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Effects of inertia on the slow motion of aerosol particles

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Abstract

The translational motion of a spherical particle and a circular cylindrical particle (in the direction normal to its axis) in a quiescent unbounded fluid at small but finite Reynolds number is examined theoretically. The fluid, which may be a slightly rarefied gas, is allowed to slip at the surfaces of the particles. The axisymmetric Navier–Stokes equation for the fluid flow around the sphere and the two-dimensional equation of motion for the flow surrounding the cylinder are solved by using a method of matched asymptotic expansions. The approximate expressions for the drag force exerted by the fluid on the sphere and the cylinder are obtained analytically. For both cases of a sphere and a circular cylinder, the normalized drag force is found to increase monotonically with the Reynolds number and to decrease monotonically with the dimensionless slip coefficient (or the Knudsen number). The resulting formulas presented here include the previous results for a no-slip rigid sphere, a perfect-slip fluid sphere, and a no-slip circular cylinder as special cases. © 2000 Elsevier Science Ltd. All rights reserved.

Keywords: Aerosol particle; Inertial effect; Slip surface; Drag force

1. Introduction

The area of the relative motion of a solid particle or fluid drop in a continuous medium at low-Reynolds numbers has continued to receive much attention from investigators in the fields of chemical, biomedical, and environmental engineering and science. The majority of the moving phenomena are fundamental in nature, but permit one to develop rational understanding of many practical systems and industrial processes such as sedimentation, centrifugation, floatation, coagulation, spray drying, and aerosol processing. The theoretical study of this subject has grown out of the classic work of Stokes (1851) for a translating rigid sphere in a viscous fluid at a vanishingly small Reynolds number. Hadamard (1911) and Rybczynski (1911) have independently extended this result to the translation of a fluid sphere. Assuming continuous velocity and continuous tangential stress across the interface of fluid phases, they found that the force exerted on a spherical drop of radius a by the surrounding fluid of viscosity η is given by

$$F = 2\pi\eta a U \frac{3\eta^* + 2}{\eta^* + 1}, \tag{1}$$

where U is the migration velocity of the drop and η^* is the ratio of the viscosity of the interior to that of exterior fluid. Since the fluid viscosities are arbitrary, Eq. (1) degenerates to the case of translation of a solid sphere (Stokes' law) when $\eta^* \to \infty$ and to the case of motion of a gas bubble with spherical shape in the limit $\eta^* \to 0$.

In many practical applications of the relative motion of a solid or fluid particle in a viscous fluid, the Reynolds number is small but finite, and the effect of inertia on the particle motion cannot be entirely ignored. Using a singular perturbation method, Taylor and Acrivos (1964) analyzed the translation of a fluid drop, which is allowed to deform slightly from its spherical shape, at small but finite Reynolds number. When the effect of the deformation on the drop is not considered, their result for the drag force exerted on the drop is

$$F = 2\pi\eta a U \frac{3\eta^* + 2}{\eta^* + 1}\left[1 + \frac{1}{8}\left(\frac{3\eta^* + 2}{\eta^* + 1}\right)Re + \frac{1}{40}\left(\frac{3\eta^* + 2}{\eta^* + 1}\right)^2 Re^2 \ln Re + O(Re^2)\right]. \tag{2}$$

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In this expansion formula, the Reynolds number is defined by

$$Re = \frac{aU}{\nu}, \tag{3}$$

where ν is the kinematic viscosity of the surrounding fluid. In the limit $\eta^* \to \infty$, Eq. (2) reduces to the result obtained by Proudman and Pearson (1957) for the translation of a solid sphere.

When one tries to solve the Navier–Stokes equation, it is usually assumed that no slippage arises at the solid–fluid interfaces. Actually, this is an idealization of occurrence of the transport processes. The phenomenon that the adjacent fluid (especially if the fluid is a slightly rarefied gas) can slip frictionally over a solid surface has been confirmed, both experimentally and theoretically (Kennard, 1938; Ying & Peters, 1991; Hutchins, Harper & Felder, 1995). Presumably, any such slipping would be proportional to the local tangential stress next to the solid surface (Basset, 1961; Happel & Brenner, 1983), at least as long as the velocity gradient is small. The constant of proportionality, β^{-1}, may be termed a 'slip coefficient'. The quantity η/β is a length, which can be pictured by noting that the fluid motion is the same as if the solid surface was displaced inward by a distance η/β with the velocity gradient extending uniformly right up to no-slip velocity at the surface. Basset (1961) has found that the drag force acting on a translating rigid sphere with a slip-flow boundary condition at its surface (e.g., a settling aerosol sphere) in the limit of zero Reynolds number is

$$F = 6\pi\eta a U \frac{\beta a + 2\eta}{\beta a + 3\eta}. \tag{4}$$

When $\beta \to \infty$, there is no slip at the particle surface and Eq. (4) degenerates to Stoke's law. In the limiting case of $\beta = 0$, there is a perfect slip at the particle surface (the particle acts like a spherical gas bubble) and Eq. (4) is consistent with Eq. (1) (taking $\eta^* = 0$).

