable, if, for any automorphism (g, σ) of (Γ, τ), $\delta(g) = \delta(\sigma)$. A graph Γ is said to be orientable, if there is a labelling τ of Γ such that (Γ, τ) is orientable.

Remark (2.2): If (Γ, τ) is orientable for some labelling τ, then (Γ, τ') is orientable for any labelling τ'.

2.1 G_k-orientation on $C(\Gamma, M)$

An orientation on $C(\Gamma, M)$ is said to be coherent with respect to the G_k-action, if, for each permutation σ in G_k with $\delta(\sigma) = -1$, the map given by the action of σ is orientation reversing. (Thus the σ-action is orientation preserving, if $\delta(\sigma) = 1$.) For simplicity, such an orientation is also called a **G_k-orientation.**

Proposition (2.3) If the graph Γ is orientable, then the space $C(\Gamma, M)$ has exactly two different G_k-orientation. ∎

The proof of (2.3) is in Section 3, shortly after Definition (3.1).

The G_k-orientation is the orientation we need. We use the G_k-oriented configuration spaces to define the degree for the map Φ.

2.2 Degree of the map Φ

Now suppose Γ is orientable and we fix a G_k-orientation on $C(\Gamma, M)$. We also choose the standard orientation for S^2. Then the map $\Phi = (\prod_k f) \circ \Psi :$ $C(\Gamma, M) \longrightarrow \prod_k S^2$ is a smooth map between two oriented spaces.

For a generic point $\mathbf{y}$ in $\prod_k S^2$, let $d(\Phi, \Gamma, \mathbf{y})$ be the summation of signs of inverse images of $\mathbf{y}$ under the map Φ.

Precisely, for each x in $\Phi^{-1}(\mathbf{y})$, let $\epsilon(\Phi, x)$ be $+1$, if the map Φ is orientation preserving on a neighborhood of x; let $\epsilon(\Phi, x)$ be -1, if the map Φ is orientation reversing on a neighborhood of x; let $\epsilon(\Phi, x)$ be 0, if otherwise.

And, $d(\Phi, \Gamma, \mathbf{y})$ is defined as the summation of $\epsilon(\Phi, x)$ over all x in $\Phi^{-1}(\mathbf{y})$.

The following is our **main theorem**:

> If some finite (G_k-oriented) trivalent graphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_l$ form a cocycle in the graph cohomology defined in [6], then the summation of $d(\Phi, \Gamma_i, \mathbf{y})$ over these graphs Γ_i, $i = 1, 2, \cdots, l$, is an integer independent of the generic point $\mathbf{y}$ in $\prod_k S^2$.

The proof of the above statement is to show that the boundaries of these spaces $C(\Gamma_i, M)$, $i = 1, 2, \cdots, l$, are either degenerate into codimension 2 subspaces in $\prod_k S^2$ under the map Φ, or cancel each other by the "cocycle" condition. **The precise statements are in Section 4.3 and Section 5.**

3 Connected component of $C(\Gamma, M)$

To describe the connected component of $C(\Gamma, M)$, we introduce the space $EQ(\Gamma)$.

An equivalence of two graphs is a simplicial isomorphism of the two graphs.

Definition (3.1): Suppose Γ is a graph. Let $EQ(\Gamma)$ denote the space of all equivalences $g : \Gamma \longrightarrow \Gamma'$ from Γ to a graph Γ' in M.

Because an injective map $g : V(\Gamma) \longrightarrow M$ determines an equivalence from Γ to a graph in M completely, $EQ(\Gamma)$ is diffeomorphic to the configuration space $C_m(M)$ of m distinct points in M, where m is the number of vertices in Γ. $C_m(M)$ is a subset of M^m. Thus $EQ(\Gamma)$ is path-connected.

For any labelling $\tau : \{1, 2, \cdots, 2k\} \longrightarrow V(\Gamma)$, there is a diffeomorphism $\phi_\tau : EQ(\Gamma) \longrightarrow C(\Gamma, M)$ defined by:

For $g \in EQ(\Gamma)$, $\phi_\tau(g) = (g(\Gamma), g \circ \tau)$.

The map ϕ_τ sends $EQ(\Gamma)$ diffeomorphically onto a connected component of $C(\Gamma, M)$. For different labellings τ and τ', the maps ϕ_τ and $\phi_{\tau'}$ are different; they have the same image space, if and only if, there is an automorphism (g_1, σ_1) of (Γ, τ) such that $\tau \circ \sigma_1 = \tau'$ (in this situation, we say that the two labellings τ and τ' **differ by an automorphism**).

Let $Aut(\Gamma, \tau)$ denote the group of all automorphisms of (Γ, τ) and $|\Gamma|$ denote the number of automorphisms in $Aut(\Gamma, \tau)$. Then the number of connected components in $C(\Gamma, M)$ is equal to $\frac{2^k \times k!}{|\Gamma|}$.

Proof of Proposition (2.3):

Using the diffeomorphisms ϕ_τ, we may define a G_k-orientation easily.

Suppose Γ is orientable. The space $EQ(\Gamma)$ is always orientable, we fix an orientation on $EQ(\Gamma)$, also fix a labelling τ_0 of Γ. Now we can define an orientation on $C(\Gamma, M)$ as follows:

> For any labelling $\tau = \sigma \cdot \tau_0$, we choose the orientation on the connected component $\phi_\tau(EQ(\Gamma))$ such that ϕ_τ is orientation reversing if $\delta(\sigma) = -1$.

It is easy to see that the orientation defined above is a well-defined G_k-orientation needed in **Proposition (2.3)** and there are exactly two G_k-orientations dependent on the choices of the orientation on $EQ(\Gamma)$.

3.1 The codimension 1 boundaries of $C(\Gamma, M)$

As above, the space $C(\Gamma, M)$ is a disjoint union of finite number of configuration spaces. These configuration spaces have a well-known compactification constructed by Fulton and MacPherson[8]. We shall assume that $C(\Gamma, M)$ has been substituted by this compactification. The codimension 1 boundaries of $C(\Gamma, M)$ can be described as follows:

(i) First we consider the boundaries of $EQ(\Gamma)$. For each subset A of $V(\Gamma)$ containing at least two vertices, there is a codimension 1 boundary $EQ(\Gamma; A)$ of $EQ(\Gamma)$ associated with the subset A; this boundary $EQ(\Gamma; A)$ is related to the collapsing of the vertices in A to a point in M, but not to infinite point x_0. The "finite part" codimension 1 boundary of $EQ(\Gamma)$ is the union of $EQ(\Gamma; A)$ for all subset A of $V(\Gamma)$ containing more than one vertex.

(ii) For a labelling τ of Γ, the connected component $\phi_\tau(EQ(\Gamma))$ of $C(\Gamma, M)$ has codimension 1 boundaries $\phi_\tau(EQ(\Gamma; A))$, associated with each subset A of the vertex set $V(\Gamma)$.

(iii) Similarly, when B is a non-empty subset of $V(\Gamma)$ and the vertices in B

approach to the infinite point x_0 of M, there is a codimension 1 boundary $EQ(\Gamma; B, x_0)$ of $EQ(\Gamma)$ associated with B and x_0. The "infinite part" codimension 1 boundary of $EQ(\Gamma)$ is the union of $EQ(\Gamma; B, x_0)$ for all non-empty subset B of $V(\Gamma)$. And, $\phi_\tau(EQ(\Gamma; B, x_0))$ is part of the codimension 1 boundary of $\phi_\tau(EQ(\Gamma))$ as above.

When the two labelling τ and τ' differ by an automorphism (g, σ), $\phi_\tau(EQ(\Gamma, g(A))) = \phi_{\tau'}(EQ(\Gamma, A))$; this space is a codimension 1 boundary of $\phi_\tau(EQ(\Gamma))$ ($= \phi_{\tau'}(EQ(\Gamma))$).

Remark (3.2): For a subset A of $V(\Gamma)$, let $A(\Gamma)$ denote the subgraph of Γ in A; precisely, $A(\Gamma)$ has vertex set A and the edge set $\{\{v, w\}$, which is an edge of Γ, $v, w \in A$ $\}$ (the multiplicity of each edge is the same as that in Γ). The purpose of choosing the G_k-orientation is to cancel the boundaries $\phi_\tau(EQ(\Gamma; A))$ for the subset A satisfying that $A(\Gamma)$ contains bivalent vertices. (The cancellations happen in $\prod_k C_2(M)$. It is meaningless to say that the cancellation can happen in $C(\Gamma, M)$.)

Remark (3.3): Suppose A is a subset of $V(\Gamma)$ containing at least three vertices and $A(\Gamma)$ has a univalent vertex. Then the associated codimension 1 boundary is mapped by $\Psi : C(\Gamma, M) \longrightarrow \prod_k C_2(M)$ into a degenerate boundary (with dimension $\leq 2k - 2$, in case that Γ is a trivalent graph.). When A contains at least two vertices and $A(\Gamma)$ is disconnected, the associated boundary is also degenerate (in $\prod_k C_2(M)$).

