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Extension of Real Numbers Topological Space to Fuzzy Numbers Topological Space

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Abstract

The fuzzy numbers, fuzzy points on $R(=(-\infty,+\infty))$ are all made vaguely from the real numbers, and fuzzy intervals are all made vaguely from the intervals of real numbers. The fuzzy sets on R that we considered here are fuzzy numbers, fuzzy intervals, fuzzy points and their arbitrary intersections,unions. Let F_N denote the family of these fuzzy sets. We construct a fuzzy topological space on F_N which has some connection with the usual topological space (R, T_R).

Keywords: fuzzy topology, fuzzy topological space, fuzzy number, fuzzy point, fuzzy interval.

1.Introduction

Since the fuzzy numbers, fuzzy points on R are all made vaguely from the real numbers, and fuzzy intervals are all made vaguely from the intervals of real numbers. Hence there are some relations between fuzzy numbers, fuzzy points, fuzzy intervals and real numbers. By using these relations, we may induce a fuzzy topological space for real line R.

In §2, let the fuzzy sets on R considered be fuzzy numbers, fuzzy intervals, fuzzy points and their arbitrary intersections,unions. Let this family be denoted by F_N. Since the intersections of fuzzy intervals, fuzzy points can be expressed by the unions of fuzzy intervals, fuzzy points, empty set, in this paper, we shall learn that fuzzy numbers can be written as a countable union of fuzzy intervals. Therefore all the fuzzy sets in F_N are fuzzy intervals, fuzzy points and their unions.

Let $\mathcal{B}^* = \{(a,b),\ (a,\infty),\ (-\infty,b),\ (-\infty,\infty) \mid \forall a < b,\ \ a,b \in R\}$ be the set of all open intervals in R. Let T_R be the family of all open sets in R and obtain the crisp topological space (R, T_R).

Corresponding to this $\mathcal{B}^*$, let $\mathcal{B} = \{(a_\lambda, b_\lambda),\ (a_\lambda,\infty),\ (-\infty, b_\lambda),\ (-\infty,\infty) \mid \forall\ 0 < \lambda \leq 1,\ a < b,\ a,b \in R\}$. Similar as the topology defined on R, we may have T_F, the family of all open fuzzy sets made from F_N.

Recall that the fuzzy sets we considered are those in F_N. Since these fuzzy sets in F_N are fuzzy intervals, or their unions, it is possible to use the sets in $\mathcal{B}$ to define the open fuzzy sets in F_N. Thus we have the fuzzy topological space (R, T_F) for R.

Let $\mathcal{B}_1$ be the family of all level 1 open fuzzy intervals, i.e.

$$\mathcal{B}_1 = \{(a_\lambda, b_\lambda),\ (a_\lambda,\infty),\ (-\infty, b_\lambda),\ (-\infty,\infty) \mid \lambda = 1,\ \forall\ a < b,\ a,b \in R\}$$

and $F_p(1)$ be the family of all level 1 open fuzzy points, i.e. $F_p(1) = \{a_\lambda \mid \lambda = 1,\ \forall\ a \in R\}$. Then we will have an one-one onto mapping $F_p(1) \longleftrightarrow R$, $\mathcal{B}_1 \leftrightarrow \mathcal{B}^*$. Let F_1 be the family of all level 1 fuzzy intervals, level 1 fuzzy points and their unions. We can then define the open fuzzy sets from sets in $\mathcal{B}_1$ and induce a fuzzy topology T_p and a fuzzy topological space (R, T_p) for R. If we restrict the fuzzy sets on R to sets in F_1, there is an one-one onto mapping between T_p and T_R which leads to the result that the topological space (R, T_p) is isomorphic to the crisp topological space (R, T_R). And also since $T_p \subset T_F$, T_p is isomorphic to T_R, we can treat T_R as a subfamily of T_F or conversely, say (R, T_F), an extension of (R, T_R).

2. Fuzzy topology for R and fuzzy topological space

The fuzzy numbers, fuzzy points on R are all made vaguely from the real numbers. The fuzzy intervals which we will define later are also made vaguely from the intervals of real numbers. Therefore there are some connections with the real numbers. By using these connections, we shall construct a fuzzy topology on R.

Definition 2.1 The fuzzy numbers on R in general have the following forms:

$$(1^\circ)\quad \mu_{\tilde{M}_L}(x) = \begin{cases} f_L(x), & a \le x \le b \\ 0, & \text{elsewhere} \end{cases} \tag{2.1}$$

$$(2^\circ)\quad \mu_{\tilde{M}_R}(x) = \begin{cases} f_R(x), & b \le x \le c \\ 0, & \text{elsewhere} \end{cases} \tag{2.2}$$

$$(3^\circ)\quad \mu_{\tilde{M}}(x) = \begin{cases} f_L(x), & a \le x \le b \\ f_R(x), & b \le x \le c \\ 0, & \text{elsewhere} \end{cases} \tag{2.3}$$

where $f_L(x)$ is a continuous increasing function in $[a, b]$, $f_R(x)$ is a continuous decreasing function in $[b, c]$ and $f_L(a) = f_R(c) = 0$, $f_L(b) = f_R(b) = 1$.

Definition 2.2 The membership function of a fuzzy set a_λ, $0 < \lambda \le 1$ on R is defined as follows:

$$\mu_{a_\lambda}(x) = \begin{cases} \lambda, & \text{if } x = a \\ 0, & \text{if } x \ne a \end{cases} \tag{2.4}$$

we call a_λ, a level λ fuzzy point. Let the family of all level λ fuzzy points be $F_p(\lambda) = \{a_\lambda \mid \forall\, a \in R\}$, $0 < \lambda \le 1$ and let the family of all fuzzy points be $F_p = \bigcup_{0<\lambda\le 1} F_p(\lambda)$.

Definition 2.3

(a) The membership function of the fuzzy set $(a_\lambda,\ b_\lambda)$, $0 < \lambda \le 1$ on R is defined as follows:

$$\mu_{(a_\lambda,\ b_\lambda)}(x) = \begin{cases} \lambda, & \text{if } a < x < b \\ 0, & \text{elsewhere} \end{cases} \tag{2.5}$$

we call $(a_\lambda,\ b_\lambda)$, a level λ fuzzy interval. Also we can write $(a_\lambda,\ b_\lambda) = \bigcup_{a<x<b} x_\lambda$. Other level λ fuzzy intervals, $0 < \lambda \le 1$; are $[a_\lambda,\ b_\lambda]$, $[a_\lambda,\ b_\lambda)$, $(a_\lambda,\ b_\lambda]$, $(a_\lambda,\ \infty)$, $[a_\lambda,\ \infty)$, $(-\infty,\ b_\lambda)$, $(-\infty,\ b_\lambda]$, $(-\infty,\ \infty)$. Their membership functions are defined accordingly as in (2.5). For instance, the membership function of $[a_\lambda,\ \infty)$ is

$$\mu_{[a_\lambda,\ \infty)}(x) = \begin{cases} \lambda, & \text{if } x \ge a \\ 0, & \text{if } x < a \end{cases}$$

The fuzzy point b_λ can be written as $b_\lambda = (a_\lambda,\ b_\lambda]\bigcap[b_\lambda,\ c_\lambda),\ a < b < c$. Let $I(a_\lambda,\ b_\lambda)$ denote any one of the following fuzzy intervals, $(a_\lambda,\ b_\lambda)$, $[a_\lambda,\ b_\lambda]$, $[a_\lambda,\ b_\lambda)$, $(a_\lambda,\ b_\lambda]$. Let $F_I(\lambda) = \{I(a_\lambda,\ b_\lambda) \mid \forall\ a < b,\ a, b \in R\}$ be the family of all level λ fuzzy intervals, and let $F_I = \bigcup_{0<\lambda\leq 1} F_I(\lambda)$ be the family of all fuzzy intervals. When $a = -\infty$, $I(a_\lambda,\ b_\lambda) = (-\infty,\ b_\lambda)$ or $(-\infty,\ b_\lambda]$; When $b = \infty$, $I(a_\lambda,\ b_\lambda) = (a_\lambda,\ \infty)$ or $[a_\lambda,\ \infty)$; When $a = -\infty$, $b = \infty$, we assume this fuzzy interval $(a_\lambda,\ b_\lambda)$ with $\lambda = 1$; i.e. $\mu_{(a_1,\ b_1)}(x) = 1,\ \forall\ -\infty < x < \infty$. Meanwhile, the characteristic function of R is $C_R(x) = 1,\ \forall\ -\infty < x < \infty$. Therefore $\mu_{(a_1,\ b_1)}(x) = C_R(x)$, $\forall\ x$, i.e. if $\lambda = 1$, $a = -\infty$, $b = \infty$, (a_1, b_1) is exact the same as $(-\infty,\ \infty)$. That is to say, we may treat R as a level 1 fuzzy interval. Thus $R \in F_I(1)$.

The following are special cases of (2.1)~(2.3).

(b) The membership function of the fuzzy set $\tilde{A}$ on R is

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & \text{elsewhere} \end{cases} \tag{2.6}$$

we call $\tilde{A}$, a triangular fuzzy number, denoted by $\tilde{A} = (a, b, c)$, $a < b < c$. Let F_T be the family of all triangular fuzzy numbers, i.e. $F_T = \{(a,\ b,\ c) \mid \forall\ a < b < c,\ a, b, c \in R\}$.

(c) The membership function of the fuzzy set $\tilde{B}$ on R is

$$\mu_{\tilde{B}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \leq x \leq b \\ 0, & \text{elsewhere} \end{cases} \tag{2.7}$$

we call $\tilde{B}$, a left triangular fuzzy number. Let $F_L = \{(a,\ b,\ c) \mid \forall\ a < b = c,\ a, b, c \in R\}$ be the family of all left triangular fuzzy numbers.

(d) The membership function of the fuzzy set $\tilde{C}$ on R is

$$\mu_{\tilde{C}}(x) = \begin{cases} \frac{c-x}{c-b}, & b \leq x \leq c \\ 0, & \text{elsewhere} \end{cases} \tag{2.8}$$

we call $\tilde{C}$, a right triangular fuzzy number. Let $F_R = \{(a,\ b,\ c) \mid \forall\ a = b < c,\ a, b, c \in R\}$ be the family of all right triangular fuzzy numbers.

Let

$$F = F_T \cup F_L \cup F_R = \{(a,\ b,\ c) \mid \forall\ a \leq b \leq c,\ \text{except } a = b = c\ a, b, c \in R\}.$$

For convenience, we shall call all the members in F, and those defined in (2.1)~(2.3) by fuzzy numbers.