In Eq. (4), the slip coefficient has been determined experimentally for various cases and found to agree with the general kinetic theory of gases. It can be evaluated from the relation

$$\beta^{-1} = \frac{C_m l}{\eta}, \tag{5}$$

where l is the mean free path of a gas molecule, and C_m is a dimensionless constant of the gas-kinetic slip, which is semi-empirically related to the momentum accommodation coefficient f_m at the solid surface by $C_m \approx (2 - f_m)/f_m$ (Kennard, 1938). Although C_m surely depends upon the nature of the surface, examination of the experimental data suggests that it will be in the range 1.0–1.5 (Davis, 1972; Talbot, Cheng, Schefer, and Willis, 1980). Note that the slip-flow boundary condition is not only applicable for a gas–solid surface in the continuum regime (Knudsen number $Kn = l/a \ll 1$), but also appears to be valid for some cases even into the molecular flow regime ($Kn \geqslant 1$). The factor $(\beta a + 2\eta)/(\beta a + 3\eta)$ in Eq. (4) is equivalent to the so-called Cunningham correction factor for the slip effect.

In the present work we wish to study the inertial effects on the relative motion of a slip spherical particle and of a slip circular cylindrical particle (in the direction perpendicular to its axis) in a viscous fluid at small but finite Reynolds numbers. The method of matched asymptotic expansions (Proudman and Pearson, 1957; Illingworth, 1963; Van Dyke, 1975) is used to solve the problem. Analytical results in the form of expansion formulas for the drag force exerted by the fluid on the sphere and the cylinder as functions of the slip coefficient are given in Eqs. (23) and (43), respectively.

2. Motion of a slip sphere

In this section we consider a rigid spherical particle of radius a translating with a velocity U in an incompressible Newtonian fluid at rest at infinity. The fluid may slip frictionally at the surface of the particle. The spherical coordinate system r', θ, ϕ, with its origin at the particle center, is chosen to translate with the particle. In view of the axial symmetry of the flow, it is possible to express the steady equation of motion, the Navier–Stokes equation, in terms of the Stokes stream function $\psi'(r, \mu)$ as

$$E_r^4\psi = -Re\frac{1}{r^2}\left[\frac{\partial(\psi, E_r^2\psi)}{\partial(r, \mu)} + 2E_r^2\psi L_r\psi\right], \tag{6}$$

where the dimensionless variables $\psi = \psi'/a^2U$ and $r = r'/a$, μ is used to denote $\cos\theta$ for brevity, the operators E_r^2 and L_r are defined by

$$E_r^2 = \frac{\partial^2}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2}{\partial \mu^2}, \tag{7a}$$

$$L_r = \frac{\mu}{1-\mu^2}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \mu}, \tag{7b}$$

and $\partial(\psi, E_r^2\psi)/\partial(r, \mu)$ is the Jacobian of ψ and $E_r^2\psi$ with respect to r and μ. The Stokes stream function is related to the velocity components v'_r and v'_θ by

$$v_r = \frac{1}{r^2}\frac{\partial\psi}{\partial\mu}, \tag{8a}$$

$$v_\theta = \frac{1}{r(1-\mu^2)^{1/2}}\frac{\partial\psi}{\partial r}, \tag{8b}$$

where the dimensionless velocity components $v_r = v'_r/U$ and $v_\theta = v'_\theta/U$.