4 Degeneracy of "infinite part" boundary

We describe the compactification in the infinite part at first. The method we use is analogous to the compactification given by Poirier[10]. The essential work is on the compactification for $EQ(\Gamma)$.

For any finite non-empty set B, let $(\mathbb{R}^3)^B_*$ denote the set of all functions from B to $\mathbb{R}^3$ which do not send every points in B to the origin 0 and C^B denote the quotient space of $(\mathbb{R}^3)^B_*$ quotiented by the rescaling relation, that is, for $h, h' \in (\mathbb{R}^3)^B_*$, $h \sim_r h'$ if there exists a positive number λ such that $h(b) = \lambda h'(b)$ for all $b \in B$. Thus C^B is a smooth compact manifold diffeomorphic to the unit sphere in $(\mathbb{R}^3)^B$. Also let $\overline{M}^{V(\Gamma)}$ denote the space of all functions from $V(\Gamma)$ to $\overline{M}$, it is also a compact smooth manifold. (Note: $\overline{M}$ is the one-point compactification of M.)

We also need to extend C^B to a similar space which contains C^B. That is, for any finite non-empty set B, let $(\mathbb{R}^n)^B_*$ denote the set of all functions from B to $\mathbb{R}^n$ which do not send every points in B to the origin 0 and D^B denote the quotient space of $(\mathbb{R}^n)^B_*$ quotiented by the rescaling relation. Here, we identify $\mathbb{R}^3$ as the subspace $\mathbb{R}^3 \times \{0\}$ of $\mathbb{R}^n$, and hence, C^B is a subspace of D^B.

Furthermore, consider the set $E(\Gamma)$ of all edges of Γ and for each $E = \{v, w\}$ in $E(\Gamma)$, let $(\mathbb{R}^n)^E_\circ$ denote the space of all non-constant functions from E to $\mathbb{R}^n$, and let $\overline{D}^E$ denote the quotient space of $(\mathbb{R}^n)^E_\circ$, quotiented by the translation and rescaling relations, that is, for $h, h' \in (\mathbb{R}^n)^E_\circ$, $h \sim_{t.r.} h'$ if there exist a positive number λ and an element $y \in \mathbb{R}^n$ such that $h(x) = \lambda h'(x) + y$ for all $x \in E$. $\overline{D}^E$ is diffeomorphic to the sphere S^{n-1} of $(n-1)$-dimension. This space $\overline{D}^E$ can be generalized to the space $\overline{D}^A$ for any set A containing at least two points, that is, the space of all non-constant functions from A to $\mathbb{R}^n$ quotiented by the translation and rescaling relation; it is also diffeomorphic to a sphere of suitable dimension.

$\overline{M}^E$ shall denote the space of all functions from E to $\overline{M}$. The product of $\overline{M}^E$ and $\overline{D}^E$ can be thought as the quotient space of a space of functions from E to $\overline{M} \times \mathbb{R}^n$.

Now, let

$$\mathcal{H} = \overline{M}^{V(\Gamma)} \times \prod_B D^B \times \prod_E (\overline{M}^E \times \overline{D}^E) ,$$

where B run over all non-empty subsets of $V(\Gamma)$ and E run over all the edges of Γ.

Consider the embedding $\Theta : EQ(\Gamma) \longrightarrow \mathcal{H}$

$$\Theta(g) = (g, \{g|_B\}_B, \{(g|_E, g|_E)\}_E), \quad \text{for all } g \in EQ(\Gamma)$$

where g is thought as a function from $V(\Gamma)$ to M, $g|_B$ is a function in $(\mathbb{R}^n)^B_*$ and $(g|_E, g|_E)$ is a function from E to $\overline{M} \times \mathbb{R}^n$ (the first $g|_E$ is considered as a function to $\overline{M}$ and the second $g|_E$ is a function to $\mathbb{R}^n$). Although M has been embedded in $\mathbb{R}^n$, to get the condition that $g|_B \in (\mathbb{R}^n)^B_*$ for all B, we still need the following convention.

Convention (4.1): M is embedded in $\mathbb{R}^n$ such that the origin is not in M.

4.1 Infinite boundary of $EQ(\Gamma)$

The boundary of $\Theta(EQ(\Gamma))$ in $\mathcal{H}$ could contains the infinite part boundary of $EQ(\Gamma)$.

In the following, we try to find some "open" submanifolds which are boundary of $\Theta(EQ(\Gamma))$ in $\mathcal{H}$.

For any non-empty set B, let $\widehat{C}^B$ denote the open subset of C^B consisting of all functions $h : B \longrightarrow \mathbb{R}^3$ such that $h(x) \neq 0$ for all $x \in B$ and $h(v) \neq h(w)$ for all edges $\{v, w\}$ of Γ.

Furthermore, for any finite set A_0, let $\widehat{M}^{A_0}$ denote the set of all functions $h : A_0 \longrightarrow M$ such that $h(v) \neq h(w)$ for all edges $\{v, w\}$ of Γ. If A_0 is empty, $\widehat{M}^{A_0}$ is a set consisting of one point.

$\mathcal{I} = (A_0, A_1, A_2, \cdots, A_r)$ is said to be an increasing family of $V(\Gamma)$, if $A_0 \subset A_1 \subset \cdots \subset A_r = V(\Gamma)$, and $A_0 \neq A_1 \neq \cdots \neq A_{r-1} \neq A_r$. A_0 could be empty or not. We also consider the following sets associated with the increasing family, $A'_1 = A_1 - A_0$, $A'_2 = A_2 - A_1$, $\cdots$, $A'_r = A_r - A_{r-1}$ and

$$C(\mathcal{I}) = \widehat{M}^{A_0} \times \widehat{C}^{A'_1} \times \widehat{C}^{A'_2} \cdots \times \widehat{C}^{A'_r} .$$

$\widehat{M}^{A_0}$ is an open set of $\overline{M}^{A_0}$. A'_i is non-empty, for $i = 1, 2, \cdots, r$. The space $C(\mathcal{I})$ is an open manifold of dimension $3m - r$, where m is the number of vertices in Γ.

There is a naturally defined embedding $\rho : C(\mathcal{I}) \longrightarrow \mathcal{H}$ as follows:

Suppose $\xi = (g_0, g_1, g_2, \cdots, g_r)$ is an element in $C(\mathcal{I})$, that is, $g_0 : A_0 \longrightarrow M$, $g_1 : A'_1 \longrightarrow \mathbb{R}^3$, $g_2 : A'_2 \longrightarrow \mathbb{R}^3$, $\cdots$, $g_r : A'_r \longrightarrow \mathbb{R}^3$; and $g_0 \in \widehat{M}^{A_0}$, $g_i \in \widehat{C}^{A'_i}$, for all $i = 1, 2, \cdots, r$. We define $\rho(\xi)$ in the following steps:

(i) Let $\overline{g}_0 : V(\Gamma) \longrightarrow \overline{M}$ be the map $\overline{g}_0(v) = g_0(v)$, for $v \in A_0$, and $\overline{g}_0(v) = x_0$, for $v \in V(\Gamma) - A_0$, where x_0 is the infinite point of $\overline{M}$.

(ii) For each non-empty subset B of $V(\Gamma)$, there is an integer j, $0 \leq j \leq r$, such that $B \subset A_j$ and $B - A_{j-1}$ is non-empty; and we define $g_B : B \longrightarrow \mathbb{R}^n$ by: $g_B(v) = g_j(v)$, if $v \in B - A_{j-1}$, and $g_B(v) = 0$, if $v \in A_{j-1} \cap B$. By the definition of $\widehat{C}^{A'_j}$, g_j does not send any element of A'_j to 0, and hence g_B represents an element in D^B. **Note:** When $j = 0$, A_{-1} is the empty set.

(iii) For each edge $E = \{v, w\}$, we define $g_E : E \longrightarrow \overline{M} \times \mathbb{R}^n$ by: As in (ii), choose j such that $E \subset A_j$ and $E - A_{j-1}$ is non-empty. If $j = 0$ ($v, w \in A_0$), $g_E(v) = (g_0(v), g_0(v))$ and $g_E(w) = (g_0(w), g_0(w))$; if $1 \leq j \leq r$ and $v \in A'_j$, $g_E(v) = (x_0, g_j(v))$; if $1 \leq j \leq r$ and $v \in A_0$, $g_E(v) = (g_0(v), 0)$; if $1 \leq j \leq r$ and $v \in A_{j-1} - A_0$, $g_E(v) = (x_0, 0)$; the same formula hold for w. We can check that $g_E(v)$ and $g_E(w)$ do not have the same value in the second component $\mathbb{R}^n$. Thus g_E represents an element in $\overline{M}^E \times \overline{D}^E$.

(iv) $\rho(\xi) = (\overline{g}_0, \{g_B\}_B, \{g_E\}_E)$,
where B run over all non-empty subset of $V(\Gamma)$, E run over all edges of Γ, and $\{g_B\}_B$ represents an element in $\prod_B D^B$ (similar for $\{g_E\}_E$).