Let $I(a,b)$ be the real number interval corresponding to $I(a_\lambda, b_\lambda)$. For instance, if $I(a_\lambda, b_\lambda) = [a_\lambda, b_\lambda)$, then $I(a,b) = [a,b)$. And let $R_I = \{I(a,\ b) | \ \forall\ a < b,\ a, b \in R\}$. Since the characteristic function for (a,b) is $C_{(a,b)}(x) = 1$, if $a < x < b$; and $C_{(a,b)}(x) = 0$, if $x \le a$ or $x \ge b$. Therefore we have $\mu_{(a_1,\ b_1)}(x) = C_{(a,b)}(x),\ \forall\ x \in R$. Similarly $\mu_{I(a_1,\ b_1)}(x) = C_{I(a,b)}(x),\ \forall\ x \in R$. Also there is an one-one onto mapping

$$I(a_1, b_1) \leftrightarrow I(a,b), \quad \forall I(a_1, b_1) \in F_I(1), \quad \forall I(a,b) \in R_I.$$

Hence $F_I(1)$ is equivalent to R_I, denoted by $F_I(1) \cong R_I$. That is, the level 1 fuzzy interval $I(a_1, b_1)$ and the real number interval $I(a,b)$ are the same thing but with different expressions.

We note that there are some connections between $F_p(1)$ and R too. The characteristic function for real number a is $C_a(x) = 1$, if $x = a$; and $C_a(x) = 0$, if $x \neq a$. And the membership function of level 1 fuzzy point $a_1 (\in F_p(1))$ is $\mu_{a_1}(x) = 1$, if $x = a$; and $\mu_{a_1}(x) = 0$, if $x \neq a$. Therefore $\mu_{a_1}(x) = C_a(x),\ \forall\ x \in R$. Also there is an one-one onto mapping $a_1 (\in F_p(1)) \leftrightarrow a (\in R)$. Hence $F_p(1)$ is equivalent to R, denoted by $F_p(1) \cong R$. That is, the level 1 fuzzy point a_1 and the real number a are same thing but with different expressions. Between F_p and R, F_I and R_I, we have a many-one onto mapping. For each $\lambda \in (0,1]$,

$$a_\lambda (\in F_p(\lambda)) \rightarrow a (\in R), \quad I(a_\lambda, b_\lambda)(\in F_I(\lambda)) \rightarrow I(a,b)(\in R_I).$$

The fuzzy sets on R considered in the following are all fuzzy numbers, fuzzy points, fuzzy intervals and their arbitrary intersections, unions. Let this family be denoted by F_N.

In order to establish a fuzzy topology for R, we need to know the following definitions.

Definition 2.4 (Chang [2], definition 2.2) A fuzzy topology is a family T of fuzzy sets in X satisfies the following conditions:

(a) $\phi,\ X \in T$,

(b) If $\tilde{A},\ \tilde{B} \in T$, then $\tilde{A} \cap \tilde{B} \in T$,

(c) If $\tilde{A}_i \in T$, $\forall i \in I$, where I is any index set, then $\bigcup_{i \in I} \tilde{A}_i \in T$.

T is called a fuzzy topology for X and the pair (X,T) is a fuzzy topological space (FTS for short). Every member of T is called $T-$open fuzzy set. A fuzzy set $\tilde{C}$ in X is $T-$closed fuzzy set iff its complement $\tilde{C}'$ is a $T-$open fuzzy set.

Definition 2.5 (Chang [2], definition 2.3) A fuzzy set $\tilde{U}$ in a FTS (X,T) is a neighborhood (nbhd for short) of a fuzzy set $\tilde{A}$ iff there exists an open fuzzy set $\tilde{O}(\in T)$ such that $\tilde{A} \subset \tilde{O} \subset \tilde{U}$.

Definition 2.6 (Chang [2], definition 3.1) A sequence of fuzzy sets, say $\{\tilde{A}_n;\ n = 1,2,\cdots,\}$ is eventually contained in a fuzzy set $\tilde{A}$ iff there exists an integer m such that whenever $n \geq m$, then $\tilde{A}_n \subset \tilde{A}$. If $\{\tilde{A}_n;\ n = 1,2,\cdots,\}$ is a sequence in FTS (X,T), then we say that this sequence converges to a fuzzy set $\tilde{A}$ iff it is eventually contained in each nbhd of $\tilde{A}$ (i.e. if $\tilde{B}$ is any nbhd of $\tilde{A}$, there is a positive integer m such that whenever $n \geq m$, $\tilde{A}_n \subset \tilde{B}$).

Since $F_p(1) \cong R$, and $F_I(1) \cong R_I$, therefore in order to establish a fuzzy topology for R relatively to the topology on R, we first go over the topology on R. Let $\mathcal{B}^*$ be the family of all open intervals on R, i.e.

$$\mathcal{B}^* = \{(a,b),\ (a,\infty),\ (-\infty,b),\ (-\infty,\infty) \mid \forall a < b,\ a,b \in R\}.$$

Define the open set on R as follows:

Definition 2.7 (corresponds to the Definition 2.8 below) A subset O of R is an open set iff for each $x \in O$ there exists an $U \in \mathcal{B}^*$ such that $x \in U \subset O$.

Let T_R be the family of all open sets of R defined in Definition 2.7. It's being proved that T_R satisfies the definition of a topology. Therefore we have a topological space (R,T_R). Since $F_p(1) \cong R$, $F_I(1) \cong R_I$, corresponding to the topology T_R induced by Definition 2.7, we shall consider the following fuzzy topology for R.

Corresponding to $\mathcal{B}^*$, let

$$\mathcal{B} = \{(a_\lambda,b_\lambda),\ (a_\lambda,\infty),\ (-\infty,b_\lambda),\ (-\infty,\infty) \mid \forall 0 < \lambda \leq 1,\ \forall a < b,\ a,b \in R\}$$

be the family of all level λ $(0 < \lambda \leq 1)$ open fuzzy intervals. Let $\mathcal{B}_1$ be the family of all level 1 open fuzzy intervals, i.e.

$$\mathcal{B}_1 = \{(a_1,b_1),\ (a_1,\infty),\ (-\infty,b_1),\ (-\infty,\infty) \mid \forall a < b,\ a,b \in R\} \subset \mathcal{B}.$$

From the following properties 2.3 and 2.4, we will learn the facts that all the fuzzy numbers in F and those defined in (2.1)~(2.3) can be expressed by countable unions of fuzzy intervals. And we know that the elements in F_N are fuzzy points, fuzzy intervals, fuzzy numbers and their arbitrary intersections, unions. Since the intersection can be expressed by union of fuzzy points or fuzzy intervals or ϕ, then F_N contains fuzzy points, fuzzy intervals, fuzzy numbers and their arbitrary unions. Hence we can discuss problems on F_N by using the fuzzy topology T_F induced by $\mathcal{B}$.

Definition 2.8 A fuzzy set $\tilde{O}(\in F_N)$ on R is an open fuzzy set iff for each $x_\lambda \subset \tilde{O}$, there exists $\tilde{U} \in \mathcal{B}$ such that $x_\lambda \subset \tilde{U} \subset \tilde{O}$.

Let T_F be the family of all open fuzzy sets in F_N. Obviously $\mathcal{B} \subset T_F$. We now prove that T_F is a fuzzy topology as defined in Definition 2.4.

Property 2.1

(1) T_F is a fuzzy topology for R.

(2) (R, T_F) is a fuzzy topological space.

Where the fuzzy sets on R are restricted to the fuzzy sets in F_N.

Proof:

(a) ϕ is open, $R \in \mathcal{B}$, therefore ϕ, $R \in T_F$. So (a) of definition 2.4 is satisfied.

(b) If $\tilde{A}$, $\tilde{B} \in T_F$, then for each $x_\lambda \subset \tilde{A} \cap \tilde{B}$, we have $x_\lambda \subset \tilde{A}$ and $x_\lambda \subset \tilde{B}$. Since $\tilde{A}$, $\tilde{B}$ are open fuzzy sets, by Definition 2.8, there exists $\tilde{U}$, $\tilde{V} \in \mathcal{B}$ such that $x_\lambda \subset \tilde{U} \subset \tilde{A}$, and $x_\lambda \subset \tilde{V} \subset \tilde{B}$ which means

$$x_\lambda \subset \tilde{U} \bigcap \tilde{V} \subset \tilde{A} \bigcap \tilde{B}$$

Since $x_\lambda \subset \tilde{U} \bigcap \tilde{V}$, therefore $\tilde{U} \bigcap \tilde{V} \neq \phi$. The intersection of $\tilde{U}$ and $\tilde{V}$ is an open fuzzy interval. For instance,

$$(a_\alpha, b_\alpha) \cap (c_\beta, d_\beta) = (c_\alpha, b_\alpha) \in \mathcal{B} \text{ if } a < c < b < d,\ 0 < \alpha \leq \beta \leq 1.$$

Therefore $\tilde{U} \cap \tilde{V} \in \mathcal{B}$, i.e. $\tilde{A} \cap \tilde{B} \in T_F$. Definition 2.4(b) is satisfied.

(c) If $\tilde{A}_j \in T_F$, $j \in I$(any index set), then for each $x_\lambda \subset \bigcup_{j \in I} \tilde{A}_j$, there exists some $n \in I$ such that $x_\lambda \subset \tilde{A}_n$. Since $\tilde{A}_n$ is an open fuzzy set, there is an $\tilde{U} \in \mathcal{B}$ such that $x_\lambda \subset \tilde{U} \subset \tilde{A}_n \subset \bigcup_{j \in I} \tilde{A}_j$. Therefore $\bigcup_{j \in I} \tilde{A}_j \in T_F$. Definition 2.4 (c) is satisfied.

Now let $X = R$, $T = T_F$ in definition 2.4, it will satisfy all the three conditions of Definition 2.4. Therefore T_F is a fuzzy topology and hence (R, T_F) is a fuzzy topological space. From property 2.1, when $X = R$, $T = T_F$, Definition 2.4 is fulfilled. Therefore Definition 2.5 and 2.6 hold for $X = R$ and $T = T_F$. $\tilde{C} \in F_X$ is a closed fuzzy set iff its complement $\tilde{C}'$ is an open fuzzy set.

In $F_I(1)$, intersection of two level 1 fuzzy intervals is a level 1 fuzzy point or union of level 1 fuzzy points or ϕ. $F_p(1) \cong R$, $F_I(1) \cong R_I$. Similarly to T_F, we can induce a fuzzy topology to T_p by $\mathcal{B}_1$. There is an one-one onto mapping between $\mathcal{B}_1$ and $\mathcal{B}^*$. Also, it leads to the one-one onto mapping between T_p and T_R. For example, if $a < b < c < d$,

$$(a_1, b_1) \cup (c_1, d_1)(\in T_p) \leftrightarrow (a, b) \cup (c, d)(\in T_R).$$

Since $\mathcal{B}_1 \subset \mathcal{B} \subset T_F$, therefore T_p is a subfamily of T_F. Hence T_R may be considered as a subfamily of T_F, or to say, T_F is an extension of T_R.

Example 2.1

(1) The complement set of level λ, $(0 < \lambda < 1)$, fuzzy interval $[a_\lambda, b_\lambda]$ is

$$[a_\lambda, b_\lambda]' = (-\infty, a_1) \cup [a_{1-\lambda}, b_{1-\lambda}] \cup [b_1, \infty) \notin T_F.$$

Therefore $[a_\lambda, b_\lambda]$ is not closed.

(2) The complement set of level 1 fuzzy interval $[a_1, b_1]$ is

$$[a_1, b_1]' = (-\infty, a_1) \cup (b_1, \infty) \in T_F.$$

Therefore $[a_1, b_1]$ is a closed fuzzy interval.