The boundary conditions require that there be no relative normal flow at the surface of the sphere and that the tangential velocity of the fluid relative to the sphere at a point on its surface be proportional to the tangential stress prevailing at that point. Thus,

$$r = 1: \quad v_r = 0, \tag{9a}$$

$$v_\theta = \frac{1}{\alpha} r \frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right), \tag{9b}$$

$$r \to \infty: \quad v_r = -\cos\theta, \tag{10a}$$

$$v_\theta = \sin\theta, \tag{10b}$$

where the dimensionless slip parameter $\alpha = \beta a/\eta$ (i.e., $\alpha^{-1} = C_m Kn$ is a dimensionless slip coefficient). Once (6)–(10) are solved for the stream function, the drag force exerted by the fluid on the spherical boundary $r = 1$ can be determined from

$$F = -\pi\eta a U \int_{-1}^{1} r^4 \frac{\partial}{\partial r}\left(\frac{E_r^2 \psi}{r^2}\right) \mathrm{d}\mu. \tag{11}$$

To obtain a solution for the fluid flow, we follow the approach of matched asymptotic expansions used by Proudman and Pearson (1957) and distinguish between the inner flow near the particle, where the Stokes variable r is $O(1)$, and the outer flow where the Oseen variable $R = Re\, r$ is $O(1)$. In the Stokes region of the flow, the solution for the stream function is assumed to take the expansion form

$$\psi = f_0(Re)\psi_0(r,\mu) + f_1(Re)\psi_1(r,\mu) + f_2(Re)\psi_2(r,\mu) + \ldots, \tag{12}$$

where $f_0(Re) = 1$ and $f_{n+1}(Re)/f_n(Re) \to 0$ as $Re \to 0$. The first term of this expansion satisfies the Stokes equation, $E_r^4\psi_0 = 0$, and the boundary conditions (9) and (10), and is given by (Basset, 1961)

$$\psi_0 = \frac{1}{4}\left(2r^2 - 3\frac{\alpha+2}{\alpha+3}r + \frac{\alpha}{\alpha+3}\frac{1}{r}\right)(1-\mu^2). \tag{13}$$

The Stokes expansion (12) is invalid far from the particle where r is of order Re^{-1}. We therefore introduce the contracted variables $R = Re\, r$ and $\Psi = Re^2\psi$, in terms of which the governing Eq. (6) becomes

$$E_R^4 \Psi = -\frac{1}{R^2}\left[\frac{\partial(\Psi, E_R^2\Psi)}{\partial(R,\mu)} + 2E_R^2\Psi L_R \Psi\right], \tag{14}$$

where E_R^2 and L_R are the same operators as those defined by Eqs. (7a) and (7b), but with r replaced by R. The Oseen expansion in the outer region is now expressed as the form

$$\Psi = \Psi_0(R,\mu) + F_1(Re)\Psi_1(R,\mu) + F_2(Re)\Psi_2(R,\mu) + \ldots, \tag{15}$$

where $F_{n+1}(Re)/F_n(Re) \to 0$ as $Re \to 0$. This expansion should be matched to the Stokes expansion (12) at small values of R, and its leading term is a uniform stream specified by the boundary condition (10),

$$\Psi_0 = \tfrac{1}{2} R^2(1-\mu^2). \tag{16}$$

When the expansion (15) for Ψ is substituted into the Navier–Stokes equation (14) written in Oseen variables, the terms involving $F_1(Re)$ show that Ψ_1 satisfies Oseen's equation

$$E_R^4\Psi_1 = \frac{1-\mu^2}{R}\frac{\partial E_R^2\Psi_1}{\partial\mu} + \mu\frac{\partial E_R^2\Psi_1}{\partial R}. \tag{17}$$

The solution to the above equation, which must vanish as $R \to \infty$ and at $\mu = \pm 1$, is

$$\Psi_1 = -\frac{3(\alpha+2)}{2(\alpha+3)}(1+\mu)[1 - \mathrm{e}^{-R(1-\mu)/2}]. \tag{18}$$

The matching requirement between the Oseen and Stokes expansions at small values of R, which has been satisfied to result in solution (18), yields that $F_1(Re) = Re$.

When the first approximation to the right-hand side of the Stokes form of the governing (6) is calculated from the leading term given by Eq. (13), one can find that $f_1(Re) = Re$, and the equation for ψ_1 may be written as

$$E_r^4\psi_1 = -\frac{9(\alpha+2)}{4(\alpha+3)}\left(\frac{2}{r^2} - 3\frac{\alpha+2}{\alpha+3}\frac{1}{r^3} + \frac{\alpha}{\alpha+3}\frac{1}{r^5}\right)\mu(1-\mu^2). \tag{19}$$

