行政院國家科學委員會補助專題研究計畫成果報告

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※ 交互作用粒子系統的流力極限 (III) ※
※ HYDRODYNAMIC LIMIT OF ※
※ INTERACTING PARTICLE SYSTEMS (III) ※
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計畫主持人：張志中

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行政院國家科學委員會專題研究計畫成果報告

交互作用粒子系統的流力極限 (III)

Hydrodynamic Limit of Interacting Particle Systems (III)

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一、中文摘要

本計畫中我們研究弱非對稱互斥過程的相對熵的估計。我們導出 d 維週期格子點空間 T_N^d 上的弱非對稱互斥過程在任一時刻對於利用離散化的流力極限方程式的解所建構出來的局部平衡態機率分佈的相對熵以及一個 N 倍的非對稱過程所對應的 Dirichlet form 的積分的總和的上界為 $CN^{d-1}\log N$，其中 C 為一常數，N 為流力極限參數。

關鍵詞：弱非對稱互斥過程、局部平衡態機率分佈、相對熵

Abstract

In this project we study the relative entropy of d dimensional weakly asymmetric simple exclusion processes (WASEP) on T_N^d with respect to the local equilibrium state associated to the profile given by the solution of the discretized hydrodynamic equation. We prove that at any time t>0, the sum of the relative entropy of WASEP and a term of order N, which consists of the integral of the Dirichlet form associated to the asymmetric part of the generator of WASEP, are bounded above by $CN^{d-1}\log N$. Here C is a positive constant and N is the parameter of the hydrodynamic limit.

Keywords: weakly asymmetric simple exclusion processes, local equilibrium state, relative entropy

二、緣由與目的

There are two powerful methods in studying the behaviors of scaling limit of stochastic particle systems. The first is the Dirichlet method developed by S.R.S. Varadhan, and the second is the relative entropy method developed by H.T. Yau. The latter has been extended to investigate the scaling limit of quantum systems recently. It is noted that both methods utilize the entropy bound to achieve a good control on interested quantities. Therefore, it is important for both methods to find, or construct, a nice reference measure, and then show that the underlying evolving stochastic or quantum system and the process governed by the reference measure is close to each other in the sense of relative entropy. More precisely, one needs to demonstrate that the relative entropy per site of the underlying system with respect to the reference process vanishes when taking the scaling limit. Once this fact has been established, ideally every estimate can be calculated under the reference measure, which is practically easier, instead of under the distribution of the original system. In this sense, to study any limiting behavior of the system, it is vital to derive a good control on the relative entropy of the underlying system with respect to a convenient reference process. In this project we derive an improved upper bound for future study.

三、結果與討論

The result stated in this report is taken from a joint work with C. Landim and T.Y. Lee [1]. Let $\mathbb{T}^d$ be the d-dimensional torus $[0,1)^d$. For each positive integer N, denote by $\mathbb{T}_N^d$ the discrete d-dimensional torus with N^d points : $\mathbb{T}_N^d = (\mathbb{Z}/N\mathbb{Z})^d$. Points of $\mathbb{T}^d$ are denoted by the letter u while sites of $\mathbb{T}_N^d$ are denoted by the letters x, y, z. Consider the weakly asymmetric simple exclusion process (WASEP) on $\mathbb{T}_N^d$ generated by

$$L_N f(\eta) = N^2 L_{1,N} f(\eta) + N L_{2,N} f(\eta),$$

where

$$\begin{aligned} L_{1,N} f(\eta) &= \sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d [f(\sigma^{x,x+e_j}\eta) - f(\eta)], \\ L_{2,N} f(\eta) &= \sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d p_j[1-\eta(x+e_j)] \times \\ &\quad \times \eta(x)[f(\sigma^{x,x+e_j}\eta) - f(\eta)]\,. \end{aligned}$$

In these formula, $\{p_j,\ 1\le j\le d\}$ is a collection of nonnegative constants, $\{e_j,\ 1\le j\le d\}$ stands for the canonical basis of $\mathbb{R}^d$ and $\sigma^{x,y}\eta$ for the configuration obtained from η by exchanging the occupation variables $\eta(x)$, $\eta(y)$:

$$(\sigma^{x,y}\eta)(z) = \begin{cases} \eta(z) & \text{if } z\ne x,\ y, \\ \eta(x) & \text{if } z = y, \\ \eta(y) & \text{if } z = x. \end{cases}$$

