行政院國家科學委員會補助專題研究計畫成果報告

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計畫主持人：吳光鐘

共同主持人：

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行政院國家科學委員會專題研究計畫成果報告

史磋法於二維異向動態彈力問題之延伸

Extension of Stroh Formalism to Two-Dimensional Anisotropic Elastodynamic Problems

計畫編號：NSC 89-2212-E-002-007

執行期限： 88 **年** 8 **月** 1 **日至** 89 **年** 7 **月** 31 **日**

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一、中文摘要

本計畫將史磋法延伸於二維異向動彈力問題。傳統動彈力分析方法是以積分轉換為基礎，本計畫發展之方法則無此需要，應用上甚為方便。利用新法，本計劃求得無限域反平面應變問題之動態格林函數。此格林函數反比於時間及一有效動態剪力模數。其所對應在通過源點之平面上之曳引力為零。以無限域格林函數為基礎，本計畫並求得楔形體，半無線域及無限板之格林函數。

關鍵詞：異向彈性、彈性波傳、格林函數。

Abstract

The project extends the Stroh formalism to two-dimensional anisotropic elastodynamic problems. Conventional methods of analysis are based on integral transforms. The method developed in this project does not require such procedure. The dynamic Green’s function due to an impulse in an infinite anisotropic medium under anti-plane deformation is derived using the extended Stroh formalism method. The Green’s function is inversely proportional to the time *t* and an effective dynamic shear modulus. It is shown that the tractions on the planes passing through the source point vanish identically. Based on the free-space Green’s function, the Green’s functions for wedges, semi-infinite media and strips are obtained.

Keywords: Anisotropic Elasticity, Elastic Wave Propagation, Green Function.

2. Introduction

For a wide class of dynamic problems the solutions are self-similar, i.e., they are homogeneous functions of the time t and position **x**. This class of problems includes such important cases as Green's function and Lamb's problem. Extensive self-similar solutions have been obtained for isotropic solid. Much less attention, however, has been given to more complicated problems for anisotropic media.

The dynamic Green's functions are usually studied by integral transform methods. A formal solution in the transform space is first obtained. Considerable effort is then involved in the inversion from the transform space to the physical space by Cagniard's method [2]. In contrast, the general dynamic solution is expressed in terms of a complex function directly in the method developed by Smirnorv [1] and no inversion of integral is required. Furthermore, the form of the general solution is similar to that in the static case so that many solution techniques for elastostatics can be employed to treat dynamic problems..

In this project Smirnov's method is employed to derive dynamic Green's function for infinite anisotropic media under anti-plane deformation. The free-space Green’s function is used to construct those for wedges, semi-infinite spaces and strips. For semi-infinite spaces and strips, either free boundary conditions or clamped boundary conditions are considered.

3. Formulation

The equation of motion in anti-plane shear deformation is :

(1)

$$\text{«}_{11}u_{,11} + 2\text{«}_{12}u_{,12} + \text{«}_{22}u_{,22} = \text{°}\text{&}$$

where $\text{«}_{\text{~}}$ are the shear moduli. Let a variable μ be defined implicitly by

(2)

$$\mu = y_1 + p(\mu)y_2$$

where $y_k = x_k / t$, k = 1,2, and p is a function of μ to be determined later. Let us assume that $u = f(\mu)$. Substitution of (2) into (1) shows that $u(\mu)$ is a solution of (1) if

(3)

$$\text{«}_{11} + 2\text{«}_{12}p + \text{«}_{22}p^2 = \text{°}\mu^2$$

By substituting (2), (3) can be rewritten as

(4)

$$\tilde{\text{«}}_{11} + 2\tilde{\text{«}}_{12}p + \tilde{\text{«}}_{22}p^2 = 0$$

where $\tilde{\text{«}}_{\text{~}}(y_1, y_2) = \text{«}_{\text{~}} - \text{°}y_{\text{~}}y$, ~, $=1,2$, are the dynamic elastic constants. (4) yields the explicit expression for p as

(5)

$$p = \frac{-\tilde{\text{«}}_{12} \pm i\sqrt{W}}{\tilde{\text{«}}_{22}}$$

where $W(y_1, y_2) = \tilde{\text{«}}_{12}^2 - \tilde{\text{«}}_{11}\tilde{\text{«}}_{22}$.

It can be shown that the bulk shear wave front curve is given by $W(y_1, y_2) = 0$.

It follows that the roots of p appear as a complex conjugate pair in the wave region bounded by the wavefront curve $W(y_1, y_2) < 0$ and are real outside.

In the wave region the general solution can be represented as

(6)

$$u_{,1} = 2\,\mathrm{Re}[\frac{f(\mu)}{\Delta'}]$$

(7) $\text{&} = -2\,\mathrm{Re}[\frac{\mu\ f(\mu)}{\Delta'}]$

where p is taken to be the root with the positive imaginary part and $\Delta' = t - p'(\mu)x_2$. The stresses $\text{²}_1 \equiv \text{²}_{13}$, $\text{²}_2 \equiv \text{²}_{23}$ can be expressed as

(8) $\text{²}_1 = 2\,\mathrm{Re}[(\text{°}\mu^2 - p(\mu)b(\mu))\frac{f'(\mu)}{\Delta'}]$

(9) $\text{²}_2 = 2\,\mathrm{Re}[b(\mu)\frac{f'(\mu)}{\Delta'}]$

where $b(\mu) = \text{«}_{12} + \text{«}_{22}p(\mu)$. From (8) and (9), the traction on the radial plane, $\text{²}_{\text{–}}$, can be expressed as

(10)

$$\text{²}_{\text{–}} = -\frac{2}{y}\mathrm{Im}[\tilde{\text{«}}\mu\frac{f'(\mu)}{\Delta'}]$$

where $\tilde{\text{«}}(y_1, y_2) = \sqrt{\tilde{\text{«}}_{11}\tilde{\text{«}}_{22} - \tilde{\text{«}}_{12}^2}$ is the effective dynamic shear modulus. The expression for Δ' may be written as

$$(11)\ \Delta' = it\frac{\tilde{«}}{b}$$

It follows) that $\Delta' = 0$ upon the arrival of the bulk wave.