The general solution of this equation vanishing at $r = 1$ and $\mu = \pm 1$ is

$$\begin{aligned}\psi_1 = &-\frac{3(\alpha+2)}{32(\alpha+3)}\left[2r^2 - 3\frac{\alpha+2}{\alpha+3}r + \frac{\alpha(\alpha+6)}{(\alpha+3)(\alpha+5)} - \frac{\alpha}{\alpha+3}\frac{1}{r} + \frac{\alpha(\alpha+4)}{(\alpha+3)(\alpha+5)}\frac{1}{r^2}\right]\mu(1-\mu^2)\\ &+ \sum_{n=1}^{\infty}\left\{A_n\left[(2n-1)r^{n+3} - (2n+1)\left(1-\frac{2}{\alpha}\right)r^{n+1} + 2\left(1-\frac{2n+1}{\alpha}\right)r^{-n+2}\right]\right.\\ &\left.+ B_n\left[2\left(1+\frac{2n+1}{\alpha}\right)r^{n+1} - (2n+1)\left(1+\frac{2}{\alpha}\right)r^{-n+2} + (2n-1)r^{-n}\right]\right\}\int_{\mu}^{1} P_n(\mu)\,\mathrm{d}\mu,\end{aligned} \tag{20}$$

where $P_n(\mu)$ is the Legendre polynomial of order n. When Eq. (20) is expressed in terms of R and multiplied by Re^2 (in order to compare it with Ψ), it must not contain any term of order greater than 1, and so all the coefficients A_n and B_n must be zero except B_1. By matching the terms that are $O(1)$ with the expansion of the Oseen term

Ψ_1 given by Eq. (18) for small R, one can obtain B_1, and the solution for ψ_1 is

$$\psi_1 = \frac{3(\alpha+2)}{32(\alpha+3)}\left[2r^2 - 3\frac{\alpha+2}{\alpha+3}r + \frac{\alpha}{\alpha+3}\frac{1}{r} - \left(2r^2 - 3\frac{\alpha+2}{\alpha+3}r + \frac{\alpha(\alpha+6)}{(\alpha+3)(\alpha+5)} - \frac{\alpha}{\alpha+3}\frac{1}{r} + \frac{\alpha(\alpha+4)}{(\alpha+3)(\alpha+5)}\frac{1}{r^2}\right)\mu\right](1-\mu^2). \quad (21)$$

The last term in the square brackets arises from the right-hand side of Eq. (19) and so represents the effect of inertia on the inner flow. Because this term is an odd function of μ, it makes no contribution to the hydrodynamic drag on the sphere. Since the remainder part in Eq. (21) is $[3(\alpha+2)/8(\alpha+3)]\psi_0$, the drag on the sphere to the second approximation is $\{1+[3(\alpha+2)/8(\alpha+3)]Re\}$ times Basset's estimate given by Eq. (4).

The third term $f_2(Re)\psi_2$ in Eq. (12) can be found as follows. After some analysis it emerges that $f_2(Re) = Re^2 \ln Re$, and hence that ψ_2, which satisfies $E_r^4\psi_2 = 0$, must be a finite multiple of Basset's solution (13). By noticing that there is no term in $Re^2 \ln Re$ in the Oseen expansion, one can obtain the solution for ψ_2 as

$$\psi_2 = \frac{9}{160}\left(\frac{\alpha+2}{\alpha+3}\right)^2\left(2r^2 - 3\frac{\alpha+2}{\alpha+3}r + \frac{\alpha}{\alpha+3}\frac{1}{r}\right)(1-\mu^2). \quad (22)$$

According to the first three terms of the Stokes expansion, the drag force exerted on the sphere can be calculated by using Eq. (11), and the result is

$$F = 6\pi\eta a U\left(\frac{\alpha+2}{\alpha+3}\right)\left[1 + \frac{3}{8}\left(\frac{\alpha+2}{\alpha+3}\right)Re + \frac{9}{40}\left(\frac{\alpha+2}{\alpha+3}\right)^2 Re^2 \ln Re + O(Re^2)\right]. \quad (23)$$

When $\alpha \to \infty$, the above formula reduces to the result obtained by Proudman and Pearson (1957) for the translation of a no-slip sphere. Also, Eq. (23) is consistent with Eq. (2) (taking $\eta^* = 0$) in the limit $\alpha = 0$ (representing the case of motion of a spherical gas bubble).