It is straightforward to find that $\rho : C(\mathcal{I}) \longrightarrow \mathcal{H}$ is a smooth embedding and $\rho(C(\mathcal{I}))$ is in the closure of $\Theta(EQ(\Gamma))$ in $\mathcal{H}$.

4.2 Extension of Φ to the infinite part boundary

In the following, we fix a labelling $\tau : \{1, 2, \cdots, 2k\} \longrightarrow V(\Gamma)$ of Γ and the diffeomorphism $\phi_\tau : EQ(\Gamma) \longrightarrow C(\Gamma, M)$ onto a connected component of $C(\Gamma, M)$.

For each i, $1 \leq i \leq k$, let E_i denote the edge $\{\tau(2i-1), \tau(2i)\}$. There is a natural diffeomorphism $\alpha_i : \overline{M}^{E_i} \times \overline{D}^{E_i} \longrightarrow \overline{M}^{\{1,2\}} \times \overline{D}^{\{1,2\}}$ defined by: For any map $h : E_i \longrightarrow \overline{M} \times \mathbb{R}^n$, $\alpha_i(h)(1) = h(\tau(2i-1))$ and $\alpha_i(h)(2) = h(\tau(2i))$.

To study the closure of images of $EQ(\Gamma)$ in $\mathcal{H}$ projecting to $\overline{M}^{E_i} \times \overline{D}^{E_i}$, it is enough to study the closure of image of $EQ(\{1,2\})$ in $\overline{M}^{\{1,2\}} \times \overline{D}^{\{1,2\}}$; and it is the standard compactification of $C_2(M)$.

Identify $M \times M$ with $M^{\{1,2\}}$. Thus $C_2(M)$ is a subset of $\overline{M}^{\{1,2\}}$, also naturally embedded in $\overline{M}^{\{1,2\}} \times \overline{D}^{\{1,2\}}$ by the map, analogous to ρ. Let $CC_2(M)$ denote the closure of $C_2(M)$ in $\overline{M}^{\{1,2\}} \times \overline{D}^{\{1,2\}}$.

In the following, we shall use $CC_2(M)$ to denote the compactification of $C_2(M)$ and assume the map f to be a map from $CC_2(M)$ to S^2.

We also consider the projection $\pi_i : \mathcal{H} \longrightarrow \overline{M}^{E_i} \times \overline{D}^{E_i}$. Then the restriction of $\prod_{1 \leq i \leq k} (\alpha_i \circ \pi_i)$ to $\Theta(EQ(\Gamma))$ is equal to $\Psi \circ \phi_\tau$. Therefore, we succeed in extending the map $\Psi : C(\Gamma, M) \longrightarrow \prod_k C_2(M)$ to the compactification of $C(\Gamma, M)$; the extension of Ψ is a map from the compactification of $C(\Gamma, M)$ to the space $\prod_k CC_2(M)$.

4.3 Statements of degeneracy results

We should go to the main purpose of this section to study the behavior of the map $\Phi = (\prod_k f) \circ \Psi$ on the boundary of $C(\Gamma, M)$, and show that the restriction of Φ to the "infinite part" boundary has image of higher codimensional in $\prod_k S^2$.

By the dimensional reason, the only codimension 1 boundary associated with the increasing family $\mathcal{I} = (A_0, A_1, \cdots, A_r)$ is of the case that $r = 1$, that is, $A_0 \subset A_1 = V(\Gamma)$ and $A_0 \neq V(\Gamma)$. And, there are still two different situations as follows:

(i) There is no edge connecting a point in A_0 and a point in $A_1' = A_1 - A_0$, that is, Γ is a disjoint union of two subgraphs Γ_0 and Γ_1, and, $V(\Gamma_0) = A_0$, $V(\Gamma_1) = A_1'$. This case includes the situation that A_0 is empty and $A_1 = V(\Gamma)$.

(ii) There is an edge $E_0 = \{v_0, v_1\}$ such that v_0 is in A_0 and v_1 is in A_1'.

In Case (i), we can show the following property which implies the degeneracy of the associated boundary.

Proposition (4.2) If $g_1, g_1' : A_1' \longrightarrow \mathbf{R}^3$ are equivalent under the translation relation, that is, there exists an element $y \in \mathbf{R}^3$ such that $g_1(v) = g_1'(v) + y$, for all $v \in A_1'$, then

$$\Phi(\phi_\tau(g_0, g_1)) = \Phi(\phi_\tau(g_0, g_1')) ,$$

for all $g_0 \in M^{A_0}$ and for all labelling τ of Γ.
Therefore, the map $\Phi \circ \phi_\tau : \widehat{M}^{A_0} \times \widehat{C}^{A_1'} \longrightarrow \prod_k S^2$ lifts to a map on the quotient space $\widehat{M}^{A_0} \times (\widehat{C}^{A_1'} / \sim_t)$, and hence, the dimension of $\Phi(\phi_\tau(\widehat{M}^{A_0} \times \widehat{C}^{A_1'}))$ is less than or equal to $(3m - 4)$, where m is the number of vertices in Γ. ∎

(Note: $\phi_\tau : EQ(\Gamma) \longrightarrow C(\Gamma, M)$ originally is a diffeomorphism onto a connected component of $C(\Gamma, M)$, and, here, we use the same notation ϕ_τ to denote the map extended to their corresponding boundaries.)

In Case (ii), we also have the following propositions which are enough to show the degeneracy of the associated boundaries, when Γ is a **trivalent graph**.

A vertex v in a graph is said to be free, if v is not an end-point of any edge of the graph. For a subset A of $V(\Gamma)$, $A(\Gamma)$ denotes the subgraph of Γ which has vertices the points in A and has edges the edges of Γ with end-points in A, for details see Remark 3.2.

Proposition (4.3) Suppose Γ is a trivalent graph. Also assume that $A_1'(\Gamma)$ has a free vertex or a univalent vertex. Then the image of the boundary associated with the increasing family $\mathcal{I} = (A_0, A_1)$, under the map $\Phi : C(\Gamma, M) \longrightarrow \prod_k S^2$, is of dimension $\leq (2k-2)$. ∎

Proposition (4.4) Suppose Γ is a trivalent graph. Also assume that $A_1'(\Gamma)$ has only the trivalent vertices and the bivalent vertices. Then the image of the boundary associated with the increasing family $\mathcal{I} = (A_0, A_1)$, under the map $\Phi : C(\Gamma, M) \longrightarrow \prod_k S^2$, is of dimension $\leq (2k-2)$. ∎

It is obvious to see that if Γ is trivalent and the increasing family $\mathcal{I} = (A_0, A_1)$ satisfies the condition in Case (ii), then $A_1'(\Gamma)$ satisfies the assumption of Proposition (4.3), or Proposition (4.4). Thus there is no boundary of "infinite part" which has codimension 1 image in $\prod_k S^2$.

The proofs of the above three propositions will be in **Section 6**.

5 Classification of codimension 1 boundary of $C(\Gamma, M)$

In the compactification described in Section 4, we consider only the edges E of Γ for the part $\prod_E(\overline{M}^E \times \overline{D}^E)$ in $\mathcal{H}$. Now, we need to consider a more general set, the subsets of $V(\Gamma)$ which contain at least two vertices, and also the nested families which is the generalization of the increasing families.

5.1 Nested family

As above, x_0 denote the infinite point of $\overline{M}$. Let $V(\Gamma)^*$ denote the set $V(\Gamma) \cup \{x_0\}$. A family $\mathcal{N}$ of subsets of $V(\Gamma)^*$ is said to be a nested family, if any set in $\mathcal{N}$ contains at least two points and any two sets in $\mathcal{N}$ are either disjoint or one of the two sets contains the other one. ($V(\Gamma)$, or $V(\Gamma)^* = V(\Gamma) \cup \{x_0\}$, may be in $\mathcal{N}$ or not.)

For any nested family $\mathcal{N}$, there are two subfamilies, the infinite subfamily $\mathcal{N}_\infty = \{A : A \in \mathcal{N}$ and A contains the point x_0 $\}$ and the finite subfamily

$\mathcal{N}_f = \{A : A \in \mathcal{N}$ and A is not a subset of any set in $\mathcal{N}_\infty$ $\}$; both are nested families. But the union of the two subfamilies may not equal to $\mathcal{N}$; the set contained in some set in $\mathcal{N}_\infty$ and not containing x_0 is not in $\mathcal{N}_f \cup \mathcal{N}_\infty$. The sets in $\mathcal{N}_\infty$ have a linear order, and they are related to the increasing family in Section 4.

A based nested family $(\mathcal{N}, \eta)$ is a nested family $\mathcal{N}$ together with a function $\eta : \mathcal{N} \longrightarrow \overline{M}$ satisfying the following conditions:
(1) $\eta(A) = \eta(A')$, for any $A, A' \in \mathcal{N}$, $A \subset A'$;
(2) $\eta(A)$ is in M, if $A \in \mathcal{N}_f$;
(3) $\eta(A) = x_0$, if $A \in \mathcal{N}_\infty$.
Such a function η shall be called a base function of $\mathcal{N}$.

