

EXPLICIT BIRATIONAL GEOMETRY OF 3-FOLDS AND 4-FOLDS OF GENERAL TYPE, III

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ABSTRACT. Nonsingular projective 3-folds V of general type can be naturally classified into 18 families according to the *pluricanonical section index* $\delta(V) := \min\{m | P_m \geq 2\}$ since $1 \leq \delta(V) \leq 18$ due to our previous series (I, II). Based on our further classification to 3-folds with $\delta(V) \geq 13$ and an intensive geometrical investigation to those with $\delta(V) \leq 12$, we prove that $\text{Vol}(V) \geq \frac{1}{1680}$ and that the pluricanonical map Φ_m is birational for all $m \geq 61$, which greatly improves known results. An optimal birationality of Φ_m for the case $\delta(V) = 2$ is obtained. As an effective application, we study projective 4-folds of general type with $p_g \geq 2$ in the last section.

1. Introduction

One of the fundamental aspects of birational geometry is to understand the behavior of the natural pluricanonical map Φ_m of any variety for any $m \in \mathbb{Z}_{>0}$. The induced fibrations possibly reduce the studies to lower dimensional situations. Varieties of general type, which are those with birational pluricanonical maps Φ_m for sufficiently large m , are therefore considered as the basic building blocks of varieties.

For varieties of general type, a key problem is to find an effective integer $m > 0$ so that Φ_m is birational. The remarkable theorem of Hacon and McKernan [17], Takayama [31], and Tsuji [32] says that there is a constant $c(n)$ so that Φ_m is birational for all n -dimensional varieties of general type and for all $m \geq c(n)$. However, these constants are explicitly known only when $n \leq 3$.

In fact, the problem is almost equivalent to find a practical lower bound of the canonical volume which computes the rate of growth of plurigenus, or equivalent to find m_0 such that plurigenus P_{m_0} is sufficiently large. One may also refer to the nice survey article Hacon–McKernan [18] for various boundedness results in birational geometry.

The motivation of this series is to study birational geometry of 3-folds and higher dimensional varieties of general type. The main purpose is to investigate the following:

The first author was partially supported by NCTS/TPE and National Science Council of Taiwan. The second author was supported by National Natural Science Foundation of China (#11171068, #11121101, #11231003) and Doctoral Fund of Ministry of Education of China (#20110071110003).

Open problem 1.1. *Find optimal constants $v_3 \in \mathbb{Q}_{>0}$ and $b_3 \in \mathbb{Z}_{>0}$ so that, for all nonsingular projective 3-folds V of general type,*

- i. $\text{Vol}(V) \geq v_3$ and
- ii. Φ_m is birational for all $m \geq b_3$.

Recall that we have proved the following:

Theorem 1.2. ([7, Theorems 1.1, 1.2]) *Let V be a nonsingular projective 3-fold of general type. Then*

- 1. $\text{Vol}(V) \geq \frac{1}{2660}$;
- 2. *there exists a positive integer $m_0(V) \leq 18$ so that $P_{m_0} \geq 2$;*
- 3. *the pluricanonical map Φ_m is birational onto its image for all $m \geq 73$.*

For more results on explicit birational geometry of 3-folds of general type, one may refer to our previous papers [6, 7].

In order to formulate our main statements of this article, we need to recall some general results and introduce some definition. Given a projective variety V of general type, there exists a minimal model X birational to V (cf. [2]). Thanks to the Riemann-Roch formula and Vanishing Theorem, $\text{Vol}(V) = K_X^{\dim X}$. Notice that in dimension three or higher, a minimal model may have singularities. Hence $K_X^{\dim X}$ is just a positive rational number.

A minimal model has at worst terminal singularities. In dimension three, terminal singularities was classified by Mori. A three dimensional terminal singularity is one of the following: a terminal quotient singularity of type $\frac{1}{r}(1, -1, b)$ for some b relatively prime to r which we usually denote it as (b, r) for short, an isolated cDV point, a quotient of an isolated cDV point. It is well-known to experts that a three dimensional terminal point can be deformed into a collection of terminal quotient singularities, which is called *basket of singularities*. An important feature of three dimensional birational geometry is the Singular Riemann-Roch formula due to Reid [29]

$$\chi(X, mK_X) = \frac{m(m-1)(2m-1)K_X^3}{12} + (1-2m)\chi(X, \mathcal{O}_X) + l_m,$$

where l_m denotes the contribution of singularities which can be computed by baskets. It follows that all plurigenera and hence canonical volume of a minimal 3-fold X are completely determined by $P_2(X)$, $\chi(X, \mathcal{O}_X)$ and baskets of singularities B_X , of which we called such a triple *the weighted basket* of X . For the basic properties of weighted baskets, one may refer to [6, Section 3]. Since our problems are birational in nature, the studies of nonsingular threefold V is equivalent to the studies of its minimal model X . In particular, we may and do

consider the weighted basket of V as the weighted basket of its minimal model X .¹