Given a real function $\gamma : \mathbb{T}^d \to [0,1]$, denote by $\nu^N_{\gamma(\cdot)}$ the Brenoulli distribution on $\Omega_N = \{0,1\}^{\mathbb{T}_N^d}$ with marginals $\nu^N_{\gamma(\cdot)}[\eta(x)] = \gamma(x/N)$. Fix a time $T>0$, denote by $\mathbb{P}^N_{\gamma(\cdot)}$ the probability measure of the process generated by L_N on the path space $D([0,T];\Omega_N)$ with initial measure $\nu^N_{\gamma(\cdot)}$. The hydrodynamic limit of WASEP governed by $\mathbb{P}^N_{\gamma(\cdot)}$ is stated in the following proposition.

Proposition 2.1 *The empirical density fields $N^{-d}\sum_{x\in\mathbb{T}_N^d}\eta_t(x)\delta_{x/N}$ converges in $\mathbb{P}^N_{\gamma(\cdot)}$-probability to the deterministic function $m(t,\theta)$, which is the unique weak solution of the hydrodynamic equation*

$$\begin{aligned} \frac{\partial m}{\partial t}(t,\theta) &= \Delta m - \sum_{j=1}^d p_j \frac{\partial}{\partial\theta_j} m(1-m), \\ m(0,\theta) &= \gamma(\theta), \end{aligned} \tag{2.1}$$

where $\Delta = \Delta_d$ stands for the d dimensional Laplacian.

In this paper we shall assume that the initial data γ is so regular that the above hydrodynamic equation has a smooth classical solution. Corresponding to the differential equation (2.1), and in view of the generator L_N, we also consider the semi-discretized difference equation (system)

$$\begin{aligned} \frac{d\rho^N}{dt}(t,x) &= (\Delta_N\rho^N)(t,x) - \sum_{j=1}^d p_j \times \\ &\quad \times \left(\nabla_j^N W_j^N\right)(t,x-e_j)\,, \\ \rho^N(0,x) &= \gamma(x/N)\,,\ x\in\mathbb{T}_N^d\,. \end{aligned} \tag{2.2}$$

Here Δ_N represents the discrete Laplacian, ∇_j^N the discrete differentiation in the j-th direction :

$$\begin{aligned} (\Delta_N\psi)(x) &= N^2\sum_{j=1}^d[\psi(x+e_j) - 2\psi(x) \\ &\quad + \psi(x-e_j)], \\ (\nabla_j^N\psi)(x) &= N[\psi(x+e_j) - \psi(x)], \end{aligned}$$

and $W_j^N(\cdot,x)$ the (symmetric) current from site x to site $x+e_j$:

$$W_j^N(t,x) = \rho^N(t,x)[1-\rho^N(t,x+e_j)].$$

Equation (2.2) will be referred to as simply the discrete or difference (hydrodynamic) equation. For convenience, from now on we shall omit the superscript N of the solution of the discrete equation ρ^N and write ρ only.

Now we follow [3], [2] to estimate the relative entropy of the process at time t with respect to the local equilibrium measure associated to the solution of the discrete equation (2.2).

Fix a product invariant reference measure $\nu_\alpha = \nu_\alpha^N$ for some density α in $(0,1)$. For each $t \geq 0$, denote by $\psi_t = \psi_t^N$ the Radon-Nikodym density of the product measure $\nu_{\rho(t,\cdot)}^N$ with respect to ν_α:

$$\begin{aligned} \psi_t(\eta) &= \frac{1}{Z_t^N} \exp\Big\{ \sum_{x\in \mathbb{T}_N^d} \lambda(t,x/N)\eta(x)\Big\} \\ &= \frac{d\nu_{\rho(t,\cdot)}^N}{d\nu_\alpha} \,, \end{aligned} \tag{2.3}$$

where

$$\lambda(t,x/N) = \log\Big\{\frac{\rho(t,x/N)[1-\alpha]}{[1-\rho(t,x/N)]\alpha}\Big\}$$

and Z_t^N is the normalizing constant

$$Z_t^N = \exp\Big\{ -\sum_{x\in\mathbb{T}_N^d} \log\frac{1-\rho(t,x/N)}{1-\alpha}\Big\}\,.$$