5.2 The boundary associated a nested family

For each based nested family $(\mathcal{N}, \eta)$, we assign it the two sets $C(\mathcal{N}, \eta)$ and $C(\mathcal{N})$ as follows:

(i) For any A in $\mathcal{N}_f$, or slightly general, not in $\mathcal{N}_\infty$, a proper function h associated with A is a function on A satisfying the following condition:

For any two vertices a_1, a_2 in A, $h(a_1) = h(a_2)$, if and only if, there exists a set A' in $\mathcal{N}$ such that $a_1, a_2 \in A'$ and A' is strictly contained in A.

(ii) For any A in $\mathcal{N}_\infty$, if it is not the minimal element in $\mathcal{N}_\infty$, there exists a unique set $\widetilde{A} \in \mathcal{N}_\infty$ such that $\widetilde{A}$ is the largest set strictly contained in A; if it is the minimal element in $\mathcal{N}_\infty$, let $\widetilde{A} = \{x_0\}$. If $A \neq V(\Gamma)^*$, a proper function h associated with A is a function from $V(\Gamma) - \widetilde{A}$ to $\mathbb{R}^3$ satisfying the following: $h(a) = 0$, for $a \in V(\Gamma) - A$, $h(a') \neq 0$, for $a' \in A - \widetilde{A}$ and $h(a_1) \neq h(a_2)$, for $a_1 \neq a_2 \in A - \widetilde{A}$.

If $A = V(\Gamma)^* \in \mathcal{N}_\infty$, a proper function associated with A is a function $h : (V(\Gamma) - \widetilde{A}) \cup \{0\} \longrightarrow \mathbb{R}^3$ satisfying the conditions above for the proper function associated with $A \neq V(\Gamma)^*$ and an additional condition that $h(0) = 0$. (The inclusion of 0 in the domain of h is quite artificial and "uncomfortable".)

(iii) For any point x in $\overline{M}$, let T_x denote the tangent space of $\overline{M}$ at x. T_{x_0} is the space $\mathbb{R}^3 \times \{0\}$, or simply denoted by $\mathbb{R}^3$. For any set A in $\mathcal{N}$, let $(T_{\eta(A)})_p^A$ denote the set of all proper functions h associated with A. And let $C^{(A,\eta)}$ denote the quotient space of $(T_{\eta(A)})_p^A$, quotiented by the translation and rescaling relation, that is, $h \sim_{t.r.} h'$, if there exist a positive number λ and a vector y in $T_{\eta(A)}$ such that $h'(v) = \lambda h(v) + y$.

Note: Let $B = A - \widetilde{A}$, then $C^{(A,\eta)}$ is exactly the same as $\widehat{C}^B$ defined in Section 4; the translation relation has no effect in this situation.

(iv) Consider all the spaces $C^{(A,\eta)}$, $A \in \mathcal{N}$, and the product of all these spaces, $\prod_{A\in\mathcal{N}} C^{(A,\eta)}$, denoted the product space by $C'(\mathcal{N}, \eta)$.

(v) Let $V_0(\mathcal{N})$ denote the set $V(\Gamma) - \cup\mathcal{N}$, where $\cup\mathcal{N}$ is the union of the sets in $\mathcal{N}$; when there is no ambiguity, we just denote $V_0(\mathcal{N})$ simply by V_0. And, let $M_\eta^{V_0}$ denote the set of all functions $g : V_0 \longrightarrow M$ such that $g(x) \neq g(y)$, for all $x \neq y$ in V_0, and, $g(x) \neq \eta(A)$, for all $x \in V_0$ and $A \in \mathcal{N}$.

(vi) Let $C(\mathcal{N}, \eta) = M_\eta^{V_0} \times C'(\mathcal{N}, \eta)$ and $C(\mathcal{N})$ denote the space of union of $C(\mathcal{N}, \eta)$, for all possible proper base functions η of $\mathcal{N}$. (η is proper, if it has different values on different maximal set in $\mathcal{N}$.) Thus $C(\mathcal{N})$ is a fibre bundle over the space $\mathcal{B}(\mathcal{N})$ of all proper base functions of $\mathcal{N}$.

The notion of nested family is a generalization of the notion of increasing family. But there is no embedding of $C(\mathcal{N})$ in $\mathcal{H}$. We must extend the structure of $\mathcal{H}$.

For any subset A of $V(\Gamma)$ with at least two elements, consider the space

$\overline{D}^A$, similar to $\overline{D}^E$ defined in Section 4, and let

$$\mathcal{H}' = \overline{M}^{V(\Gamma)} \times \prod_B D^B \times \prod_A (\overline{M}^A \times \overline{D}^A) ,$$

where B run over all non-empty subsets of $V(\Gamma)$ and A run over all subsets A of $V(\Gamma)$ which contain at least two elements.

We also have an embedding $\Theta' : EQ(\Gamma) \longrightarrow \mathcal{H}'$ similar to $\Theta : EQ(\Gamma) \longrightarrow \mathcal{H}$, and for each nested family $\mathcal{N}$, we can define an embedding $\rho' : C(\mathcal{N}) \longrightarrow \mathcal{H}'$ as the same way as in Section 4. Then the union of $\rho'(C(\mathcal{N}))$, over all nested families $\mathcal{N}$, shall form the total boundary of $EQ(\Gamma)$ in $\mathcal{H}'$.

5.3 Definition of $\rho' : C(\mathcal{N}) \longrightarrow \mathcal{H}'$

The definition of Θ' is exactly the same as that of Θ and we omit it. The definition of ρ' is also similar to that of ρ, but is more complicated.

We introduce some notations at first. Let $\mathcal{N}^c$ denote the family of sets, $(\mathcal{N}-\mathcal{N}_\infty)\cup\{V(\Gamma)-A : A \in \mathcal{N}_\infty\}\cup\{V(\Gamma)\}$. $\mathcal{N}^c$ satisfies one of the properties for nested families that any two sets in $\mathcal{N}^c$ are either disjoint or one of the two sets containing the other one. For any non-empty subset B of $V(\Gamma)$, let A_B denote the smallest set in $\mathcal{N}^c$ which contains B.

As mentioned above, the sets in $\mathcal{N}_\infty$ have a linear order, let $A^0 \supset A^1 \supset A^2 \supset \cdots \supset A^{r-1}$ be the all sets in $\mathcal{N}_\infty$. Thus $A^0 = \cup\mathcal{N}_\infty$. Furthermore, let $A_{-1} = \cup\mathcal{N}_f$, the union of sets in $\mathcal{N}_f$, $A_r = V(\Gamma)$, and $A_i = V(\Gamma) - A^i$, $i = 0, 1, 2, \cdots, r-1$. Then $(A_{-1}, A_0, A_1, \cdots, A_r)$ is a generalized type of increasing family of sets in $V(\Gamma)$; the sets A_{-1} and A_0 could be empty. (If $r = 0$, that is, there is no set in $\mathcal{N}_\infty$, then we have only A_{-1} and $A_0 = V(\Gamma)$.)

The set A_B associated with a non-empty set B can be separated into the following situations:

(1) $A_B \subset A_{-1}$ ($A_B \in \mathcal{N}_f$),

(2) $A_B = A_0$,

(3) $A_B = A_i$, $1 \le i \le r$,

(4) otherwise, that is, A_B is contained in A^0, but does not contain x_0.

An element $\xi = (g_0, \eta, \{g_A\}_{A \in \mathcal{N}})$ in $C(\mathcal{N})$ consists of a one-to-one function $g_0 : V_0 \longrightarrow M$, a proper base function η on $\mathcal{N}$, together with a family of proper functions g_A associated with the set A, for all $A \in \mathcal{N}$. And, we define $\rho'(\xi)$ as follows:

(i) If $V(\Gamma)^*$ is not in $\mathcal{N}$, we use g_0 and η to define a "proper" function $g_{V(\Gamma)^*} : V(\Gamma)^* \longrightarrow \overline{M}$ as follows: $g_{V(\Gamma)^*}(x_0) = x_0$; $g_{V(\Gamma)^*}(v) = g_0(v)$, if $v \in V_0$; and, $g_{V(\Gamma)^*}(v) = \eta(A)$, if $v \in A$, for some $A \in \mathcal{N}$. In the following, we think the map $g_{V(\Gamma)^*}$ as a proper function on $V(\Gamma)^*$. If $V(\Gamma)^*$ is in $\mathcal{N}$, we already have the map $g_{V(\Gamma)^*} : V(\Gamma)^* \longrightarrow \mathbf{R}^3$; and we can interpret it as a map from $V(\Gamma)^*$ to $\overline{M}$ by sending every point to $\eta(V(\Gamma)^*) = x_0$, whenever necessary.