The normalized drag force on the sphere,

$$C_D = \frac{F}{6\pi\eta a U}\left(\frac{\alpha+3}{\alpha+2}\right), \quad (24)$$

as calculated from Eq. (23), is plotted versus the Reynolds number in the range $0 < Re < 1$ in Fig. 1 with α as a parameter, and is plotted versus α in a broad range in Fig. 2 with Re as a parameter. Although the results for small values of α are included in these figures for completeness, it is understood that a small value of α is related to a large value of Kn and does not apply to the continuum regime of the fluid flow (this implies a free molecule regime for which the Navier–Stokes equation is not valid). As expected, the inertial effect of the fluid flow increases the normalized drag on the particle. It can be seen that C_D is a monotonic decreasing function of the dimensionless slip coefficient α^{-1} for a given Reynolds number (consistent with the fact that the drag force on the particle decreases significantly as the free molecule

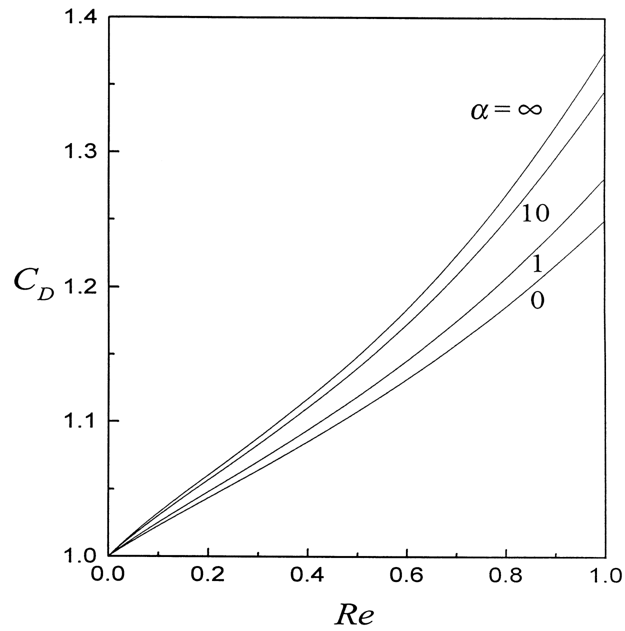

Fig. 1. Plots of the normalized drag force C_D on a slip sphere versus the Reynolds number Re for various values of the slip parameter α.

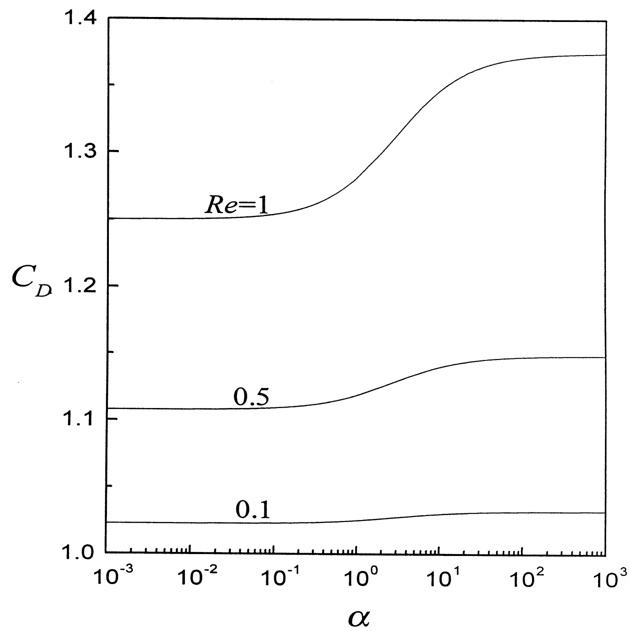

Fig. 2. Plots of the normalized drag force C_D on a slip sphere versus the slip parameter α with the Reynolds number Re as a parameter.

regime is approached). Note that C_D is a sensitive function of the parameter α only over the range of α equal to 0.4–20, which is equivalent to the range of Kn equal to 0.04–2. When $Re = 1$, the effect of inertia on C_D in the case of $\alpha = 1$ is 25% smaller than that in the case of $\alpha \to \infty$.