4. Dynamic Green Functions

3.1 Infinite Space

Consider a unit force $F = H(t)¢(x_1)¢(x_2)$ applied suddenly at the origin in an infinite space. Here H is the step function and $¢$ the Dirac delta function. The corresponding $f'(\mu)$ is determined as

$$(12)\ f'(\mu) = \frac{1}{4®i\mu\, b(\mu)}$$

The corresponding velocity field is given by

$$(13)\ \dot{u} = G_0$$

where

$$G_0 = \frac{1}{2®t}\mathrm{Re}[\frac{1}{\tilde{«}(y_1, y_2)}] \quad (14)$$

The velocity is also the *displacement* due to a unit impulse $I = ¢(t)¢(x_1)¢(x_2)$ and G_0 is the dynamic Green's function. The corresponding $^2{}_2$ is

$$(15)\ ^2{}_2 = -\frac{1}{2®t}\mathrm{Re}[\frac{b}{\tilde{«}\mu}]$$

3.2 Wedge

Since the traction on the radial plane ^{2}– obtained by substituting (12) into (10) is identically zero, the function $f'(\mu)$ corresponding to a line force suddenly applied on the tip of a wedge with angle ~ takes the following form:

$$f'(\mu) = \frac{}{4®i\mu\, b(\mu)} \quad (16)$$

where is a constant. The constant is determined by matching the limit of (15) as $t \to \infty$ to the static result [3]. The result is

$$= \frac{2®}{\sim} \quad (17)$$

The Green's function, G_w, in this case is thus given by

$$G_w(y_1, y_2; \sim) = \frac{2®}{\sim} G_0(y_1, y_2) \quad (18)$$

In particular the Green's function for a line impulse on the surface of a half-space is

$$G_w(y_1, y_2; ®) = 2\, G_0(y_1, y_2) \quad (19)$$

3.3 Half-Space

Consider two line forces F_1 and F_2 applied suddenly at $x_1 = 0, x_2 = ¶$ and $x_1 = -2¥¶, x_2 = -¶$,respectively. Here $¥ = «_{12} / «_{22}$. From (15), the corresponding stress $^2{}_2$ is given by

$$^2{}_2 = -\frac{1}{2®t}(\mathrm{Re}[\frac{b^{(1)}}{\tilde{«}^{(1)}\mu^{(1)}}]F_1 + \mathrm{Re}[\frac{b^{(2)}}{\tilde{«}^{(2)}\mu^{(2)}}]F_2) \quad (20)$$

where

$$\mu^{(1)} = y_1 + (y_2 - ¦)\, p^{(1)}(\mu^{(1)})\,,$$
$$\mu^{(2)} = y_1 + 2¥¦ + (y_2 + ¦)\, p^{(2)}(\mu^{(2)})$$

$$b^{(k)} = «_{12} + «_{22}\, p^{(k)}(\mu^{(k)}), \quad k = 1{,}2$$
$$\tilde{«}^{(1)}(y_1, y_2) = \tilde{«}(y_1, y_2 - ¦),$$
$$\tilde{«}^{(2)}(y_1, y_2) = \tilde{«}(y_1 + 2¥¦, y_2 + ¦)$$

and $¦ = ¶ / t$. It follows that the Green's function G_f^h due to a line impulse at the same location in the half-space is

$$G_f^h(x_1, x_2, t) = G_0(y_1, y_2 - ¦) + G_0(y_1 + 2¥¦, y_2 + ¦)$$

where G_0 is the free-space Green's function given by (14).

The Green's function G_c^h due to a line impulse in a half-space with clamped surface can be derived similarly by considering a unit line force at $x_1 = 0, x_2 = ¶$ and another unit line force in the opposite direction at $x_1 = -2¥¶, x_2 = -¶$. The result is

$$G_c^h(x_1, x_2, t) = G_0(y_1, y_2 - ¦) - G_0(y_1 + 2¥¦, y_2 + ¦)$$

3.4 Strips

Consider a strip described by $-\infty \le x_1 \le \infty,\ 0 \le x_2 \le h$. The Green's function for a line impulse in the strip at $x_1 = 0, x_2 = ¶ < h$ can be constructed based on the method of images discussed in Section 3.3. If the surfaces of the strip are traction-free, the Green's function is

$$G_{ff}^s = \sum_{k=-\infty}^{\infty} [G_0(y_1 - 2k¥v, y_2 - ¦ - 2kv) + G_0(y_1 + 2¥¦ + 2k¥v, y_2 + ¦ + 2kv)]$$

where $v = h / t$. If the surfaces are clamped, the Green's function is

$$G_{cc}^s = \sum_{k=-\infty}^{\infty} [G_0(y_1 - 2k¥v, y_2 - ¦ - 2kv) - G_0(y_1 + 2¥¦ + 2k¥v, y_2 + ¦ + 2kv)]$$

If the surface at $x_2 = 0$ is fixed and the surface at $x_2 = h$ is free, the Green's function is

$$G_{cf}^s = \sum_{k=-\infty}^{\infty} (-1)^k [G_0(y_1 - 2k¥v, y_2 - ¦ - 2kv) - G_0(y_1 + 2¥¦ + 2k¥v, y_2 + ¦ + 2kv)]$$

4. References

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