(ii) For each non-empty subset B of $V(\Gamma)$, consider the associated set A_B and define a map h_B as follows:
Case (1): $A_B \in \mathcal{N}_f$, $h_B : B \longrightarrow T_{\eta(A_B)}$ is the restriction of g_{A_B} to B.
Case (2): $A_B = A_0$, $h_B = g_{V(\Gamma)^*}|_B : B \longrightarrow M \subset \mathbf{R}^n$, as a map from B to $\mathbf{R}^n$.
Case (3): $A_B = A_i$, $1 \le i \le r$, $h_B = g_{A^{i-1}}|_B$.
Case (4): Otherwise, $h_B = g_{A_B}|_B : B \longrightarrow T_{\eta(A_B)}$.
Note: $g_{A^{i-1}}|_B$ is a map from $V(\Gamma) - A^i = A_i$ to $\mathbf{R}^3$.

We need also another map $h'_B : B \longrightarrow \mathbf{R}^n$ defined by:
if $A_B \in \mathcal{N}_f$, then $h'_B(v) = \eta(A)$, for all $v \in B$;
if $A_B = A_0$, then $h'_B = h_B$;
if $A_B = A_i$, $1 \le i \le r$, then $h'_B = h_B$;
if otherwise, that is, there exists a unique integer i, $0 \le i \le r-1$, such that $A_B \subset A^i$ and $A_B \subset A_{i+1}$, then $h'_B = g_{A^i}|_B : B \longrightarrow \mathbf{R}^3$.
Note: g_{A^i} is a map $A_{i+1} \longrightarrow \mathbf{R}^3$.

(iii) $\rho'(\xi) = ((g_{V(\Gamma)^*})|_{V(\Gamma)}, \{h'_B\}_B, \{((g_{V(\Gamma)^*})|_A, h_A)\}_A)$, where B run over all non-empty subsets of $V(\Gamma)$ and A run over all subsets of $V(\Gamma)$ containing at least two vertices.

The functions $(g_{V(\Gamma)^\bullet})|_{V(\Gamma)}$ and $(g_{V(\Gamma)^\bullet})|_A$ are considered as maps to $\overline{M}$; h'_B and h_A are considered as maps to $\mathbb{R}^n$.

It is not hard to see that ρ' is a well-defined smooth embedding and $\rho'(C(\mathcal{N}))$ is contained in the closure of $\Theta'(EQ(\Gamma))$ in $\mathcal{H}'$.

To prove the closure of $\Theta'(EQ(\Gamma))$ in $\mathcal{H}'$ to be a qualified compactification of $EQ(\Gamma)$, we should show that every point in the closure of $\Theta'(EQ(\Gamma))$ is in $\rho'(C(\mathcal{N}))$ for some nested family $\mathcal{N}$, and, for any nested family $\mathcal{N}$, every point in the closure of $\rho'(C(\mathcal{N}))$, is also in $\rho'(C(\mathcal{N}_1))$, for some other nested family $\mathcal{N}_1$ which is finer than $\mathcal{N}$, that is, $\mathcal{N} \subset \mathcal{N}_1$. But, to prevent the paper from becoming too lengthy, we do not prove these famous facts.

5.4 The codimension 1 boundaries for "trivalent graph"

The graph Γ considered in the following part of this section is **trivalent.**

By the dimensional reason, if $\rho'(C(\mathcal{N}))$ is a codimension 1 boundary of $EQ(\Gamma)$, then $\mathcal{N}$ contains exactly one set, say A. And there are several cases we should discuss:

(i) **The regular part** That is, A is exactly equal to an edge of Γ. This kind of boundaries are the only boundaries which appear in the graph cohomology and are cancelled by the cycle condition.

(ii) **The infinite part** That is, A contains the infinite point x_0, then $\rho'(C(\mathcal{N}))$ is the same as $\rho(C(A_0, A_1))$ for $A_0 = V(\Gamma) - A$ and $A_1 = V(\Gamma)$. In the above section, such "infinite" boundaries are shown to be degenerate by choosing the map $f : C_2(M) \longrightarrow S^2$ with fine behaviors on an "end" of $C_2(M)$, this "end" subspace has the same homology as $C_2(M)$. (For details see Section 1.1, or [14].)

(iii) **The non-anomalous part** That is, A does not contain x_0; and furthermore, either A consists of two vertices which do not form an edge of Γ, or, A contains at least three vertices and the associated subgraph $A(\Gamma)$ is not a connected component of Γ. When Γ is a trivalent graph, there is a standard argument to prove that the codimension 1 boundary associated with the set A is mapped by the extension of $\Psi : C(\Gamma, M) \longrightarrow \prod_k C_2(M)$ to a space of dimension $\leq (2k-2)$, or to a union of boundaries which cancel by pairs (the same discussion also in Remark 3.2 and 3.3).

(iv) **The anomalous part** That is, A does not contain x_0 and $A(\Gamma)$ is a connected component of Γ. Except the non-interesting case, we may assume A containing at least two vertices. And, by the second property of the map f stated in Section 1 that f is fibrewise orthogonal on the boundary $S(TM)$ of $C_2(M)$, we can show easily that the boundary of $C(\Gamma, M)$ associated with A is mapped by the extension of $\Phi : C(\Gamma, M) \longrightarrow \prod_k S^2$ to a space of dimension $\leq (2k-4)$. Thus, the degeneracy is too large to say that it is an anomalous one (details see the following).

5.5 Discussion on the anomalous boundary

We show the degeneracy of the anomalous boundary here.

We follow the notations in (iv) above. Suppose Γ_0 is the complement of $A(\Gamma)$ in Γ. Then $C(\mathcal{N}) = \cup_\eta M_\eta^{V_0} \times C^{(A,\eta)}$, where $V_0 = V(\Gamma) - A$, which is exactly the vertex set of Γ_0. The extension of $\Phi : C(\Gamma, M) \longrightarrow \prod_k S^2$ to the boundary $\rho'(C(\mathcal{N}))$ is similar to that for $\rho(\widehat{M}^{A_0} \times \widehat{C}^{A'_1})$ defined in Section 4.2; but the later case is a product space which is better than that in the anomalous case, in which the space is a fibre bundle over the space $\mathcal{B}(\mathcal{N})$ of all base functions η. Thus we consider the product space $EQ(\Gamma_0) \times (\cup_\eta C^{(A,\eta)})$. Because the space $M_\eta^{V_0}$ can be considered as a subspace of $EQ(\Gamma_0)$, $C(\mathcal{N})$ is a subspace of $EQ(\Gamma_0) \times (\cup_\eta C^{(A,\eta)})$.

Now we describe the extension of Φ to $EQ(\Gamma_0) \times (\cup_\eta C^{(A,\eta)})$. For convenience, choose a labelling τ of Γ such that the first r edges $E_1, E_2, \cdots, E_r$ are in $A(\Gamma)$ and the last $k-r$ edges $E_{r+1}, E_{r+2}, \cdots, E_k$ are in Γ_0. Let S_η denote the 2-sphere of $T_{\eta(A)}$. Then there is a map $\Psi_1 : C^{(A,\eta)} \longrightarrow \prod_r S_\eta$ defined by:

$$\Psi_1(h) = (\frac{h(\tau(2i)) - h(\tau(2i-1))}{|h(\tau(2i)) - h(\tau(2i-1))|})_{1 \le i \le r} ,$$

for all h in $C^{(A,\eta)}$.

$\prod_r S_\eta$ is a subset of $\prod_r CC_2(M)$. Thus the composite map $\Phi_1 = (\prod_r f) \circ \Psi_1$ maps $C^{(A,\eta)}$ into $\prod_r S^2$. By the fibrewise orthogonal property, the image of Φ_1 is independent of η. Let $\overline{\Phi}_1 : \cup_\eta C^{(A,\eta)} \longrightarrow \prod_r S^2$ denote the union of the maps Φ_1 over all base functions η. Thus the image of $\overline{\Phi}_1$ in $\prod_r S^2$ is the same as the image of Φ_1 in $\prod_r S^2$ for each η. By the trivalency of Γ, $A(\Gamma)$ is also trivalent; and hence, $3|A| = 2r$, dim $C^{(A,\eta)} =$ dim $(T_{\eta(A)})^A_p - 4 = 3|A| - 4 = 2r - 4$, where $|A|$ is the number of elements in A. Thus the dimension of image of $\overline{\Phi}_1$ in $\prod_r S^2$ is at most $2r-4$.

Consider also the map $\Psi_0 : EQ(\Gamma_0) \longrightarrow \prod_{k-r} C_2(M)$ defined by:

$$\Psi_0(g) = (g(\tau(2i-1)), g(\tau(2i)))_{r+1 \le i \le k} ,$$

for all g in $EQ(\Gamma_0)$.

Let $\Phi_0 = (\prod_{k-r} f) \circ \Psi_0 : EQ(\Gamma_0) \longrightarrow \prod_{k-r} S^2$, and consider the product of the above two maps, $\overline{\Phi}_1 \times \Phi_0 : (\cup_\eta C^{(A,\eta)}) \times EQ(\Gamma_0) \longrightarrow \prod_k S^2$. It is the extension we need.