Property 2.2 Let $\tilde{A}_1 \subset \tilde{A}_2 \subset \cdots \subset \tilde{A}_n \subset \cdots \subset \tilde{A}$ be a sequence of increasing fuzzy sets in F_X. If $\lim_{n\to\infty} \mu_{\tilde{A}_n}(x) = \mu_{\tilde{A}}(x)$, $\forall\ x$, then sequence $\{\tilde{A}_n;\ n = 1, 2, \cdots\}$ will converge to $\tilde{A}$ and denoted by $\tilde{A} = \lim_{n\to\infty} \tilde{A}_n = \bigcup_{n=1}^{\infty} \tilde{A}_n$.

Proof: Let $\tilde{B}$ be any nbhd of $\tilde{A}$, then by definition 2.5, there exists $\tilde{O} \in T_F$ such that $\tilde{A} \subset \tilde{O} \subset \tilde{B}$. Let m be any fixed positive number, then for all $n \geq m$, $\tilde{A}_n \subset \tilde{A} \subset \tilde{B}$. Thus by definition 2.6, sequence $\{\tilde{A}_n;\ n = 1, 2, \cdots\}$ converges to $\tilde{A}$.

Property 2.3

(a) The fuzzy number $\tilde{B} = (a, b, b)$ in F_L can be written as a countable union of fuzzy intervals $\tilde{B}_{k,n}$, i.e.

$$\tilde{B} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{B}_{k,n}$$

where $\tilde{B}_{k,n}$ is defined in the proof.

(b) The fuzzy number $\tilde{C} = (b, b, c)$ in F_R also can be written as a countable union of fuzzy intervals $\tilde{C}_{k,n}$, i.e.

$$\tilde{C} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{C}_{k,n}$$

where $\tilde{C}_{k,n}$ is defined in the proof.

(c) The fuzzy number $\tilde{A} = (a, b, c)$ in F_T can be written as a countable union of fuzzy intervals, i.e.

$$\tilde{A} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} (\tilde{B}_{k,n} \cup \tilde{C}_{k,n})$$

Proof:

(a) For any fixed positive integer n, $n = 1, 2, \cdots$, we now partition $[a, b]$ into n equal subintervals by points $p^{(k,n)} = a + \frac{k(b-a)}{n}$, $k = 0, 1, \cdots, n$. Then

$$[a, b] = [p^{(0,n)}, p^{(1,n)}) \cup [p^{(1,n)}, p^{(2,n)}) \cup \cdots \cup [p^{(n-1,n)}, p^{(n,n)}]$$

Let fuzzy intervals $\tilde{B}_{k,n}$ be

$$\tilde{B}_{k,n} = [p^{(k-1,n)}_{\alpha_{k-1,n}}, p^{(k,n)}_{\alpha_{k-1,n}}),\ k = 1, 2, \cdots, n-1; \qquad (*1)$$

and

$$\tilde{B}_{n,n} = [p^{(n-1,n)}_{\alpha_{n-1,n}}, p^{(n,n)}_{\alpha_{n-1,n}}],\quad n = 1, 2, \cdots; \qquad (*2)$$

where

$$\alpha_{k,n} = \frac{p^{(k,n)} - a}{b - a} = \frac{k}{n},\ k = 0, 1, \cdots, n-1;\ n = 1, 2, \cdots.$$

Insert Fig.1 Here

It is obvious that

$$\bigcup_{k=1}^{n} \tilde{B}_{k,n} \subset \tilde{B}, \ \forall n \text{ and } \tilde{B}_{k,n} \cap \tilde{B}_{l,n} = \emptyset, \text{ if } k \neq l. \qquad (*3)$$

Therefore

$$0 \leq \mu_{\tilde{B}}(x) - \mu_{\cup_{k=1}^{n} \tilde{B}_{k,n}}(x) < \frac{1}{n}, \ \forall\, a \leq x \leq b, \text{ and } n = 1, 2, \cdots.$$

Thus we have

$$\lim_{n \to \infty} \mu_{\bigcup_{k=1}^{n} \tilde{B}_{k,n}}(x) = \mu_{\tilde{B}}(x), \ \forall x. \qquad (*4)$$

Now, for each m, $m = 1, 2, \cdots$, let s_m be the least common multiple of $1, 2, \cdots, m$; i.e. $s_m = L.C.M.\{1, 2, \cdots, m\}$. Then there is a positive integer t such that $s_m = tm$. For instance, $m = 1 \Rightarrow s_1 = 1$; $m = 2 \Rightarrow s_2 = 2$; $m = 3 \Rightarrow s_3 = 6$; $\cdots$ etc. and $\lim_{m \to \infty} s_m = \infty$.

Same as in $(*1)$ and $(*2)$, let $s_m + 1$ points of s_m equal subintervals of $[a, b]$ be

$$\{a = p^{(0,s_m)}, p^{(1,s_m)}, p^{(2,s_m)}, \cdots, p^{(s_m,s_m)} = b\}$$

Then we have these fuzzy intervals $\tilde{B}_{k,s_m}$, $k = 1, 2, \cdots, s_m$. If $m < n$, then

$$s_n = L.C.M.(1, 2, \cdots, m, m+1, \cdots, n) = L.C.M.(s_m, m+1, \cdots, n).$$

Hence there is a positive integer $w(m, n)$ such that $s_n = w(m, n) s_m$, i.e. $\frac{s_n}{s_m} = w(m, n)$ is a positive integer. And the numbers that divide $[a, b]$ into s_n equal subintervals and s_m equal subintervals are

$$p^{(l,s_n)} = a + \frac{l(b-a)}{s_n}, \quad l = 0, 1, \cdots, s_n. \qquad (*5)$$

$$p^{(k,s_m)} = a + \frac{k(b-a)}{s_m}, \quad k = 0, 1, \cdots, s_m. \qquad (*6)$$

respectively.

In $(*5)$ and $(*6)$, for those $l = \frac{s_n}{s_m} k = w(m, n) k$, $k = 0, 1, \cdots, s_m$, we have $l = 0, w(m, n), 2w(m, n), \cdots, s_n$. Then

$$p^{(l,s_n)} = a + \frac{l(b-a)}{s_n} = a + \frac{k(b-a)}{s_m} = p^{(k,s_m)}, \quad k = 0, 1, \cdots, s_m,$$

Specially,

$$p^{(0,s_n)} = a = p^{(0,s_m)} \text{ and } p^{(s_n,s_n)} = b = p^{(s_m,s_m)}$$

Thus

$$\{p^{(k,s_m)} | k = 0, 1, \cdots, s_m\} \subset \{p^{(l,s_n)} | l = 0, 1, \cdots, s_n\} \text{ for } m < n.$$

Therefore the sequence of fuzzy sets $\bigcup_{k=1}^{s_m} \tilde{B}_{k,s_m}$, $m = 1, 2, \cdots$ is increasing as m increases, also

$$\bigcup_{k=1}^{s_m} \tilde{B}_{k,s_m} \subset \tilde{B}, \ \forall\, m \quad \text{and} \quad \lim_{m\to\infty} \mu_{\bigcup_{k=1}^{s_m} \tilde{B}_{k,s_m}}(x) = \mu_{\tilde{B}}(x), \ \forall\, x$$

This is the same result as in $(*4)$. Hence from property 2.2 we have

$$\tilde{B} = \bigcup_{m=1}^{\infty} \bigcup_{k=1}^{s_m} \tilde{B}_{k,s_m} \qquad (*7)$$

If $s_r \leq u$, u is a positive integer, then $s_1 < s_2 < \cdots < s_r \leq u$, where $s_1 = 1$, $s_2 = 2$, $s_3 = 6, \cdots$; and

$$\begin{aligned}
&\bigcup_{m=1}^{r} \bigcup_{k=1}^{s_m} \tilde{B}_{k,s_m} \\
&= (\bigcup_{k=1}^{s_1} \tilde{B}_{k,s_1}) \cup (\bigcup_{k=1}^{s_2} \tilde{B}_{k,s_2}) \cup \cdots \cup (\bigcup_{k=1}^{s_r} \tilde{B}_{k,s_r}) \\
&\subset (\bigcup_{k=1}^{s_1} \tilde{B}_{k,s_1}) \cup (\bigcup_{k=1}^{s_2} \tilde{B}_{k,s_2}) \cup \cdots \cup (\bigcup_{k=1}^{s_r} \tilde{B}_{k,s_r}) \cup \cdots \cup (\bigcup_{k=1}^{u} \tilde{B}_{k,u}) \\
&\subset \bigcup_{n=1}^{u} \bigcup_{k=1}^{n} \tilde{B}_{k,n} \subset \tilde{B}, \quad \forall\, r,\ u \text{ if } s_r \leq u \text{ (by } (*3) \text{)}.
\end{aligned}$$

Since $\lim_{r\to\infty} s_r = \infty$, let $r \to \infty$, hence by $(*7)$ we have

$$\tilde{B} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{B}_{k,n}$$

(b) As in part (a), for fixed n, we divide $[b, c]$ into n equal subintervals by points

$$q^{(k,n)} = b + \frac{k(c-b)}{n}, \quad k = 0, 1, \cdots, n;$$

then

$$[b, c] = [q^{(0,n)}, q^{(1,n)}] \cup (q^{(1,n)}, q^{(2,n)}] \cup \cdots \cup (q^{(n-1,n)}, q^{(n,n)}].$$

Let the fuzzy intervals $\tilde{C}_{k,n}$ be

$$\tilde{C}_{1,n} = [q^{(0,n)}_{\beta_{1,n}}, q^{(1,n)}_{\beta_{1,n}}], \tilde{C}_{k,n} = (q^{(k-1,n)}_{\beta_{k,n}}, q^{(k,n)}_{\beta_{k,n}}], k = 2, 3, \cdots, n; \ n = 1, 2, \cdots$$

and

$$\beta_{k,n} = \frac{c - q^{(k,n)}}{c - b} = \frac{n-k}{n}, \; k = 1, 2, \cdots, n; \; n = 1, 2, \cdots$$

Using the same argument as in part (a), we can prove

$$\tilde{C} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{C}_{k,n}$$

(c) Follows from (a) and (b).

Property 2.4

(a) The fuzzy number $\tilde{M}_L$ can be written as a countable union of fuzzy intervals, i.e.

$$\tilde{M}_L = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{M}_{Lkn}$$

where

$$\tilde{M}_{Lkn} = [p^{(k-1,n)}_{\alpha_{k-1,n}}, p^{(k,n)}_{\alpha_{k-1,n}}), k = 1, 2, \cdots, n-1; \tilde{M}_{Lnn} = [p^{(n-1,n)}_{\alpha_{n-1,n}}, p^{(n,n)}_{\alpha_{n-1,n}}]$$

and

$$p^{(k,n)} = a + \frac{k(b-a)}{n}, \; k = 0, 1, \cdots, n; \quad \alpha_{k,n} = f_L(p^{(k,n)}), \; k = 0, 1, \cdots, n-1.$$

(b) The fuzzy number $\tilde{M}_R$ can also be written as a countable union of fuzzy intervals, i.e.

$$\tilde{M}_R = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{M}_{Rkn}$$

where

$$\tilde{M}_{R1n} = [q^{(0,n)}_{\beta_{1,n}}, q^{(1,n)}_{\beta_{1,n}}]; \quad \tilde{M}_{Rkn} = (q^{(k-1,n)}_{\beta_{k,n}}, q^{(k,n)}_{\beta_{k,n}}], k = 2, 3, \cdots, n$$

and

$$q^{(k,n)} = b + \frac{k(c-b)}{n}, \; k = 0, 1, \cdots, n; \quad \beta_{k,n} = f_R(q^{(k,n)}), \; k = 1, 2, \cdots, n.$$

(c) The fuzzy number $\tilde{M}$ can also be written as a countable union of fuzzy intervals, i.e.

$$\tilde{M} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} (\tilde{M}_{Lkn} \cup \tilde{M}_{Rkn}),$$

where $\tilde{M}_{Lkn}$, $\tilde{M}_{Rkn}$ are as in (a) and (b) respectively.