It follows from the definition of λ and from the hydrodynamic equation (2.1) that λ is the solution of

$$\partial_t\lambda = \Delta\lambda + [1-2\rho]\sum_{i=1}^d \Big\{[\partial_{u_i}\lambda]^2 - p_i\partial_{u_i}\lambda\Big\}\,. \tag{2.4}$$

We prove in [1] that the entropy of $\nu_{\rho_0(\cdot)}^N P_t^N$ with respect to $\nu_{\rho(t,\cdot)}^N$ is bounded by $CN^{d-1}\log N$ for some finite constant C. The statement of this result requires some notation.

Let $f_t = f_t^N$ be the Radon-Nikodym derivative of the state of the process at time t with respect to the reference measure ν_α:

$$f_t = \frac{d\nu_{\rho_0(\cdot)}^N P_t^N}{d\nu_\alpha}\,.$$

Since the initial measure is the local equilibrium state associated to the profile $\rho_0(\cdot)$, the density f_t is the solution of the Kolmogorov backward equation

$$\begin{cases} \partial_t f_t = & L_N^* f_t\,, \\ f_0 = & \psi_0\,, \end{cases} \tag{2.5}$$

where L_N^* is the adjoint of the generator L_N in $L^2(\nu_\alpha)$. A straightforward computation gives that $L_N^* = N^2 L_{1,N} + N L_{2,N}^*$, where

$$\begin{aligned} (L_{2,N}^* f)(\eta) &= \sum_{x\in\mathbb{T}_N^d}\sum_{j=1}^d p_j\eta(x+e_j)[1-\eta(x)] \times \\ &\quad \times [f(\sigma^{x,x+e_j}\eta) - f(\eta)]\,. \end{aligned}$$

Denote by $H_N(t)$ the entropy of $\nu_{\rho_0(\cdot)}^N P_t^N$ with respect to $\nu_{\rho(t,\cdot)}^N$:

$$H_N(t) = H(\nu_{\rho_0(\cdot)}^N P_t^N | \nu_{\rho(t,\cdot)}^N)\,,$$

and by $D_{2,N}$ the Dirichlet form:

$$D_{2,N}(f) = -\int \sqrt{f} L_{2,N}\sqrt{f}\,\nu_\alpha(d\eta)\,.$$

The main result of this project is the following estimate.

Proposition 2.2 *Fix $T > 0$. There exists a finite constant C_0, depending only on the initial profile ρ_0 and on T, such that*

$$\begin{aligned} & H_N(t) + N\int_0^t ds\, D_{2,N}(f_s) \\ \leq\ & C_0 N^{d-1}\log N \end{aligned} \tag{2.6}$$

for all $0 \leq t \leq T$.

Proof: It follows from (2.5) and from an elementary computation that the time derivative of $H_N(t)$ is given by

$$\begin{aligned} \partial_t H_n(t) &= -D_N(f_t|\psi_t) \\ &+ \int \psi_t^{-1}\Big(L_N^* - \partial_t\Big)\psi_t\, f_t\, d\nu_\alpha\,, \end{aligned} \tag{2.7}$$

where $D_N(f_t|\psi_t)$ stands for the positive convex lower semicontinuous functional given by

$$\int \psi\Big\{L_N\frac{f}{\psi} - \frac{f}{\psi}L_N\log\frac{f}{\psi}\Big\}\,\nu_\alpha(d\eta)\,.$$

The two terms on the right hand side of (2.7) can be estimated separately. See details in [1]. The proposition follows now from Gronwall inequality. □

四、計畫成果自評

We consider the entropy bound obtained in this project a very good result. This estimate will be used in the future to study several important behaviors of interacting particle systems, such as the non equilibrium fluctuations and the next order correction of hydrodynamic limit.

五、參考文獻

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