The dimension of the image of Φ_0 in $\prod_{k-r} S^2$ is at most $2(k-r)$. Thus the dimension of the image of $\overline{\Phi}_1 \times \Phi_0$ is at most $2k-4$.

5.6 Discussion on the non-anomalous boundary

A subset A of $V(\Gamma)$ is said to satisfy the splitting condition, if $A(\Gamma)$ splits into two subgraphs Γ_1 and Γ_2 such that the intersection of the two subgraphs has at most one vertex and both subgraphs contain more than one vertices. Thus each edge of $A(\Gamma)$ is either in Γ_1 or in Γ_2.

Proposition (5.1) Suppose A is a subset of $V(\Gamma)$ and A satisfies the splitting condition. $\mathcal{N}$ is the nested family $\{A\}$ consisting of a single set. Then codimension 1 boundary $EQ(\Gamma; A) = \rho'(C(\mathcal{N}))$ associated with A is mapped by the extension of $\Psi : C(\Gamma, M) \longrightarrow \prod_k C_2(M)$ into a subspace of $\prod_k C_2(M)$ with dimension at most $2k-2$. Therefore, it is a degenerate boundary. ∎

Proof of (5.1): We start with counting the dimensions. As in Section 5.5, $C(\mathcal{N}) = \cup_\eta M_\eta^{V_0} \times C^{(A,\eta)}$. For any finite set B, let $|B|$ denote the number of elements in B. dim $C(\mathcal{N}) =$ dim $M_\eta^{V_0}+$ dim $C^{(A,\eta)}+$ dim $B(\mathcal{N})$ $= 3|V_0| + (3|A| - 4) + 3 = 3|V(\Gamma)| - 1$.

Also similar to the argument in Section 5.5, consider the map $\Psi_1 : C^{(A,\eta)} \longrightarrow \prod_r S_\eta$ and the related maps $\Psi_1' : C^{(A_1,\eta)} \longrightarrow \prod_{r_1} S_\eta$, $\Psi_1'' : C^{(A_2,\eta)} \longrightarrow \prod_{r_2} S_\eta$, where A_i is the set of vertices of the subgraph Γ_i and r_i is the number of edges in Γ_i, for $i = 1, 2$; Γ_1 and Γ_2 are the subgraphs of $A(\Gamma)$ in the splitting assumed.

Because of the splitting condition, there is a natural map $\Pi : C^{(A,\eta)} \longrightarrow C^{(A_1,\eta)} \times C^{(A_2,\eta)}$ such that $\Psi_1 = (\Psi_1' \times \Psi_1'') \circ \Pi$, where $\Psi_1' \times \Psi_1''$ is the product map from $C^{(A_1,\eta)} \times C^{(A_2,\eta)}$ to $\prod_r S_\eta$

By counting the dimensions also, we have
dim $(C^{(A_1,\eta)} \times C^{(A_2,\eta)}) = (3|A_1| - 4) + (3|A_2| - 4) = 3(|A_1| + |A_2|) - 8$
$\leq 3(|\bar{A}| + 1) - 8 = 3|A| - 5$.

Compare with the dimension of $C(\mathcal{N})$, its image in $\prod_k C_2(M)$ can be reduced by one; that is the result we need.

This completes the proof of (5.1).

Proposition (5.1) proves the degeneracy of the boundary associated with

A, when $A(\Gamma)$ has a univalent vertex, or, when $A(\Gamma)$ has a free vertex and with more than two vertices. When A has exacly two vertices and is not a edge of Γ, it is also a degenerate case obviously.

5.7 The case of "bivalent vertex"

Thus the only case left is that $A(\Gamma)$ has a bivalent vertex, and we shall show that all the different boundaries associated with A in different components $\phi_\tau(EQ(\Gamma))$ of $C(\Gamma, M)$ cancel themselves by pairs.

As mentioned above, $C(\Gamma, M)$ has different components associated with different labellings of Γ. Now we define an involution in the set of labellings of Γ as flows:

Assume $\tau : \{1, 2, \cdots, 2k\} \longrightarrow V(\Gamma)$ is a labelling. $E_i = \{\tau(2i-1), \tau(2i)\}$, $i = 1, 2, \cdots, k$. For each bivalent vertex v in $A(\Gamma)$, let $b(v)$ denote the integer such that $E_{b(v)}$ is the unique edge containing v and not in $A(\Gamma)$; and let v_0 be the bivalent vertex in $A(\Gamma)$ with minimum value of b. Suppose w_1 and w_2 are the two possible vertices in A such that $E_{i_1} = \{v_0, w_1\}$ and $E_{i_2} = \{v_0, w_2\}$ are the only two edges in $A(\Gamma)$ and containing v_0. Let $\overline{\tau} : \{1, 2, \cdots, 2k\} \longrightarrow V(\Gamma)$ denote the labelling which interchanges the i_1-th and the i_2-th edges; the precise definition of $\overline{\tau}$ should be given in the following cases:

(Case 1) If $\tau(2i_1) = v_0$ and $\tau(2i_2) = v_0$, then $\overline{\tau}(2i_1 - 1) = v_0$, $\overline{\tau}(2i_1) = w_2$, $\overline{\tau}(2i_2 - 1) = v_0$ and $\overline{\tau}(2i_2) = w_1$.

(Case 2) If $\tau(2i_1) = w_1$ and $\tau(2i_2) = v_0$, then $\overline{\tau}(2i_1 - 1) = w_2$, $\overline{\tau}(2i_1) = v_0$, $\overline{\tau}(2i_2 - 1) = v_0$ and $\overline{\tau}(2i_2) = w_1$.

(Case 3) If $\tau(2i_1) = v_0$ and $\tau(2i_2) = w_2$, then $\overline{\tau}(2i_1 - 1) = v_0$, $\overline{\tau}(2i_1) = w_2$, $\overline{\tau}(2i_2 - 1) = w_1$ and $\overline{\tau}(2i_2) = v_0$.

(Case 4) If $\tau(2i_1) = w_1$ and $\tau(2i_2) = w_2$, then $\overline{\tau}(2i_1 - 1) = w_2$, $\overline{\tau}(2i_1) = v_0$, $\overline{\tau}(2i_2 - 1) = w_1$ and $\overline{\tau}(2i_2) = v_0$.

It is easy to check that the map sending τ to $\overline{\tau}$ is an involution and this involution is dependent on the subset A.

Now consider the two connected components $X = \phi_\tau(EQ(\Gamma))$ and $\overline{X} = \phi_{\overline{\tau}}(EQ(\Gamma))$ in $C(\Gamma, M)$ and the associated boundaries $\partial(A) = \phi_\tau(EQ(\Gamma, A))$ and $\overline{\partial}(A) = \phi_{\overline{\tau}}(EQ(\Gamma, A))$. Although X and $\overline{X}$ are different components, the boundaries $\partial(A)$ and $\overline{\partial}(A)$ do have the same image in $\prod_k CC_2(M)$. Precisely, let $\zeta : \partial(A) \longrightarrow \overline{\partial}(A)$ be the map defined as follows:

As above, $EQ(\Gamma; A) = \cup_\eta M_\eta^{V_0} \times C^{(A,\eta)}$. At first, we consider an involution on the space $C^{(A,\eta)}$.

For any base function η of $\mathcal{N} = \{A\}$ and any $h : A \longrightarrow T_{\eta(A)}$ in $C^{(A,\eta)}$, let $\overline{h} : A \longrightarrow T_{\eta(A)}$ defined by: $\overline{h}(v_0) = h(w_1) + h(w_2) - h(v_0)$, and, $\overline{h}(v) = h(v)$, for $v \neq v_0$.

Now, for any $(\eta, g, h) \in \cup_\eta M_\eta^{V_0} \times C^{(A,\eta)}$, $g \in M_\eta^{V_0}$,

$\zeta(\phi_\tau(\eta, g, h)) = \phi_{\overline{\tau}}(\eta, g, \overline{h})$.

Then we have

Lemma (5.2): For any $\xi \in \partial(A)$, $\zeta(\xi)$ and ξ have the same image in $\prod_k CC_2(M)$.

Proof:

Similar to the discussion in Section 5.5, the extension of Ψ on the boundary $\partial(A) = \phi_\tau(EQ(\Gamma, A))$ can be splitted into the product of two maps, one is a function of (η, g); the other one is a function of h and with value in $\prod_r S_\eta$ (thus also dependent on η). Roughly, we may write the second function in the form as in Section 5.5, that is, $\Psi_1^\tau : C^{(A,\eta)} \longrightarrow \prod_r S_\eta$,

$$\Psi_1^\tau(h) = \left(\frac{h(\tau(2j_i)) - h(\tau(2j_i - 1))}{|h(\tau(2j_i)) - h(\tau(2j_i - 1))|}\right)_{1 \leq i \leq r} ;$$

where $j_i, i = 1, 2, \cdots, r$, are the integers for which E_{j_i} is in $A(\Gamma)$ and the S_η's are in the correspoding components of $CC_2(M)$.