Next, we would like to define the *pluricanonical section index* (or, in short, the *ps-index*)

$$\delta(V) := \min\{m \mid m \in \mathbb{Z}_{>0}, P_m(V) \geq 2\},$$

which is clearly a birational invariant. By Theorem 1.2, we have $\delta(V) \leq 18$ for any 3-fold V of general type. Note that 3-folds V with $\delta(V) = 1$ (i.e., $p_g(V) \geq 2$) have been intensively studied in [10, 11] where optimal results are realized. Threefolds of general type with $\delta(V) \geq 2$ are far from being clear. Sometimes we use the symbol $\delta(X)$ directly since X is birationally equivalent to V .

Example 1.3. The “worst” known minimal 3-fold is the weighted hyper-surface $X := X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23)$ (cf. [15]) which has the invariants: $\delta(X) = 10$ and $\text{Vol}(X) = K_X^3 = \frac{1}{420}$. Also Φ_{26} is not birational.

In this paper, we mainly investigate projective 3-folds of general type with $\delta(V) \geq 2$. Our main results are as follows.

Theorem 1.4. (=Theorem 5.1) *Let V be a nonsingular projective 3-fold of general type with $\delta(V) \geq 13$. Then its weighted basket $\mathbb{B} = \{B_V, P_2(V), \chi(\mathcal{O}_V)\}$ belongs to one of the types in Tables F-0, F-1, F-2 in Appendix and the following is true:*

- (1) $\delta(V) = 18$ if and only if $\mathbb{B}(V) = \{B_{2a}, 0, 2\}$;
- (2) $\delta(V) \neq 16, 17$;
- (3) $\delta(V) = 15$ if and only if $\mathbb{B}(V)$ belongs to one of the types in Table F-1;
- (4) $\delta(V) = 14$ if and only if $\mathbb{B}(V)$ belongs to one of the types in Table F-2;
- (5) $\delta(V) = 13$ if and only if $\mathbb{B}(V) = \{B_{41}, 0, 2\}$,

where B_{2a} and B_{41} can be found in Table F-0

Some other results for 3-folds with large $\delta(V)$ are given in Section 4. For example, one has

Corollary 1.5. (=Corollary 4.8) *Let V be a nonsingular projective 3-fold of general type with $\text{Vol}(V) < \frac{1}{336}$. Then $\delta(V) \geq 8$.*

We also prove the following:

Theorem 1.6. *Let V be a nonsingular projective 3-fold of general type. Then*

- (1) Φ_m is birational for all $m \geq 61$;

¹Even though minimal models are not necessarily unique, it is known that two birational minimal models are connected by flops (cf. [21]). Together with the fact that a 3-dimensional flop preserves singularity types (cf. [23]), it follows that baskets of V is independent of choices of minimal models.

- (2) $\text{Vol}(V) \geq \frac{1}{1680}$. Furthermore, $\text{Vol}(V) = \frac{1}{1680}$ if and only if $\mathbb{B}(V) = \{B_{7a}, 0, 2\}$ or $\{B_{36a}, 0, 2\}$, where B_{7a} and B_{36a} can be found in Table F-2

A direct by-product of our method is the following:

Corollary 1.7. *Let V be a nonsingular projective 3-fold of general type with $p_g(V) = 1$. Then*

- (1) $\text{Vol}(V) \geq \frac{1}{75}$;
- (2) Φ_m is birational for all $m \geq 18$.

In the second part of this paper we prove some optimal results on 3-folds with $\delta(V) = 2$.

Theorem 1.8. *Let V be a nonsingular projective 3-fold of general type with $\delta(V) \leq 2$. Then*

- (1) Φ_m is birational for all $m \geq 11$;
- (2) If Φ_{10} is not birational, then $0 \leq \chi(\mathcal{O}_V) \leq 3$ and $|2K_V|$ is composed of a rational pencil of $(1, 2)$ surfaces. Furthermore, $\#\{\mathbb{B}(V)\} < +\infty$ and the initial basket B^0 of B_V belongs to one of the types in Tables II-1, II-2, II-3 in the Appendix.

The following examples show that our results in Theorem 1.8 are optimal.