Proof: Similar as in Property 2.3.

There are many-one and onto mappings between F_p and R. F_I and R_I. From Definition 2.5 and 2.6. we may have the following definitions.

Definition 2.9

(a) The ϵ-nbhd of fuzzy point a_λ is the open fuzzy interval $((a-\epsilon)_\lambda, (a+\epsilon)_\lambda)$.

Since $a_\lambda \subset ((a-\epsilon)_\lambda, (a+\epsilon)_\lambda)(\in T_F)$, Definition 2.5 is satisfied.

(b) The ϵ-nbhd of fuzzy interval $I(a_\lambda, b_\lambda)$ is the open fuzzy interval $((a-\epsilon)_\lambda, (a+\epsilon)_\lambda)$.

Since $I(a_\lambda, b_\lambda) \subset ((a-\epsilon)_\lambda, (a+\epsilon)_\lambda)(\in T_F)$, Definition 2.5 is satisfied.

(c) The ϵ-nbhd of real number a is the open interval $(a-\epsilon, a+\epsilon)$.

Remark: For each λ fixed, $0 < \lambda \leq 1$, there is a one-one onto mapping between (a) and (c).

Definition 2.10

(a) In F_p, x_λ approaches to a_λ $(\in F_p)$ iff $\forall\ \epsilon > 0$, $x_\lambda \subset ((a-\epsilon)_\lambda, (a+\epsilon)_\lambda)$.

(b) In F_I, $I(x_\lambda, y_\lambda)$ approaches to $I(a_\lambda, b_\lambda)$ $(\in F_I)$ iff $\forall\ \epsilon > 0$, $I(x_\lambda, y_\lambda) \subset ((a-\epsilon)_\lambda, (a+\epsilon)_\lambda)$.

Note here $I(x_\lambda, y_\lambda)$ and $I(a_\lambda, b_\lambda)$ have the same form, that is, if $I(x_\lambda, y_\lambda) = [x_\lambda, y_\lambda)$, then $I(a_\lambda, b_\lambda) = [a_\lambda, b_\lambda)$.

(c) In R, x approaches to $a(\in R)$ iff $\forall\ \epsilon > 0$, $x \in (a-\epsilon, a+\epsilon)$.

Remark: For each λ fixed. $0 < \lambda \leq 1$, there is a one-one onto mapping between (a) and (c).

Definition 2.11

(a) In F_p, sequence $x_\lambda^{(n)}$, $n = 1, 2, \cdots$ converges to $a_\lambda(\in F_p)$ iff $\forall\ \epsilon > 0$, there exists a natural number m such that whenever $n \geq m$, $x_\lambda^{(n)} \subset ((a-\epsilon)_\lambda, (a+\epsilon)_\lambda)$

(b) In F_I, sequence $I(x_\lambda^{(n)}, y_\lambda^{(n)})$, $n = 1, 2, \cdots$ converges to $I(a_\lambda, b_\lambda)$ iff $\forall\ \epsilon > 0$, there exists a natural number m such that whenever $n \geq m$, $I(x_\lambda^{(n)}, y_\lambda^{(n)}) \subset ((a-\epsilon)_\lambda, (b+\epsilon)_\lambda)$

(c) In R, sequence $x^{(n)}$, $n = 1, 2, \cdots$ converges to $a(\in R)$ iff $\forall\ \epsilon > 0$, there exists a natural number m such that whenever $n \geq m$, $x^{(n)} \in (a-\epsilon, a+\epsilon)$.

Remark: For each λ fixed, $0 < \lambda \leq 1$, there is a one-one onto mapping between (a) and (c).

Note that for each λ, $0 < \lambda \leq 1$, $F_p(\lambda) \cong R$, and $F_I(\lambda) \cong R_I$. Therefore we have the following property.

Property 2.5

(a) In R, x approaches to a iff in F_p, x_λ approaches to a_λ.

(b) In R, sequence $x^{(n)}$, $n = 1, 2, \cdots$ converges to a iff in F_p, sequence $x_\lambda^{(n)}$, $n = 1, 2, \cdots$ converges to a_λ.

Proof: It follows from Definitions 2.10 and 2.11 and the equivalence between F_p and R.

Property 2.6

(a) In R, x approaches to a, and y approaches to b, $x < y$, $a < b$ iff in F_p, x_λ approaches to a_λ, and y_λ approaches to b_λ $\Rightarrow$ In F_I, (x_λ, y_λ) approaches to (a_λ, b_λ).

(b) In R, sequence $x^{(n)}$, $n = 1, 2, \cdots$ converges to a and sequence $y^{(n)}$, $n = 1, 2, \cdots$ converges to b, $x^{(n)} < y^{(n)}$, $\forall n$ and $a < b$ iff in F_p, sequence $x_\lambda^{(n)}$, $n = 1, 2, \cdots$ converges to a_λ and $y_\lambda^{(n)}$, $n = 1, 2, \cdots$ converges to b_λ $\Rightarrow$ In F_I, $(x_\lambda^{(n)}, y_\lambda^{(n)})$ converges to (a_λ, b_λ).

Proof:

(a) In R, by Definition 2.10 (c), we have for each $\epsilon > 0$,

$$x \in (a - \epsilon, a + \epsilon) \text{ and } y \in (b - \epsilon, b + \epsilon) \forall \epsilon > 0$$

$$\Leftrightarrow x_\lambda \subset ((a - \epsilon)_\lambda, (a + \epsilon)_\lambda) \text{ and } y_\lambda \subset ((b - \epsilon)_\lambda, (b + \epsilon)_\lambda) \forall \epsilon > 0 \quad \text{(by property 2.5 (a))}$$

$$\Rightarrow (x_\lambda, y_\lambda) \subset ((a - \epsilon)_\lambda, (b + \epsilon)_\lambda)$$

(b) Similar to part (a).

We shall now proceed to define the neighborhood of fuzzy numbers in F by Definition 2.5 and Property 2.3.

Definition 2.12

(a) The ϵ-nbhd of fuzzy number $\tilde{B} = (a, b, b)$ in F_L is $\bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{B}_\epsilon^{(k,n)}$

(b) The ϵ-nbhd of fuzzy number $\tilde{C} = (b, b, c)$ in F_R is $\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} \tilde{C}_{\epsilon}^{(k,n)}$

(c) The ϵ-nbhd of fuzzy number $\tilde{A} = (a, b, c)$ in F_T is

$$\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} (\tilde{B}_{\epsilon}^{(k,n)} \cup \tilde{C}_{\epsilon}^{(k,n)})$$

where

$$\begin{aligned}
\tilde{B}_{\epsilon}^{(k,n)} &= \Big((p^{(k-1,n)} - \epsilon)_{\alpha_{k-1,n}}, (p^{(k,n)} + \epsilon)_{\alpha_{k-1,n}} \Big) \in T_F \\
\tilde{C}_{\epsilon}^{(k,n)} &= \Big((q^{(k-1,n)} - \epsilon)_{\beta_{k,n}}, (q^{(k,n)} + \epsilon)_{\beta_{k,n}} \Big) \in T_F
\end{aligned}$$

and

$$\begin{aligned}
p^{(k,n)} &= a + \frac{k(b-a)}{n}, \quad q^{(k,n)} = b + \frac{k(c-b)}{n}, \quad k = 0, 1, \cdots, n; \; n = 1, 2, \cdots. \\
\alpha_{k,n} &= \frac{k}{n}, \; k = 0, 1, \cdots, n-1; \; n = 1, 2, \cdots. \\
\beta_{k,n} &= \frac{n-k}{n}, \; k = 1, 2, \cdots, n; \; n = 1, 2, \cdots.
\end{aligned}$$

(d) For the fuzzy numbers $\tilde{M}_L$, $\tilde{M}_R$, $\tilde{M}$ defined in (2.1),(2.2),(2.3) respectively, by Property 2.4, their ϵ-nbhd can be defined accordingly:

1. For $\tilde{M}_L$, we take $\alpha_k = f_L(p^{(k,n)})$, $k = 0, 1, \cdots, n-1$; $n = 1, 2, \cdots$ in (a).
2. For $\tilde{M}_R$, we take $\beta_k = f_R(q^{(k,n)})$, $k = 1, 2, \cdots, n$; $n = 1, 2, \cdots$ in (b).
3. For $\tilde{M}$, its ϵ-nbhd is the union of ϵ-nbhds of $\tilde{M}_L$ and $\tilde{M}_R$.

Remark: By Property 2.3, we have

$$\begin{aligned}
\tilde{B} &= \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} \tilde{B}_{k,n} \subset \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} \tilde{B}_{\epsilon}^{(k,n)} (\in T_F) \\
\tilde{C} &= \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} \tilde{C}_{k,n} \subset \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} \tilde{C}_{\epsilon}^{(k,n)} (\in T_F) \\
\tilde{A} &= \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} (\tilde{B}_{k,n} \cup \tilde{C}_{k,n}) \subset \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n} (\tilde{B}_{\epsilon}^{(k,n)} \cup \tilde{C}_{\epsilon}^{(k,n)}) (\in T_F)
\end{aligned}$$

They all satisfy the definition of ϵ-nbhd as stated in Definition 2.5.