3. Transverse motion of a slip circular cylinder

We now consider the steady translational motion of a long cylindrical particle of radius a normal to its axis in a quiescent fluid extending to infinity. The fluid may slip frictionally at the surface of the cylinder. This transverse motion may be treated as a two-dimensional problem in a cross section perpendicular to the cylinder. It is understood that, for two-dimensional flow past any body, there exists no solution of the creeping motion equations ($Re \to 0$) vanishing on the body that remains finite at infinity (known as Stoke's paradox) (Happel & Brenner, 1983). The polar coordinate system (ρ', ϕ) measured from the cylinder axis is chosen to translate with the cylinder. When the Lagrangian stream function ψ' is expressed in nondimensional form by the relation $\psi = (\psi'/aU)$, the equation of motion for ψ becomes

$$\nabla_\rho^4 \psi = -\frac{Re}{\rho}\frac{\partial(\psi, \nabla_\rho^2 \psi)}{\partial(\rho, \phi)}, \tag{25}$$

where the dimensionless radial coordinate $\rho = \rho'/a$, and

$$\nabla_\rho^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2}. \tag{26}$$

The Lagrangian stream function is related to the velocity components v'_ρ and v'_ϕ by

$$v_\rho = \frac{1}{\rho}\frac{\partial \psi}{\partial \phi}, \tag{27a}$$

$$v_\phi = -\frac{\partial \psi}{\partial \rho}, \tag{27b}$$

where the dimensionless variables $v_\rho = (v'_\rho/U)$ and $v_\phi = (v'_\phi/U)$.

Similar to Eqs. (9a), (9b), (10a) and (10b) in the previous section, the boundary conditions for the fluid flow surrounding the cylinder are

$$\rho = 1: \quad v_\rho = 0, \tag{28a}$$

$$v_\phi = \frac{1}{\alpha}\rho \frac{\partial}{\partial \rho}\left(\frac{v_\phi}{\rho}\right), \tag{28b}$$

$$\rho \to \infty: v_\rho = -\cos\phi, \tag{29a}$$

$$v_\phi = \sin\phi. \tag{29b}$$

The drag force exerted by the fluid on the cylindrical boundary $\rho = 1$ per unit length can be evaluated by

$$F = -2\eta U \int_0^{2\pi}\left[\rho\frac{\partial^2 \psi}{\partial \rho^2}\sin\phi + \left(\frac{\partial^2 \psi}{\partial \rho \partial \phi} - \frac{1}{\rho}\frac{\partial \psi}{\partial \phi}\right)\cos\phi\right]\mathrm{d}\phi. \tag{30}$$

The stream function ψ may be expressed as a Stokes expansion

$$\psi = f_0(Re)\psi_0(\rho, \phi) + f_1(Re)\psi_1(\rho, \phi) + f_2(Re)\psi_2(\rho, \phi) + \dots \tag{31}$$

in the inner flow, and by an Oseen expansion

$$\Psi = \Psi_0(P, \phi) + F_1(Re)\Psi_1(P, \phi) + F_2(Re)\Psi_2(P, \phi) + \dots \tag{32}$$

in the outer flow, where the Oseen variables $P = Re\,\rho$ and $\Psi = Re\,\psi$. In terms of the Oseen variables, Eq. (25) becomes

$$\nabla_P^4 \Psi = -\frac{1}{P}\frac{\partial(\Psi, \nabla_P^2 \Psi)}{\partial(P, \phi)}. \tag{33}$$

The Oseen expansion is to satisfy the above equation and match the Stokes expansion at small values of P. Also, the leading term of the Oseen expansion is the uniform stream specified by the boundary condition (29),

$$\Psi_0 = -P\sin\phi. \tag{34}$$

The first term in the Stokes expansion must satisfy the Stokes equation, $L_\rho^4 V_0 = 0$, and the boundary condition (28) at $\rho = 1$. The appropriate expression for it is

$$\psi_0 = -\left[\rho\ln\rho - \frac{\alpha}{2(\alpha+2)}\rho + \frac{\alpha}{2(\alpha+2)}\rho^{-1}\right]\sin\phi, \tag{35}$$

$$f_0(Re) = \varepsilon = \left[\ln\frac{4}{Re} - \gamma + \frac{\alpha+4}{2(\alpha+2)}\right]^{-1}, \tag{36}$$

where γ is Euler's constant (= 0.5772). The condition given by Eq. (36) is necessary in order to match the Oseen expansion. Note that, unlike the three-dimensional case, the fluid velocity resulting from the solution (35) for ψ_0 diverges as $\rho \to \infty$.