With a straightforward computation, we have

$$\frac{h(\tau(2i_1)) - h(\tau(2i_1 - 1))}{|h(\tau(2i_1)) - h(\tau(2i_1 - 1))|} = \frac{\overline{h}(\overline{\tau}(2i_1)) - \overline{h}(\overline{\tau}(2i_1 - 1))}{|\overline{h}(\overline{\tau}(2i_1)) - \overline{h}(\overline{\tau}(2i_1 - 1))|} ,$$

and the same equality for changing i_1 to i_2; thus, $\Psi_l^{\tau}(h) = \Psi_l^{\overline{\tau}}(\overline{h})$. This proves the lemma.

Consider the map $F = \phi_{\overline{\tau}} \circ \phi_{\tau}^{-1} : X \longrightarrow \overline{X}$. By the definition of G_k-orientation, F is orientation preserving. The boundaries $\partial(A)$, $\overline{\partial}(A)$ of X, $\overline{X}$, respectively, take the standard boundary orientation; then the map F, restricted to the boundary, is also orientation preeserving. Finally, the orientation reversing property of the map $\zeta : \partial(A) \longrightarrow \overline{\partial}(A)$ implies that the two boundaries $\partial(A)$ and $\overline{\partial}(A)$ cancel each other.

This finishes the proof of the degeneracy of the boundary, when there is a bivalent vertex in $A(\Gamma)$.

6 Proofs of Proposition (4.2), (4.3), and (4.4)

At first, we clarify the definition of $\Phi \circ \phi_\tau$ on $\widehat{M}^{A_0} \times \widehat{C}^{A_1'}$.

In Section 4, we define the smooth embedding

$$\rho : \widehat{M}^{A_0} \times \widehat{C}^{A_1'} \longrightarrow \mathcal{H}$$

and, for each i, $1 \leq i \leq k$, $E_i = \{\tau(2i-1), \tau(2i)\}$, we have also defined the maps $\alpha_i : \overline{M}^{E_i} \times \overline{D}^{E_i} \longrightarrow \overline{M}^{\{1,2\}} \times \overline{D}^{\{1,2\}}$ and $\pi_i : \mathcal{H} \longrightarrow \overline{M}^{E_i} \times \overline{D}^{E_i}$.

Let $\beta_i : \rho(\widehat{M}^{A_0} \times \widehat{C}^{A_1'}) \longrightarrow \overline{M}^{\{1,2\}} \times \overline{D}^{\{1,2\}}$ denote the restriction of the map $\alpha_i \circ \pi_i : \mathcal{H} \longrightarrow \overline{M}^{\{1,2\}} \times \overline{D}^{\{1,2\}}$ to the subspace $\rho(\widehat{M}^{A_0} \times \widehat{C}^{A_1'})$. Because β_i sends $\rho(\widehat{M}^{A_0} \times \widehat{C}^{A_1'})$ into the closure of $C_2(M)$ in $\overline{M}^{\{1,2\}} \times \overline{D}^{\{1,2\}}$ (that is, $CC_2(M)$), we shall consider β_i as a map from $\rho(\widehat{M}^{A_0} \times \widehat{C}^{A_1'})$ to $CC_2(M)$.

The following lemma is directly from the definition of the map ρ in Section 4.1.

Lemma (6.1) Suppose (g_0, g_1) is an element in $\widehat{M}^{A_0} \times \widehat{C}^{A_1'}$, $g_0 : A_0 \longrightarrow M$, $g_1 : A_1' \longrightarrow \mathbb{R}^3$. $(\beta_i \circ \rho)(g_0, g_1)$ is an element, denoted by (h_0, h_1), in $\overline{M}^{\{1,2\}} \times \overline{D}^{\{1,2\}}$, $h_0 : \{1,2\} \longrightarrow M$, $h_1 : \{1,2\} \longrightarrow \mathbb{R}^n$. Then (h_0, h_1) can be described explicitly as follows:

For convenience, E_i is considered as a subset of $V(\Gamma)$.

(i) If E_i is contained in A_0,
$h_0(1) = g_0(\tau(2i-1))$, $h_0(2) = g_0(\tau(2i))$,
$h_1(1) = g_0(\tau(2i-1))$, $h_1(2) = g_0(\tau(2i))$.

(ii) If E_i is contained in A_1',
$h_0(1) = h_0(2) = x_0$,
$h_1(1) = g_1(\tau(2i-1))$, and $h_1(2) = g_1(\tau(2i))$.

(iii) If E_i meets both A_0 and A_1', say, $\tau(2i-1)$ is in A_0,
$h_0(1) = g_0(\tau(2i-1))$, $h_0(2) = x_0$,
$h_1(1) = 0$, and $h_1(2) = g_1(\tau(2i))$ ∎

When E_i is contained in A'_1, we may have further results. In $\mathcal{H}$, $\rho(g_0, g_1)$ represents a limit of the functions in $EQ(\Gamma)$ whose values on the vertices in A'_1 approach to the infinite point x_0. Thus, in $CC_2(M)$, $\beta_i(\rho(g_0, g_1))$ also represents the limit of pairs of points in M, which approach to (x_0, x_0).

Furthermore, by the property of the map $f : C_2(M) \longrightarrow S^2$, stated in Section 1.1,

$$f(x, y) = \frac{y - x}{|y - x|}$$

for (x, y) in a neighborhood (x_0, x_0).

Thus the limit point $\beta_i(\rho(g_0, g_1))$ also satisfies the same equality

$$f(\beta_i(\rho(g_0, g_1))) = \frac{g_1(\tau(2i)) - g_1(\tau(2i-1))}{|g_1(\tau(2i)) - g_1(\tau(2i-1))|} .$$

When E_i meets both A_0 and A'_1, for example, $\tau(2i-1)$ is in A_0 and $\tau(2i)$ is in A'_1, we also have a similar result that

$$f(\beta_i(\rho(g_0, g_1))) = \frac{g_1(\tau(2i)) - 0}{|g_1(\tau(2i)) - 0|} .$$

In summary, we have

Proposition (6.2)

(i) If E_i is contained in A'_1,

$$f(\beta_i(\rho(g_0, g_1))) = \frac{g_1(\tau(2i)) - g_1(\tau(2i-1))}{|g_1(\tau(2i)) - g_1(\tau(2i-1))|} .$$

(ii) If E_i meets both A_0 and A'_1, and $\tau(2i)$ is in A'_1,

$$f(\beta_i(\rho(g_0, g_1))) = \frac{g_1(\tau(2i))}{|g_1(\tau(2i))|} .$$

(iii) If E_i meets both A_0 and A'_1, and $\tau(2i-1)$ is in A'_1,

$$f(\beta_i(\rho(g_0, g_1))) = \frac{-g_1(\tau(2i-1))}{|g_1(\tau(2i-1))|} . \qquad \blacksquare$$

Because $(\prod_{1\le i\le k}\beta_i)\circ\rho$ is exactly the extension of $\Psi\circ\phi_\tau$ to the boundaries, we have got partial information on the map of $\Phi\circ\phi_\tau$ restricting to $\widehat{M}^{A_0}\times\widehat{C}^{A_1'}$.

6.1 Remark on the extension of the map f

As the notations in Lemma 6.1, we may describe the embedding $\theta: C_2(M)\longrightarrow\widehat{M}^{A_0}\times\widehat{C}^{A_1'}$ by: $\theta(x,y)=(h_0,h_1)$, $h_0(1)=h_1(1)=x$ and $h_0(2)=h_1(2)=y$. Now, we choose a point y in M and a point x in $\mathbb{R}^3\times\{0\}$, then the point (tx,y) is in $C_2(M)$, for t sufficiently large. Consider the limit of $\theta(tx,y)=(h_0^t,h_1^t)$, as $t\to\infty$, the limit point (h_0^∞,h_1^∞) of (h_0^t,h_1^t) are as follows: $h_0^\infty(1)=x_0$, $h_0^\infty(2)=y$; $h_1^\infty(1)=x$, $h_1^\infty(2)=0$. The corresponding values of $f: C_2(M)\longrightarrow S^2$ are

$$f(tx,y)=\frac{\pi(y)-tx}{|\pi(y)-tx|}=\frac{t^{-1}\pi(y)-x}{|t^{-1}\pi(y)-x|}\ ,$$

as $t\to\infty$, its limit is

$$\frac{0-x}{|0-x|}\ .$$

Therefore, we should define the value of f on the limit point (h_0^∞,h_1^∞) as follows:

$$f(h_0^\infty,h_1^\infty)=\frac{0-x}{|0-x|}=\frac{0-h_1^\infty(1)}{|0-h_1^\infty(1)|}\ .$$

This implies the result in Proposition (6.2) (iii). We can also get the other results in (6.2), similarly.