Definition 2.13

(a) The fuzzy number $\tilde{Y} = (x, y, z)$ in F approaches to the fuzzy number $\tilde{B} = (a, b, b)$ in F_L, iff $z = y$ and

$$\forall \epsilon > 0, \quad \tilde{Y}_{1kn} \subset \tilde{B}_\epsilon^{(k,n)}, \quad \forall k = 1, 2, \cdots n; \quad n = 1, 2, \cdots. \tag{2.9}$$

(b) The fuzzy number $\tilde{Y} = (x, y, z)$ in F approaches to the fuzzy number $\tilde{C} = (b, b, c)$ in F_R, iff $x = y$ and

$$\forall \epsilon > 0, \quad \tilde{Y}_{2kn} \subset \tilde{C}_\epsilon^{(k,n)}, \quad \forall k = 1, 2, \cdots n; \quad n = 1, 2, \cdots. \tag{2.10}$$

(c) The fuzzy number $\tilde{Y} = (x, y, z)$ in F approaches to the fuzzy number $\tilde{A} = (a, b, c)$ in F_T, iff

$$\forall \epsilon > 0, \quad \tilde{Y}_{1kn} \subset \tilde{B}_\epsilon^{(k,n)}, \quad \text{and} \quad \tilde{Y}_{2kn} \subset \tilde{C}_\epsilon^{(k,n)}, \quad \forall k = 1, 2, \cdots n; \quad n = 1, 2, \cdots. \tag{2.11}$$

Where

$$\tilde{Y}_{1kn} = [t^{(k-1,n)}_{\alpha_{k-1,n}}, t^{(k,n)}_{\alpha_{k-1,n}}) \quad k = 1, 2, \cdots, n-1; \quad n = 1, 2, \cdots . \tilde{Y}_{1nn} = [t^{(n-1,n)}_{\alpha_{n-1,n}}, t^{(n,n)}_{\alpha_{n-1,n}}]$$
$$\tilde{Y}_{21n} = [u^{(0,n)}_{\beta_{1,n}}, u^{(1,n)}_{\beta_{1,n}}], \quad \tilde{Y}_{2kn} = (u^{(k-1,n)}_{\beta_{k,n}}, u^{(k,n)}_{\beta_{k,n}}] \quad k = 2, 3, \cdots, n; \quad n = 1, 2, \cdots.$$
$$t^{(k,n)} = x + \frac{k(y-x)}{n}, \quad u^{(k,n)} = y + \frac{k(z-y)}{n}, \quad k = 0, 1, \cdots n; n = 1, 2, \cdots.$$
$$\alpha_{k,n} = \frac{k}{n}, \quad \beta_{k,n} = \frac{n-k}{n}, \quad k = 0, 1, \cdots, n, \quad n = 1, 2, \cdots;$$

Remark:

1. In (a),

$$\tilde{Y} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{Y}_{1kn} \subset \tilde{B}_\epsilon = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{B}_\epsilon^{(k,n)} \in (T_F)$$

which is the ϵ-nbhd of $\tilde{Y}$ as defined in definition 2.12 (a).

2. In (b),

$$\tilde{Y} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{Y}_{2kn} \subset \tilde{C}_\epsilon = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{C}_\epsilon^{(k,n)} \in (T_F)$$

which is the ϵ-nbhd of $\tilde{Y}$ as defined in Definition 2.12 (b).

3. In (c),

$$\tilde{Y} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} (\tilde{Y}_{1kn} \cup \tilde{Y}_{2kn}) \subset \tilde{B}_\epsilon \cup \tilde{C}_\epsilon \in (T_F)$$

which is the ϵ-nbhd of $\tilde{Y}$ as defined in Definition 2.12 (c).

For the fuzzy numbers $\tilde{M}_L$, $\tilde{M}_R$, and $\tilde{M}$, we can similarly define their concept of approach.

Definition 2.14

(a) A sequence of fuzzy numbers in F, $\tilde{Y}^{(m)} = (x_m, y_m, z_m)$, $m = 1, 2, \cdots$ converges to the fuzzy number $\tilde{B} = (a, b, b)$ in F_L, iff $z_m = y_m$, $\forall\ m = 1, 2, \cdots$ and $\forall\ \epsilon > 0$, there exists a natural number M such that whenever $m \geq M$,

$$\tilde{Y}^{(m)}_{1kn} \subset \tilde{B}^{(k,n)}_{\epsilon}, \quad \forall\ k = 1, 2, \cdots n; \quad n = 1, 2, \cdots. \tag{2.12}$$

(b) A sequence of fuzzy numbers in F, $\tilde{Y}^{(m)} = (x_m, y_m, z_m)$, $m = 1, 2, \cdots$ converges to the fuzzy number $\tilde{C} = (b, b, c)$ in F_R, iff $x_m = y_m$, $\forall m = 1, 2, \cdots$ and $\forall\ \epsilon > 0$, there exists a natural number M such that whenever $m \geq M$,

$$\tilde{Y}^{(m)}_{2kn} \subset \tilde{C}^{(k,n)}_{\epsilon}, \quad \forall\ k = 1, 2, \cdots n; \quad n = 1, 2, \cdots. \tag{2.13}$$

(c) A sequence of fuzzy numbers in F, $\tilde{Y}^{(m)} = (x_m, y_m, z_m)$, $m = 1, 2, \cdots$ converges to the fuzzy number $\tilde{A} = (a, b, c)$ in F_T, iff $\forall\ \epsilon > 0$, there exists a natural number M such that whenever $m \geq M$,

$$\tilde{Y}^{(m)}_{1kn} \subset \tilde{C}^{(k,n)}_{\epsilon} \text{ and } \tilde{Y}^{(m)}_{2kn} \subset \tilde{D}^{(k,n)}_{\epsilon}, \quad \forall\ k = 1, 2, \cdots n; \quad n = 1, 2, \cdots. \tag{2.14}$$

where

$$\tilde{Y}^{(m)}_{1kn} = [t^{(k-1,n,m)}_{\alpha_{k-1,n,m}}, t^{(k,n,m)}_{\alpha_{k-1,n,m}}) \quad k = 1, 2, \cdots, n-1; \quad n = 1, 2, \cdots . \tilde{Y}^{(m)}_{1nn} = [t^{(n-1,n,m)}_{\alpha_{n-1,n,m}}, t^{(n,n,m)}_{\alpha_{n-1,n,m}}]$$

$$\tilde{Y}^{(m)}_{21n} = [u^{(0,n,m)}_{\beta_{1,n,m}}, u^{(1,n,m)}_{\beta_{1,n,m}}], \quad \tilde{Y}^{(m)}_{2kn} = (u^{(k-1,n,m)}_{\beta_{k,n,m}}, u^{(k,n,m)}_{\beta_{k,n,m}}] \quad k = 2, 3, \cdots, n; \quad n = 1, 2, \cdots.$$

$$t^{(k,n,m)} = x_m + \frac{k(y_m - x_m)}{n}, \quad u^{(k,n,m)} = y_m + \frac{k(z_m - y_m)}{n}, \quad k = 0, 1, \cdots n; n = 1, 2, \cdots.$$

$$\alpha_{k,n,m} = \frac{k}{n}, \quad \beta_{k,n,m} = \frac{n-k}{n}, \quad k = 0, 1, \cdots, n, \quad n = 1, 2, \cdots;$$

Remark 1: In (a),

$$\tilde{Y}^{(m)} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{Y}^{(m)}_{1kn} \subset \tilde{B}_{\epsilon} = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{B}^{(k,n)}_{\epsilon} \in (T_F)$$

satisfies Definition 2.6. Similarly for (b) and (c).

Remark 2: For the fuzzy numbers $\tilde{M}_L$, $\tilde{M}_R$, and $\tilde{M}$, by Property 2.4, we can similarly define the concept of their convergence. As to the fuzzy sets of F_N not described in Definition 2.9~2.14, their ϵ-nbhd and concept of convergence may be defined accordingly by using Definition 2.5 and 2.6.

Property 2.7

(a) The fuzzy number $\tilde{Y} = (x, y, z)$ in F, approaches to the fuzzy number $\tilde{B} = (a, b, b)$ in F_L, iff $z = y$ and $\forall\ \epsilon > 0$, $|x - a| < \epsilon$ and $|y - b| < \epsilon$.

(b) The fuzzy number $\tilde{Y} = (x, y, z)$ in F, approaches to the fuzzy number $\tilde{C} = (b, b, c)$ in F_R, iff $x = y$ and $\forall\ \epsilon > 0$, $|y - b| < \epsilon$ and $|z - c| < \epsilon$.

(c) The fuzzy number $\tilde{Y} = (x, y, z)$ in F, approaches to the fuzzy number $\tilde{A} = (a, b, c)$ in F_T, iff $\forall\ \epsilon > 0$, $|x - a| < \epsilon$, $|y - b| < \epsilon$ and $|z - c| < \epsilon$.

Proof:

(a) From Definition 2.13 (a), we have

$$\tilde{Y}_{1kn} \subset \tilde{B}_{\epsilon}^{(k,n)},\ \forall\ k = 1, 2, \cdots n \text{ and } n = 1, 2, \cdots;$$

i.e.

$$a + \frac{(k-1)(b-a)}{n} - \epsilon < x + \frac{(k-1)(y-x)}{n} \quad \text{and}$$

$$x + \frac{k(y-x)}{n} < a + \frac{k(b-a)}{n} + \epsilon \quad \forall\ k = 1, 2, \cdots n \text{ and } n = 1, 2, \cdots.$$

That is

$$-(k-1)(y - x - b + a) < n(x - a + \epsilon), \forall k = 1, 2, \cdots n;\ n = 1, 2, \cdots \tag{2.15}$$

and

$$k(y - x - b + a) < n(-x + a + \epsilon),\ \forall\ k = 1, 2, \cdots n;\ n = 1, 2, \cdots \tag{2.16}$$

Let $x - a + \epsilon > 0$ and $-x + a + \epsilon > 0$, we have

$$|x - a| < \epsilon \tag{2.17}$$

Since $|x - a| < \epsilon$, if $y - x - b + a > 0$, then (2.15) holds and from (2.16) which holds for $k = 1, 2, \cdots, n$; $n = 1, 2, \cdots$. Therefore

$$n(y - x - b + a) < n(-x + a + \epsilon),\ \forall\ n = 1, 2, \cdots$$

and hence

$$y < b + \epsilon \tag{2.18}$$

If $y - x - b + a < 0$ then (2.16) holds and from (2.15), which holds for $k = 1, 2, \cdots n;\ n = 2, 3, \cdots$. Therefore we have

$$(n-1)(-y + x + b - a) < n(x - a + \epsilon),\ \forall\ n = 2, 3, \cdots$$

$$\text{or} \qquad b - \frac{1}{n-1}(x - a + n\epsilon) < y,\ \forall\ n = 2, 3, \cdots \tag{2.19}$$

Combining (2.18) and (2.19), we have

$$b - \frac{1}{n-1}(x - a + n\epsilon) < y < b + \epsilon,\ \forall\ n = 2, 3, \cdots \tag{2.20}$$

Letting $n \to \infty$, (2.20) becomes

$$b - \epsilon < y < b + \epsilon \text{ or } |y - b| < \epsilon \tag{2.21}$$

Conversely, if $|x - a| < \epsilon$ and $|y - b| < \epsilon$, i.e. $a - \epsilon < x < a + \epsilon$ and $b - \epsilon < y < b + \epsilon$. from $-\epsilon < x - a$, we add $n\epsilon$ to both sides, and get

$$(n-1)\epsilon < x - a + n\epsilon \text{ or } \epsilon < \frac{1}{n-1}(x - a + n\epsilon),\ \forall\ n = 2, 3, \cdots;$$

which implies

$$b - \epsilon > b - \frac{1}{n-1}(x - a + n\epsilon),\ \forall\ n = 2, 3, \cdots$$

Then

$$b - \frac{1}{n-1}(x - a + n\epsilon) < b - \epsilon < y < b + \epsilon,\ \forall\ n = 2, 3, \cdots$$

This is (2.20). Tracing back from there, we have $\tilde{Y}$ approaches to $\tilde{B}$.