The second term in the Oseen expansion (32) satisfies Oseen's equation

$$\left(\nabla_P^2 - \frac{\partial}{\partial X}\right)\nabla_P^2 \Psi_1 = 0, \tag{37}$$

where $X = P\cos\phi$. The general solution for $\nabla_P^2 \Psi_1$ that is antisymmetric about $\phi = 0$ and vanishes at infinity is

$$\nabla_P^2 \Psi_1 = \mathrm{e}^{P\cos\theta/2}\sum_{n=1}^{\infty} C_n K_n(P/2)\sin n\phi, \tag{38}$$

where $K_n(P/2)$ is the modified Bessel function of the second kind of order n and C_n are constants. By expanding both sides of the relation $\nabla_\rho^2\psi = Re\nabla_P^2\Psi$ for small P, one obtains that $C_1 = 1$, $C_n = 0$ for $n \geqslant 2$, and $F_1(Re) = \varepsilon$ (it can be found that $F_n(Re) = \varepsilon^n$) with ε defined by Eq. (36). Then, Eq. (37) may be integrated and matched to the Stokes expansion to give

$$\Psi_1 = \sum_{n=1}^{\infty} \{2K_1(P/2)I_n(P/2) + K_0(P/2)[I_{n-1}(P/2) + I_{n+1}(P/2)]\} \frac{P \sin n\phi}{n}, \tag{39}$$

where $I_n(P/2)$ is the modified Bessel function of the first kind of order n.

The Stokes expansion (31) turns out to be of the form

$$\psi = \sum_{n=0}^{\infty} \varepsilon^{n+1}\psi_n, \tag{40}$$

where the ψ_n satisfy the Stokes equation and are therefore proportional to ψ_0. Thus, we have

$$\psi = -\varepsilon\left(1 + \sum_{n=1}^{\infty} a_n\varepsilon^n\right)\left[\rho \ln \rho - \frac{\alpha}{2(\alpha+2)}\rho + \frac{\alpha}{2(\alpha+2)}\rho^{-1}\right] \sin \phi. \tag{41}$$

The above expansion agrees with Eq. (39) at small values of P in the terms P and $P \ln P$ to $O(\varepsilon)$ if $a_1 = 0$.

The third term in the Oseen expansion (32) must satisfy the inhomogeneous Oseen equation

$$\left(\nabla_P^2 - \frac{\partial}{\partial X}\right)\nabla_P^2\Psi_2 = -\frac{1}{P}\frac{\partial(\Psi_1, \nabla_P^2\Psi_1)}{\partial(P, \phi)} \tag{42}$$

and vanish at infinity. The constant a_2 can be determined by the matching procedure of Eq. (41) with the expansion of Ψ_2 for small P. Kaplun (1957) has carried a somewhat different process for the transverse motion of a no-slip cylinder to find a_2 equal to -0.87 approximately. Hence, the drag force exerted on the cylinder per unit length calculated by using (30) and (41) is

$$F = 4\pi\eta U[\varepsilon - 0.87\varepsilon^3 + O(\varepsilon^4)], \tag{43}$$

where ε is a function of Re defined by Eq. (36). The above formula differs from that for a no-slip cylinder (with $\alpha \to \infty$ and $\varepsilon = [\ln(4/Re) - \gamma + 1/2]^{-1}$) (Illingworth, 1963; Van Dyke, 1975) only by the function $\varepsilon(Re)$.

The normalized drag force on the cylinder,

$$C_D = \frac{F}{4\pi\eta U}, \tag{44}$$

as calculated from Eq. (43), is plotted versus the Reynolds number in the range $0 < Re < 1$ in Fig. 3 with α as a parameter, and is illustrated as a function of α in Fig. 4 with Re as a parameter. Analogous to the case of a slip

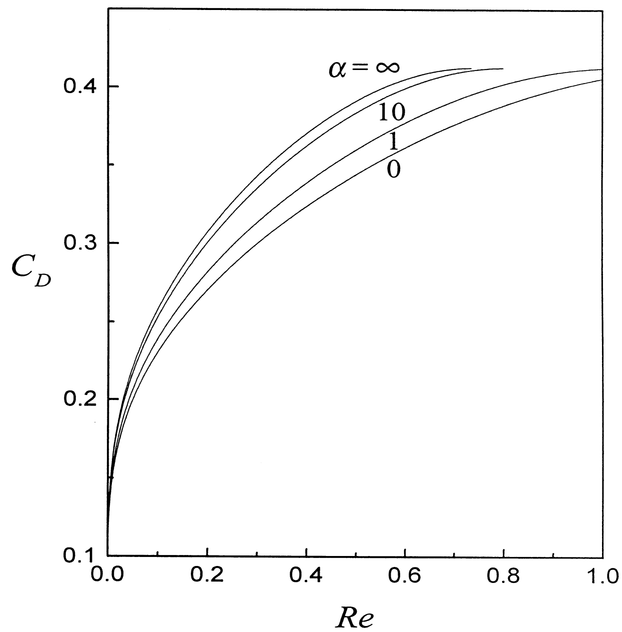

Fig. 3. Plots of the normalized drag force C_D on a slip circular cylinder versus the Reynolds number Re for various values of the slip parameter α.