6.2 Proof of Proposition (4.2)

Assume that the maps $g_1,g_1': A_1'\longrightarrow\mathbb{R}^3$ are equivalent under the translation relation and $g_0: A_0\longrightarrow M$ is a map in M^{A_0}.

For each i, $1\le i\le k$, consider the i-th edge E_i and the map $f\circ\beta_i:\mathcal{H}\longrightarrow S^2$ defined above.

By the assumption, E_i is either in A_0 or in A_1'.

When E_i is in A_1', the formula in Proposition (6.2) (i) directly implies

$$f(\beta_i(\rho(g_0,g_1)))=f(\beta_i(\rho(g_0,g_1')))\ .$$

When E_i is in A_0, $\beta_i(\rho(g_0, g_1))$ is an element in $C_2(M)$ and it is dependent only on the map g_0. Precisely, as the notation in Lemma (6.1), an element in $C_2(M)$ is completely determined by the map h_0, that is, the element $(h_0(1), h_0(2))$ in $M \times M$. Then, by Lemma (6.1) (i), we have
$\beta_i(\rho(g_0, g_1)) = \beta_i(\rho(g_0, g_1')) = ((g_0 \circ \tau)(2i-1), (g_0 \circ \tau)(2i))$,
which also implies the equality

$$f(\beta_i(\rho(g_0, g_1))) = f(\beta_i(\rho(g_0, g_1'))) \ .$$

Thus $\prod_{1 \le i \le k} (f \circ \beta_i)$ has the same value on (g_0, g_1) and (g_0, g_1'), that is,

$$\Phi(\phi_\tau(g_0, g_1)) = \Phi(\phi_\tau(g_0, g_1')) \ .$$

To prove the last statement that the dimension of $\Phi(\phi_\tau(\widehat{M}^{A_0} \times \widehat{C}^{A_1'}))$ is less than or equal to $(3m-4)$ (m is the number of vertices in Γ), it is enough to check the obvious results that the dimension of M^{A_0} is $3m_0$ and the dimension of $(\widehat{C}^{A_1'}/\sim_t)$ is $3(m - m_0) - 4$, where m_0 is the number of vertices in A_0.

This proves Proposition (4.2).

To prove Proposition (4.3), we need the following lemma.

Lemma (6.3) Suppose $\Phi = (\phi_1, \phi_2, \cdots, \phi_k) : X \longrightarrow \prod_k S^2$ is a smooth map from a smooth manifold X to $\prod_k S^2$, and there are i and j, $1 \le i < j \le k$, such that either $\phi_i(x) = \phi_j(x)$, for all $x \in X$, or $\phi_i(x) = -\phi_j(x)$, for all $x \in X$. Then the dimension of $\Phi(X)$ is less than or equal to $(2k-2)$. ∎

The proof of the lemma is straightforward and is omitted.

6.3 Proof of Proposition (4.3)

By assumption, $A_1'(\Gamma)$ has a free vertex or a univalent vertex, say, v ($v \in A_1'$). Also, by the trivalency of Γ, there are i and j, $1 \le i < j \le k$, such that $E_i = \{v, w_i\}$ and $E_j = \{v, w_j\}$, $w_i, w_j \in A_0$. Thus, v is equal to $\tau(2i-1)$ or $\tau(2i)$, and, v is also equal to $\tau(2j-1)$ or $\tau(2j)$.

(There are three edges E_i, E_j, and E_l containing the vertex v; E_i and E_j meet both A_0 and A_1', E_l could be in A_1' or meet both A_0 and A_1'.)

Then, by Proposition (6.2) (ii) and (iii),
if $v = \tau(2i-1) = \tau(2j-1)$, or, if $v = \tau(2i) = \tau(2j)$,

$$f \circ \beta_i = f \circ \beta_j \ ;$$

if $v = \tau(2i-1) = \tau(2j)$, or, if $v = \tau(2i) = \tau(2j-1)$,

$$f \circ \beta_i = -f \circ \beta_j \ .$$

By Lemma (6.3), we prove Proposition (4.3).

6.4 Proof of Proposition (4.4)

By assumption, an increasing family $\mathcal{I} = (A_0, A_1)$ is considered. The edges $E_i = \{\tau(2i-1), \tau(2i)\}$, $i = 1, 2, \cdots, k$, can be separated into three different kinds:

(1) The r edges E_{i_1}, E_{i_2}, $\cdots$, E_{i_r}, each of those edges meets both A_0 and A_1';

(2) The s edges $E_{i_{r+1}}$, $E_{i_{r+2}}$, $\cdots$, $E_{i_{r+s}}$, each of those edges is contained in A_1';

(3) The t edges E_{j_1}, E_{j_2}, $\cdots$, E_{j_t}, each of those edges is contained in A_0.

r, s and t must satisfy the equality $r + s + t = k$. Also, by assumption that $A_1'(\Gamma)$ has at least one bivalent vertex, $r \geq 1$.

We consider the product of the maps $f \circ \beta_{i_h}$, $h = 1, 2, \cdots, r+s$,

$$\prod_{1 \leq h \leq r+s} f \circ \beta_{i_h} : \widehat{M}^{A_0} \times \widehat{C}^{A_1'} \longrightarrow \prod_{r+s} S^2 \ .$$

By Proposition (6.2), for each h, $1 \leq h \leq r+s$, the value $(f \circ \beta_{i_h})(g_0, g_1)$ depends only on the map $g_1 : A_1' \longrightarrow \mathbf{R}^3$.

Therefore, we can think that $\prod_{1 \leq h \leq r+s} f \circ \beta_{i_h}$ is a map from $\widehat{C}^{A_1'}$ to $\prod_{r+s} S^2$.

Lemma (6.4) The dimension of $\widehat{C}^{A'_1}$ is equal to $r+2s-1$. ∎

Thus the image of the map $\prod_{1\le h\le r+s} f\circ\beta_{i_h}$ has codimension at least $r+1$ in the space $\prod_{r+s} S^2$, and hence, so is the map of total product $\prod_{1\le i\le k} f\circ\beta_i$. By assumption, $r\ge 1$, which implies that the image of $\prod_{1\le i\le k} f\circ\beta_i$ has codimension at least 2 in $\prod_k S^2$.

Proof of Lemma (6.4):

Assume there are m_1 bivalent vertices and m_2 trivalent vertices in $A'_1(\Gamma)$. Then the dimension of $\widehat{C}^{A'_1}$ is equal to $3m_1+3m_2-1$. And, by counting the end-points of the edges in (1) and (2), we have the equality $2(r+s)=r+3m_1+3m_2$. Thus the dimension of $\widehat{C}^{A'_1}$ is equal to $r+2s-1$. This proves the lemma.

References

[1] D. Altschuler and L. Freidel, *Vassiliev knot invariants and Chern-Simons perturbation theory to all order*, 1995, Preprint.

[2] S. Axelrod and I. Singer, *Chern-Simons perturbation theory*, in the proceedings of the XXth International conference on differential geometric methods in theoretical physics, June 3-7, 1991, New City (World Scientific, Singapore, 1992).

S. Axelrod and I. Singer, *Chern-Simons perturbation theory II*, Jour. Diff. Geom. **39** (1994) 173.

[3] D. Bar-Natan, *Perturbative Aspects of the Chern-Simons Topological Quantum Field Theory*, Ph. D. thesis, Princeton University, June 1991.

[4] D. Bar-Natan, *Perturbative Chern-Simons theory*, Journal of Knot Theory and its Ramifications **4** (4) (1995), 503-547.

[5] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology **34** (1995) 423.

[6] R. Bott and A. S. Cattaneo, *Integral Invariants of 3-Manifolds*, J. Diff. Geom. **48** (1998) 91-133.

R. Bott and A. S. Cattaneo, *Integral Invariants of 3-Manifolds. II.*

[7] R. Bott and C. Taubes, *On the self-linking of knots*, Jour. Math. Phys. **35** (10) (1994), 5247-87.

[8] W. Fulton and R. MacPherson, *Compactification of Configuration spaces*, Ann. Math. 139 (1994), 183-225.

[9] M. Kontsevich, *Feynman Diagrams and Low-Dimensional Topology*, First European Congress of Mathematics, Paris 1992, Volume II, *Progress in Mathematics* **120** (Birkhauser, 19940, 120.

[10] S. Poirier, *Rationality Results for the Configuration Space Integral of Knots*, Math. GT/9901028.

[11] V. A. Vassiliev, *Cohomology of knot spaces*, in "Theory of singularities and its applications" (ed. V. I. Arnold), Advances in Soviet Mathematics, AMS, 1990.

[12] E.Witten, *Quantum Field Theory and Jones Polynomial*, Comm. Math. Phys. **121** (1989), 351-399.

[13] S. W. Yang, *Feynman Integral, Knot Invariant and Degree Theory of Maps*, q-alg/9709029.

[14] S. W. Yang, *A Natural Framing for Asymptotically Flat Integral Homology 3-Sphere*, math. GT/9903067.

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