(b) By definition 2.13 (b), we have

$$\tilde{Y}_{2kn} \subset \tilde{C}_{\epsilon}^{(k,n)},\ \forall\ k = 1, 2, \cdots n;\ n = 1, 2, \cdots;$$

i.e.

$$b + \frac{(k-1)(c-b)}{n} - \epsilon < y + \frac{(k-1)(z-y)}{n} \quad \text{and}$$

$$y + \frac{k(z-y)}{n} < b + \frac{k(c-b)}{n} + \epsilon, \quad \forall\ k = 1, 2, \cdots n \text{ and } n = 1, 2, \cdots;$$

With the same argument used in (a), we have the necessary and sufficient conditions for $\tilde{Y}$ approaches to $\tilde{C}$ are: $|y - b| < \epsilon$ and $|z - c| < \epsilon$, $\forall \epsilon > 0$.

(c) Follows from (a) and (b).

Property 2.8

(a) A sequence of fuzzy numbers in F, $\tilde{Y}^{(m)} = (x_m, y_m, z_m)$, $m = 1, 2, \cdots$ converges to the fuzzy number $\tilde{B} = (a, b, b)$ in F_L, iff $z_m = y_m$, $\forall m = 1, 2, \cdots$ and $\forall\ \epsilon > 0$, there exists a natural number M such that whenever $m \geq M$, $|x_m - a| < \epsilon$ and $|y_m - b| < \epsilon$.

(b) A sequence of fuzzy numbers in F, $\tilde{Y}^{(m)} = (x_m, y_m, z_m)$, $m = 1, 2, \cdots$ converges to the fuzzy number $\tilde{C} = (b, b, c)$ in F_R, iff $x_m = y_m$, $\forall m = 1, 2, \cdots$ and $\forall\ \epsilon > 0$, there exists a natural number M such that whenever $m \geq M$, $|y_m - b| < \epsilon$ and $|z_m - c| < \epsilon$.

(c) A sequence of fuzzy numbers in F, $\tilde{Y}^{(m)} = (x_m, y_m, z_m)$, $m = 1, 2, \cdots$ converges to the fuzzy number $\tilde{A} = (a, b, c)$ in F_T, iff $\forall\ \epsilon > 0$, there exists a natural number M such that whenever $m \geq M$, $|x_m - a| < \epsilon$, $|y_m - b| < \epsilon$ and $|z_m - c| < \epsilon$.

Proof: Similar as the proof of Property 2.7.

Example 2.2 Let

$$\tilde{A}_m = (9 - \frac{1}{m}, 10, 12 + \frac{1}{m}), \quad m = 1, 2, \cdots$$

be a sequence of fuzzy numbers and $\tilde{A} = (9, 10, 12)$; i.e. in Property 2.8 (c),

$$x_m = 9 - \frac{1}{m}, \ y_m = 10, \ z_m = 12 + \frac{1}{m}, \ a = 9, \ b = 10, \ c = 12.$$

For each $\epsilon > 0$, take any natural number $M > \frac{1}{\epsilon}$, then for every $m \geq M$,

$$|x_m - 9| = \frac{1}{m} \leq \frac{1}{M} < \epsilon, \ |y_m - 10| = 0 < \epsilon, \ |z_m - 12| = \frac{1}{m} \leq \frac{1}{M} < \epsilon.$$

Therefore by Property 2.8 (c), this sequence $\tilde{A}_m$, $m = 1, 2, \cdots$ converges to $\tilde{A}$.

Now we have the addition $(+)$ in F by the following:

If (a, b, c), $(p, q, r) \in F_T$, then $(a, b, c)(+)(p, q, r) = (a + p, b + q, c + r) \in F_T$.

Thus we have the following property (by Property 2.7):

Property 2.9 If $(x, y, z) \in F$ approaches to $(a, b, c) \in F$ and $(s, t, w) \in F$ approaches to (p, q, r) $\in F$, then $(x, y, z)(+)(s, t, w) \in F$ approaches to $(a, b, c)(+)(p, q, r) \in F_T$.

Proof: Since (x, y, z) approaches to (a, b, c) and (s, t, w) approaches to (p, q, r), by Property 2.7 (c), we have

$$|x-a| < \frac{\epsilon}{2}, \quad |y-b| < \frac{\epsilon}{2}, \quad |z-c| < \frac{\epsilon}{2}, \quad |s-p| < \frac{\epsilon}{2}, \quad |t-q| < \frac{\epsilon}{2}, \quad |w-r| < \frac{\epsilon}{2}$$

which implies

$$|x+s-a-p| \leq |x-a| + |s-p| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

$$|y+t-b-q| \leq |y-b| + |t-q| < \epsilon \text{ and } |z+w-c-r| \leq |z-c| + |w-r| < \epsilon$$

Hence $(x, y, z)(+)(s, t, w)$ approaches to $(a, b, c)(+)(p, q, r)$.

Property 2.10 If sequence $(x_m, y_m, z_m) \in F$, $m = 1, 2, \cdots$ converges to $(a, b, c) \in F$ and sequence $(s_m, t_m, w_m) \in F$, $m = 1, 2, \cdots$ converges to $(p, q, r) \in F$, then sequence $(x_m, y_m, z_m)(+)(s_m, t_m, w_m) \in F$, $m = 1, 2, \cdots$ converges to $(a, b, c)(+)(p, q, r) \in F_T$.

Proof: Similar as the proof of Property 2.9.

Next we have the multiplication in F: For $(a, b, c) \in F$,

$$k \cdot (a, b, c) = \begin{matrix} (ka, kb, kc), & k > 0 \\ (kc, kb, ka), & k < 0 \end{matrix}$$

Property 2.11

(a) If $(x, y, z) \in F$ approaches to $(a, b, c) \in F$, then $k \cdot (x, y, z)$ approaches to $k \cdot (a, b, c)$, $k \neq 0$.

(b) If $(x_n, y_n, z_n) \in F$, $n = 1, 2, \cdots$ converges to $(a, b, c) \in F$, then $k \cdot (x_n, y_n, z_n)$, $n = 1, 2, \cdots$ converges to $k \cdot (a, b, c)$, $k \neq 0$.

Proof:

(a) If (x, y, z), $(a, b, c) \in F$, then by Property 2.7 (c), we have

$$|x-a| < \frac{\epsilon}{|k|}, \quad |y-b| < \frac{\epsilon}{|k|}, \quad |z-c| < \frac{\epsilon}{|k|}.$$

Therefore

$$|kx - ka| = |k||x - a| < \epsilon, \quad |ky - kb| = |k||y - b| < \epsilon, \quad |kz - kc| = |k||z - c| < \epsilon.$$

Thus $k(x, y, z)$ approaches to $k(a, b, c)$.

(b) Similarly as in part (a).

Definition 2.15 In FTS (R, T_F), a subfamily $\mathcal{A}$ of T_F is called a base for T_F iff for any open fuzzy set $\tilde{O}$ $(\in T_F)$, each $x_\lambda \subset \tilde{O}$, there is a member $\tilde{U}$ of $\mathcal{A}$ such that $x_\lambda \subset \tilde{U} \subset \tilde{O}$.

From the definition of T_F, we know that

$$\mathcal{B} = \{(a_\lambda, b_\lambda),\ (a_\lambda, \infty),\ (-\infty, b_\lambda),\ (-\infty, \infty)\ |\forall\ 0 < \lambda \leq 1,\ a < b,\ a, b \in R\}$$

is a base for T_F.

Let R_r be the family of all rational numbers in R. We know that R_r is dense in R, i.e. $Cl(R_r) = R$, where $Cl(R_r)$ means the closure of R_r. Let

$$\mathcal{B}_r = \{(p_\alpha, q_\alpha),\ (p_\alpha, \infty),\ (-\infty, q_\alpha),\ (-\infty, \infty)\ |\forall\ 0 < \alpha \leq 1,\ p < q,\ p, q, \alpha \in R_r\} \subset \mathcal{B}$$

Then $\mathcal{B}_r$ is a subfamily of T_F.

Property 2.12 FTS (R, T_F) has a countable base $\mathcal{B}_r$ for T_F.

Proof: For each open fuzzy set $\tilde{O} \in T_F$, since $\mathcal{B}$ is a base for T_F, by Definition 2.15, for each $x_\lambda \subset \tilde{O}$, there exists open interval, say $(a_\alpha, b_\alpha) \in \mathcal{B}$ such that $x_\lambda \subset (a_\alpha, b_\alpha) \subset \tilde{O}$, where $a < x < b$ and $\lambda \leq \alpha$. The reason is that $\mu_{x_\lambda}(y) \leq \mu_{(a_\alpha, b_\alpha)}(y)$, $\forall y$. Now since $Cl(R_r) = R$, there are p, q, $\beta \in R_r$ and $\beta \in (0, 1]$ such that $a < p < x$, $x < q < b$ and $0 < \lambda < \beta < \alpha$. Therefore

$$x_\lambda \subset (p_\beta, q_\beta) \subset (a_\alpha, b_\alpha) \subset \tilde{O}.$$

where $(p_\beta, q_\beta) \in \mathcal{B}_r$ and by Definition 2.15, $\mathcal{B}_r$ is a base for T_F. Since R_r is countable, therefore we say $\mathcal{B}_r$ is a countable base for T_F.

Definition 2.16 (Pu and Liu [4], Definition 4.1′; Azad [1], §3) The intersection of all the closed fuzzy sets containing $\tilde{A}(\in F_N)$, is called the closure of $\tilde{A}$, denoted by $Cl(\tilde{A})$. Obviously, $Cl(\tilde{A})$ is the smallest closed fuzzy set containing $\tilde{A}$; i.e.

$$Cl(\tilde{A}) = \inf\{\tilde{C}|\tilde{C} \supseteq \tilde{A}, \tilde{C}' \in T_F\} = \quad \{\tilde{C}|\tilde{C} \supseteq \tilde{A}, \tilde{C}' \in T_F\}$$

Property 2.13 (R, T_F) has the following properties:

(C_1) $\tilde{A} \subset Cl(\tilde{A})$

(C_2) $\tilde{A}$ is closed iff $Cl(\tilde{A}) = \tilde{A}$

(C_3) $Cl(Cl(\tilde{A})) = Cl(\tilde{A})$

(C_4) $\tilde{A} \subset \tilde{B} \Rightarrow Cl(\tilde{A}) \subset Cl(\tilde{B})$

(C_5) $Cl(\tilde{A} \cup \tilde{B}) = Cl(\tilde{A}) \cup Cl(\tilde{B})$

$(C_6) \bigcup_{k=1}^{\infty} Cl(\tilde{A}_k) \subset Cl(\bigcup_{k=1}^{\infty} \tilde{A}_k)$ (see Azad [1], §3)

(C_7) $Cl(\phi) = \phi$

Proof:

(C_1) Follows immediately from Definition 2.16.

(C_2) If $\tilde{A}$ is closed, then from Definition 2.16, $Cl(\tilde{A}) = \inf\{\tilde{C} | \tilde{C} \supseteq \tilde{A}, \tilde{C}' \in T_F\} = \tilde{A}$. Conversely, if $Cl(\tilde{A}) = \tilde{A}$, since $Cl(\tilde{A})$ is closed, so does $\tilde{A}$.