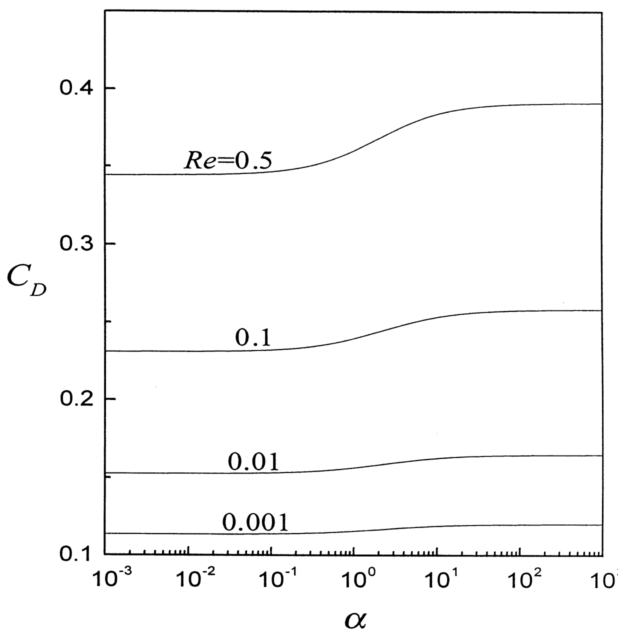

Fig. 4. Plots of the normalized drag force C_D on a slip circular cylinder versus the slip parameter α with the Reynolds number Re as a parameter.

sphere discussed in the previous section, the normalized drag C_D increases monotonically with the increase of the Reynolds number Re for a constant slip parameter α, and decreases monotonically with the increase of the dimensionless slip coefficient α^{-1} (or the Knudsen number Kn) for a fixed Re. Again, C_D is a sensitive function of α only over the range $0.4 < \alpha < 20$. When $Re = 0.5$, the inertial effect on C_D in the case of $\alpha = 1$ is about 8% smaller than that in the case of $\alpha \to \infty$ (which shows a relatively weak influence of $\alpha = 1$ in comparison with the case of a slip sphere). Note that $C_D \to 0$ as $Re \to 0$, because there is no solution of the Stokes equation for the transverse motion of a long cylinder.

Notation

a	particle radius, m
a_n	coefficients defined by Eq. (41)
C_D	normalized drag force on the particle
C_m	dimensionless frictional slip coefficient
E_r^2, E_R^2	operator defined by Eq. (7a)
F	drag force on the particle, N
f_n	terms in Stokes expansion given by Eq. (12) or (31)
F_n	terms in Oseen expansion given by Eq. (15) or (32)
I_n, K_n	modified Bessel functions of the first and second kinds, respectively, of order n
Kn	Knudsen number ($= l/a$)
l	mean free path of the gas molecules, m
L_r, L_R	operator defined by Eq. (7b)
P	equal to $Re\,\rho$
P_n	Lengendre polynomial of order n
r	dimensionless radial spherical coordinate
R	equal to $Re\,r$
Re	Reynolds number defined by Eq. (3)
U	particle velocity, m s^{-1}
v_r, v_θ	dimensionless fluid velocity components in spherical coordinates, m s^{-1}
v_ρ, v_ϕ	dimensionless fluid velocity components in polar coordinates, m s^{-1}
X	equal to $P \cos \phi$

Greek letters

α	$= \beta a/\eta$ or $(C_m Kn)^{-1}$
β	reciprocal of the slip coefficient at particle surface, kg m^{-2} s^{-1}
ε	function of Re and α defined by Eq. (36)
η	fluid viscosity, kg m^{-1} s^{-1}
θ, ϕ	angular spherical coordinates
μ	$= \cos\theta$
ρ, ϕ	dimensionless polar coordinates
ψ	dimensionless stream function of the fluid
ψ_n	coefficients in Stokes expansion given by Eq. (12) or (31)
Ψ	$= Re\,\psi$
Ψ_n	coefficients in Oseen expansion given by Eq. (15) or (32)

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