(C_3) Follows from (C_2).

(C_4) Since $\tilde{B} \subset Cl(\tilde{B})$, $\tilde{A} \subset \tilde{B}$, therefore $\tilde{A} \subset Cl(\tilde{B})$. By Definition 2.16, we have $Cl(\tilde{A}) \subset Cl(\tilde{B})$.

(C_5) Since $\tilde{A} \subset Cl(\tilde{A})$, $\tilde{B} \subset Cl(\tilde{B})$, therefore $\tilde{A} \cup \tilde{B} \subset Cl(\tilde{A}) \cup Cl(\tilde{B})$. However, both $Cl(\tilde{A})$ and $Cl(\tilde{B})$ are closed, which means $Cl(\tilde{A}) \cup Cl(\tilde{B})$. is closed. Therefore

$$Cl(\tilde{A} \cup \tilde{B}) \subset Cl(\tilde{A}) \cup Cl(\tilde{B}).$$

Conversely, we have $\tilde{A} \subset \tilde{A} \cup \tilde{B}$, $\tilde{B} \subset \tilde{A} \cup \tilde{B}$. Therefore $Cl(\tilde{A}) \subset Cl(\tilde{A} \cup \tilde{B})$ and $Cl(\tilde{B}) \subset Cl(\tilde{A} \cup \tilde{B})$ by (C_4). Hence $Cl(\tilde{A}) \cup Cl(\tilde{B}) \subset Cl(\tilde{A} \cup \tilde{B})$. Thus $Cl(\tilde{A}) \cup Cl(\tilde{B}) = Cl(\tilde{A} \cup \tilde{B})$

(C_6) Follows from Azad [1], §3.

(C_7) It holds clearly.

Example 2.3

(1) From Example 2.1, $[a_\lambda, b_\lambda]$, $(0 < \lambda < 1)$, is not closed, $[a_1, b_1]$ is a closed fuzzy interval.

(2) Since $I(a_\lambda, b_\lambda) \subset [a_\alpha, b_\alpha]$ if $\lambda \le \alpha \le 1$. From the fact that $[a_\alpha, b_\alpha]$, $(0 < \alpha < 1)$ is not closed, we have $Cl(I(a_\lambda, b_\lambda)) = [a_1, b_1]$. Also $Cl(I(a_1, b_1) = [a_1, b_1]$. This fact corresponds to the fact that $Cl(I(a, b)) = [a, b]$ in (R, T_R).

(3) If $\tilde{B} = (a, b, b)$, by Property 2.3,

$$Cl(\tilde{B}_{k,n}) = [p_1^{(k-1,n)}, p_1^{(k,n)}], \; k = 1, 2, \cdots, n;$$
$$\bigcup_{k=1}^{n} Cl(\tilde{B}_{k,n}) = \bigcup_{k=1}^{n} [p_1^{(k-1,n)}, p_1^{(k,n)}] = [a_1, b_1], \forall n.$$

By (C_5), $\bigcup_{k=1}^{n} Cl(\tilde{B}_{k,n}) = Cl(\bigcup_{k=1}^{n} \tilde{B}_{k,n})$, therefore by (C_6),

$$[a_1, b_1] = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} Cl(\tilde{B}_{k,n}) = \bigcup_{n=1}^{\infty} Cl(\bigcup_{k=1}^{n} \tilde{B}_{k,n}) \subset Cl(\bigcup_{n=1}^{\infty} (\bigcup_{k=1}^{n} \tilde{B}_{k,n})) = Cl(\tilde{B}).$$

Since $\mu_{\tilde{B}}(x) \le 1$, $\forall a \le x \le b$; $\mu_{[a_1,b_1]}(x) = 1$, $\forall a \le x \le b$ and $\mu_{\tilde{B}}(x) = 0 = \mu_{[a_1,b_1]}(x)$, $\forall x < a$ or $x > b$, therefore $\mu_{\tilde{B}}(x) \le \mu_{[a_1,b_1]}(x)$, $\forall x$; i.e. $\tilde{B} \subseteq [a_1, b_1]$. By (C_2) and (C_4), $Cl(\tilde{B}) \subseteq [a_1, b_1]$. Therefore $Cl(\tilde{B}) = [a_1, b_1]$.

Similarly, we have

(4) If $\tilde{C} = (b, b, c)$ in F_R, then $Cl(\tilde{C}) = [b_1, c_1]$.

(5) If $\tilde{A} = (a, b, c) = \tilde{B} \cup \tilde{C}$, by (C_5),

$$Cl(\tilde{A}) = Cl(\tilde{B} \cup \tilde{C}) = Cl(\tilde{B}) \cup Cl(\tilde{C}) = [a_1, c_1]$$

Also from Property 2.4, we have

$$Cl(\tilde{M}_L) = [a_1, b_1], \quad Cl(\tilde{M}_R) = [b_1, c_1], \quad Cl(\tilde{M}) = [a_1, c_1].$$

Definition 2.17 (Pu and Liu [6], definition 4.1; Azad [1], §3) In FTS (R, T_F), let $\tilde{A}$ be a fuzzy set in F_N, the union of all open fuzzy sets contained in $\tilde{A}$ is called the interior of $\tilde{A}$, denoted by $Int(\tilde{A})$ or $Int\ \tilde{A}$. Evidently $Int\ \tilde{A}$ is the largest open fuzzy set contained in $\tilde{A}$; i.e.

$$Int\ \tilde{A} = \sup\{\tilde{O} | \tilde{O} \subset \tilde{A}, \tilde{O} \in T_F\} = \quad \{\tilde{O} | \tilde{O} \subset \tilde{A}, \tilde{O} \in T_F\}.$$

Property 2.14 Let $\tilde{A}$, $\tilde{B}$, $\tilde{A}_1$, $\tilde{A}_2, \cdots$ be fuzzy sets in F_N, then

(I_1) $Int\ \tilde{A} \subseteq \tilde{A}$

(I_2) $\tilde{A}$ is open iff $Int\ \tilde{A} = \tilde{A}$

(I_3) $Int(Int\ \tilde{A}) = Int\ \tilde{A}$

(I_4) $\tilde{A} \subset \tilde{B} \Rightarrow Int\ \tilde{A} \subseteq Int\ \tilde{B}$

(I_5) $Int(\tilde{A} \cap \tilde{B}) = Int\ \tilde{A} \cap Int\ \tilde{B}$

(I_6) $\bigcup_{n=1}^{\infty} Int(\tilde{A}_n) \subseteq Int(\bigcup_{n=1}^{\infty} \tilde{A}_n)$ (see Azad [1], §3)

Proof: Similar as Property 2.13.

Example 2.4

(1) (a) $(a_\lambda, b_\lambda) \supseteq (a_\alpha, b_\alpha)$ $\forall\ 0 < \alpha \leq \lambda \leq 1$, therefore $Int\ (a_\lambda, b_\lambda) = (a_\lambda, b_\lambda)$.

(b) $I(a_1, b_1) \supseteq (a_1, b_1)$, therefore $Int\ I(a_1, b_1) = (a_1, b_1)$. This is a level 1 open fuzzy interval, coincide with the fact $Int\ I(a, b) = (a, b)$ in (R, T_R).

(2) By Property 2.3 (a), $\tilde{B} = (a, b, b)$ and

$$\tilde{B}_{k,n} = [p_{\alpha_{k-1,n}}^{(k-1,n)}, p_{\alpha_{k-1,n}}^{(k,n)}), \quad k = 1, 2, \cdots, n-1, \ n = 1, 2, \cdots$$

$$\tilde{B}_{n,n} = [p_{\alpha_{n-1,n}}^{(n-1,n)}, p_{\alpha_{n-1,n}}^{(n,n)}]; \quad n = 1, 2, \cdots$$

By (1) (a), we have

$$Int\ \tilde{B}_{k,n} = (p_{\alpha_{k-1,n}}^{(k-1,n)}, p_{\alpha_{k-1,n}}^{(k,n)}) \subset \tilde{B}_{k,n}; \quad k = 1, 2, \cdots, n; \ n = 1, 2, \cdots.$$

Let

$$\tilde{B}_\epsilon^{k,n} = ((p^{(k-1,n)} - \epsilon)_{\alpha_{k-1,n}}, \ (p^{(k,n)} + \epsilon)_{\alpha_{k-1,n}})(\supset \tilde{B}_{k,n}); \quad k = 1, 2, \cdots, n; \ n = 1, 2, \cdots,$$

Since

$$(p_{\alpha_{k-1,n}}^{(k-1,n)}, p_{\alpha_{k-1,n}}^{(k,n)}) \subset \tilde{B}_{k,n} \subset \tilde{B}_\epsilon^{k,n}; \quad k = 1, 2, \cdots, n; \ n = 1, 2, \cdots$$

then

$$\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}(p_{\alpha_{k-1,n}}^{(k-1,n)},p_{\alpha_{k-1,n}}^{(k,n)})\subset\tilde{B}\subset\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\tilde{B}_{\epsilon}^{k,n}.$$

Since

$$\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}(p_{\alpha_{k-1,n}}^{(k-1,n)},p_{\alpha_{k-1,n}}^{(k,n)})\ \text{ and }\ \bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\tilde{B}_{\epsilon}^{k,n}$$

are open fuzzy sets, by (I_2) and (I_4), we have

$$\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}(p_{\alpha_{k-1,n}}^{(k-1,n)},p_{\alpha_{k-1,n}}^{(k,n)})\subset Int\ \tilde{B}\subset\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\tilde{B}_{\epsilon}^{k,n}$$

Let $\epsilon\to 0$, we have

$$Int\ \tilde{B}=\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}(p_{\alpha_{k-1,n}}^{(k-1,n)},p_{\alpha_{k-1,n}}^{(k,n)})$$

Similarly, if $\tilde{C}=(b,b,c)$, then

$$Int\ \tilde{C}=\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}(q_{\beta_{k,n}}^{(k-1,n)},q_{\beta_{k,n}}^{(k,n)}),$$

and if $\tilde{A}=(a,b,c)$, then

$$Int\ \tilde{A}=\bigcup_{n=1}^{\infty}\bigcup_{k=1}^{n}\ (p_{\alpha_{k-1,n}}^{(k-1,n)},p_{\alpha_{k-1,n}}^{(k,n)})\cup(q_{\beta_{k,n}}^{(k-1,n)},q_{\beta_{k,n}}^{(k,n)})\ .$$

Also we can find the $Int\ M_L$, $Int\ M_R$, $Int\ M$ accordingly.

Definition 2.18 (Pu and Liu [4], Definition 4.4) A fuzzy point x_λ is called a boundary fuzzy point of a fuzzy set $\tilde{A}(\in F_N)$ iff

$$x_\lambda\subset Cl(\tilde{A})\cap Cl(\tilde{A}').$$

The union of all the boundary points of $\tilde{A}$ is called the boundary of $\tilde{A}$, denoted by $b(\tilde{A})$.

Property 2.15

(B_1) $b(\tilde{A})=b(\tilde{A}')$

(B_2) $b(\tilde{A})=Cl(\tilde{A})\cap Cl(\tilde{A}')$

(B_3) $Cl(\tilde{A})\supseteq\tilde{A}\cup b(\tilde{A})$

Proof:

(B_1) and (B_2) follows by definition.

(B_3) In Pu and Liu [4], Proposition 4.1 has proved that $Cl(\tilde{A}) \supset \tilde{A} \cup b(\tilde{A})$ and $Cl(\tilde{A}) \neq \tilde{A} \cup b(\tilde{A})$, but we shall show that "=" does hold in our Example 2.5 given in the following. The reason is that because T_F is an extension of T_R as seen previously in this paper.

Example 2.5 Suppose $a < b < c < d$, $0 < \alpha < 1$, $0 < \beta < 1$.

(a) Let $\tilde{A} = (a_\alpha, b_\alpha] \cup [c_\beta, d_\beta)$ then

$$\tilde{A}' = (-\infty, a_1] \cup (a_{1-\alpha}, b_{1-\alpha}] \cup (b_1, c_1) \cup [c_{1-\beta}, d_{1-\beta}) \cup [d_1, \infty).$$

From Example 2.1, we know that $[a_\alpha, b_\alpha]$ is not closed if $0 < \alpha < 1$, and $[a_1, b_1]$ is closed, therefore $Cl(\tilde{A}) = [a_1, b_1] \cup [c_1, d_1]$ and $Cl(\tilde{A}') = R$. So

$$b(\tilde{A}) = Cl(\tilde{A}) \cap Cl(\tilde{A}') = [a_1, b_1] \cup [c_1, d_1] = (\bigcup_{a \le x \le b} x_1) \cup (\bigcup_{c \le x \le d} x_1)$$

and $\tilde{A} \cup b(\tilde{A}) = Cl(\tilde{A})$.

(b) Let $\tilde{A} = (a_1, b_1] \cup [c_1, d_1)$ then $\tilde{A}' = (-\infty, a_1] \cup (b_1, c_1) \cup [d_1, \infty)$. Therefore $Cl(\tilde{A}) = [a_1, b_1] \cup [c_1, d_1]$ and $Cl(\tilde{A}') = (-\infty, a_1] \cup [b_1, c_1] \cup [d_1, \infty)$. So

$$b(\tilde{A}) = Cl(\tilde{A}) \cap Cl(\tilde{A}') = a_1 \cup b_1 \cup c_1 \cup d_1, \quad \tilde{A} \cup b(\tilde{A}) = [a_1, b_1] \cup [c_1, d_1] = Cl(\tilde{A}).$$

Definition 2.19 Two fuzzy sets $\tilde{A}$, $\tilde{B}$ in F_N are said to be separated iff

$$(Cl\ \tilde{A}) \cap \tilde{B} = \tilde{A} \cap (Cl\ \tilde{B}) = \phi$$

Property 2.16

(a) If $\tilde{A}$, $\tilde{B}$ are separated then $\tilde{A} \cap \tilde{B} = \phi$

(b) If $\tilde{A}$, $\tilde{B}$ are closed, and $\tilde{A} \cap \tilde{B} = \phi$, then $\tilde{A}$, $\tilde{B}$ are separated.

Proof:

(a) Since $\tilde{A} \subset Cl(\tilde{A})$, so that $\tilde{A} \cap \tilde{B} \subset (Cl\ \tilde{A}) \cap \tilde{B} = \phi \Rightarrow \tilde{A} \cap \tilde{B} = \phi$.

(b) If $\tilde{A}$ and $\tilde{B}$ are both closed,then by (C_2) of Property 2.13. we have

$$(Cl\ \tilde{A}) \cap\ \tilde{B} = \tilde{A} \cap (Cl\ \tilde{B}) = \tilde{A} \cap \tilde{B} = \phi$$

Therefore $\tilde{A}$. $\tilde{B}$ are separated.

Property 2.17 Let $\tilde{A}$ and $\tilde{B}$ be separated. If $\tilde{A} \cup \tilde{B}$ is closed, then $\tilde{A}$ and $\tilde{B}$ are both closed fuzzy sets.

Proof: Let $\tilde{A} \cup \tilde{B}$ be a closed fuzzy set, then

$$Cl(\tilde{A}) \cup Cl(\tilde{B}) = Cl(\tilde{A} \cup \tilde{B}) = \tilde{A} \cup \tilde{B}.$$

So $Cl(\tilde{A}) \subset (\tilde{A} \cup \tilde{B})$ and $Cl(\tilde{B}) \subset (\tilde{A} \cup \tilde{B})$. However

$$(Cl\ \tilde{A}) \cap\ \tilde{B} = \tilde{A} \cap (Cl\ \tilde{B}) = \phi$$

Since $\mu_{Cl\ \tilde{A}}(x) \leq \mu_{\tilde{A}}(x) \quad \mu_{\tilde{B}}(x)\ \forall x$ and for each x either $\mu_{Cl\ \tilde{A}}(x) = 0$ or $\mu_{\tilde{B}}(x) = 0$. So $Cl(\tilde{A}) \subset \tilde{A}$. Similarly $Cl(\tilde{B}) \subset \tilde{B}$. By Property 2.13 (C_1) and (C_2), we have $\tilde{A}$, $\tilde{B}$ are closed fuzzy sets.

Example 2.6 By Property 2.3 (a), (b) and Example 2.3, we have

$\tilde{B} = (a,b,b) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{B}_{k,n}$ $\qquad$ $\tilde{C} = (c,c,d) = \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{n} \tilde{C}_{k,n}$

$\tilde{B}_{k,n} = [p^{(k-1,n)}_{\alpha_{k-1,n}}, p^{(k,n)}_{\alpha_{k-1,n}}),$ $\qquad$ $k = 1, 2, \cdots, n-1,\ n = 1, 2, \cdots$

$\tilde{B}_{n,n} = [p^{(n-1,n)}_{\alpha_{n-1,n}}, p^{(n,n)}_{\alpha_{n-1,n}}];$ $\qquad$ $n = 1, 2, \cdots$

$\tilde{C}_{1,n} = [q^{(0,n)}_{\beta_{1,n}}, q^{(1,n)}_{\beta_{1,n}}],$ $\qquad$ $n = 1, 2, \cdots$

$\tilde{C}_{k,n} = (q^{(k-1,n)}_{\beta_{k,n}}, q^{(k,n)}_{\beta_{k,n}}],$ $\qquad$ $k = 2, 3, \cdots, n;\ n = 1, 2, \cdots$

$p^{(k,n)} = a + \frac{k(b-a)}{n}, \quad q^{(k,n)} = b + \frac{k(c-b)}{n},$ $\qquad$ $k = 0, 1, \cdots, n;\ n = 1, 2, \cdots$

$\alpha_{k-1,n} = \frac{k-1}{n}, \quad \beta_{k,n} = \frac{n-k}{n},$ $\qquad$ $k = 1, 2, \cdots, n;\ n = 1, 2, \cdots$

Assume $a < b < c < d$, from Example 2.3, we have $Cl(\tilde{B}) = [a_1, b_1]$, $Cl(\tilde{C}) = [c_1, d_1]$. It is clear that

$$Cl(\tilde{B}) \cap \tilde{C} = \tilde{B} \cap Cl(\tilde{C}) = \phi,$$

Therefore $\tilde{B}$, and $\tilde{C}$ are separated.

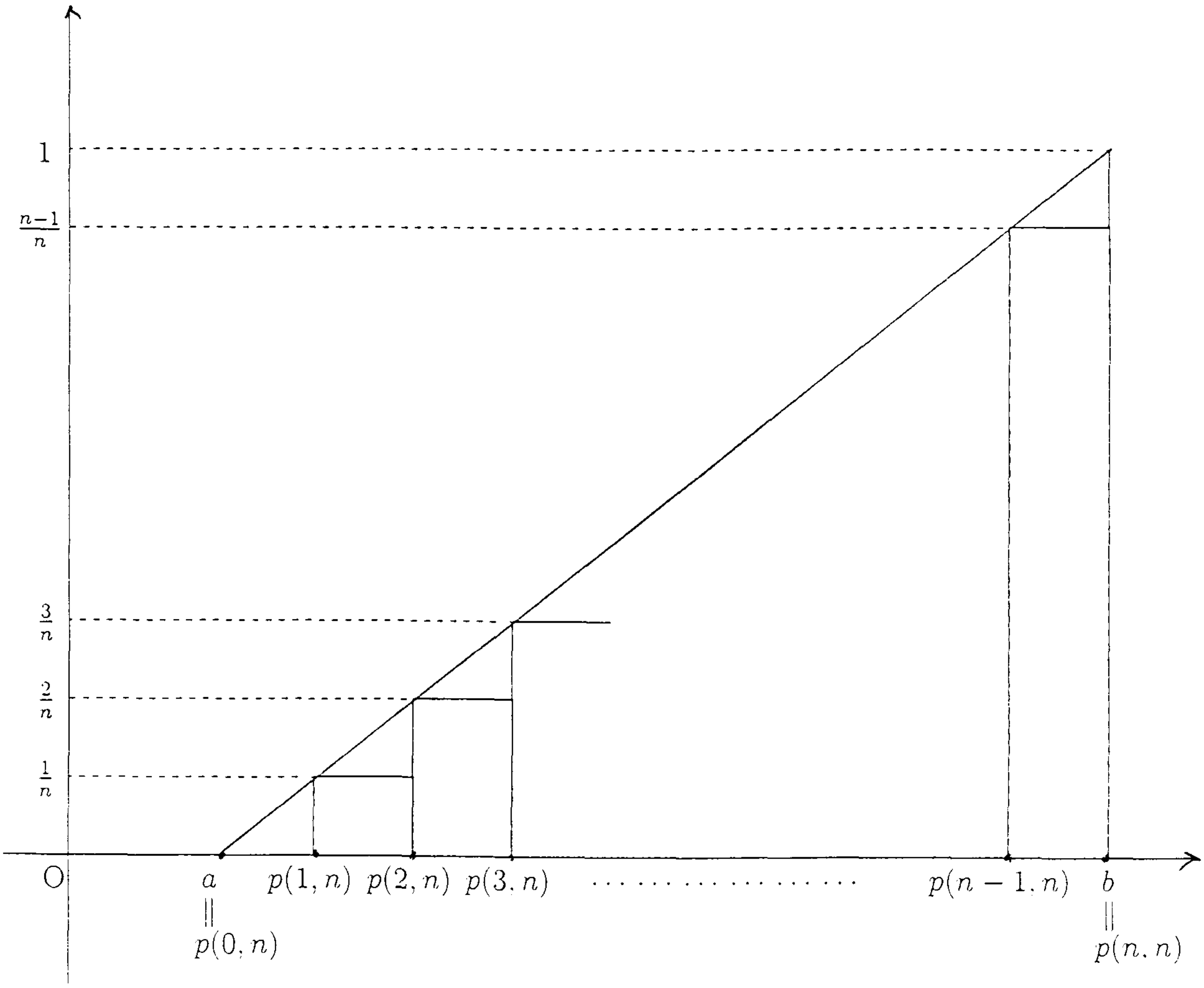

Fig.1 Fuzzy number $\tilde{A}$ and fuzzy set $\cup_{k=1}^{n}\tilde{B}_{k,n}$

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