Example 1.9. (Iano-Fletcher [15, P. 151, P. 153])

- (1) General weighted complete intersections $X_{22} \subset \mathbb{P}(1, 2, 3, 4, 11)$ and $X_{6,18} \subset \mathbb{P}(2, 2, 3, 3, 4, 9)$ both have ps-index $\delta = 2$. Since both X_{22} and $X_{6,18}$ have non-birational 10-canonical map, Theorem 1.8(1) is optimal.
- (2) The 3-fold X_{22} corresponds to No. 1 in Table II-1 with $\chi = 0$ and $X_{6,18}$ belongs to No. 11 (with $t = 1$) in Table II-1.

Remark 1.10. Theorem 1.8 is parallel to main results in [10]. We have similar statements to Theorem 1.8 for 3-folds with $\delta(V) \geq 3$. We omit them since we are not sure whether they are optimal or not.

In the last part we study projective 4-folds. The main result is the following:

Theorem 1.11. (=Theorem 8.2) *Let V be a nonsingular projective 4-fold of general type. Then,*

- (i) when $p_g(V) \geq 2$, $\Phi_{|mK_V|}$ is birational for all $m \geq 35$;
- (ii) when $p_g(V) \geq 19$, $\Phi_{|mK_V|}$ is birational for all $m \geq 18$.

This paper is organized as follows. In Section 2, we start with general setting on rational maps on varieties of general type and review some known useful inequalities. Then we list several basic lemmas on 3-folds. In Section 3, we improve our technique used in [7] to bound K_X^3

from below. Applying our basket analysis developed in [6], we obtain an effective function $v(x)$ in Section 4 so that $K_X^3 \geq v(\delta(X))$ for any given minimal 3-fold X . Section 5 is devoted to compiling the clean list for $\mathbb{B}(X)$ with $\delta(X) \geq 13$. Then, in Section 6, we are able to study the birationality of Φ_m . Section 7 is dedicated to classifying 3-folds with $\delta = 2$. Finally we study nonsingular projective 4-folds of general type with $p_g \geq 2$ in Section 8. All subsidiary tables are presented in the Appendix.

Throughout we work over any algebraically closed field k of characteristic 0. We are in favor of the following symbols:

- “ $\sim$ ” denotes linear equivalence or $\mathbb{Q}$ -linear equivalence;
- “ $\equiv$ ” denotes numerical equivalence;
- “ $|A| \preceq |B|$ ” means that $|B| \supseteq |A| + \text{fixed effective divisors}$.

2. Preliminaries

We begin with the general setting on rational maps defined by some sub-linear system of the pluricanonical system $|mK|$ on varieties of general type. Let V be any nonsingular projective variety of general type with dimension $n \geq 3$. According to the Minimal Model Program, V has a minimal model (see e.g. [22], [24], [2] and [30]). From the point of view of birational geometry, we may always consider the rational map on minimal varieties of general type. A minimal model X is a normal projective variety with a nef canonical divisor K_X and with $\mathbb{Q}$ -factorial terminal singularities.

2.1. The rational map Φ_Λ for $\Lambda \subset |m_0 K|$. Let X be a minimal projective variety of general type on which $P_{m_0}(X) \geq 2$ for a positive integer m_0 . Let $\Lambda \subset |m_0 K_X|$ be a positive dimensional linear system. Fix an effective Weil divisor $K_{m_0} \sim m_0 K_X$ on X . Take successive blow-ups $\pi: X' \rightarrow X$ along nonsingular centers, such that the following conditions are satisfied:

- (i) X' is smooth;
- (ii) the moving part of $\pi^*(\Lambda)$ is base point free and so that $g := \Phi_\Lambda \circ \pi$ is a non-constant morphism;
- (iii) $\pi^*(K_{m_0}) \cup \{\pi\text{-exceptional divisors}\}$ has simple normal crossing supports.

Sometimes we will take further blow-ups so that π satisfies some more conditions, which will be specified explicitly.

We have a morphism $g: X' \rightarrow \overline{\Phi_\Lambda(X)} \subseteq \mathbb{P}^N$. Let $X' \xrightarrow{f} \Gamma \xrightarrow{s} \overline{\Phi_\Lambda(X)}$ be the Stein factorization of g . We have the following commutative diagram:

$$\begin{array}{ccc}
X' & \xrightarrow{f} & \Gamma \\
\pi \downarrow & \searrow g & \downarrow s \\
X & \xrightarrow{\Phi_\Lambda} & \overline{\Phi_\Lambda(X)}
\end{array}$$

We may write $m_0 K_{X'} =_{\mathbb{Q}} \pi^*(m_0 K_X) + E_{\pi, m_0}$ where E_{π, m_0} is an effective π -exceptional $\mathbb{Q}$ -divisor. Denote by M_{m_0} (resp. M_Λ) the movable part of $|m_0 K_{X'}|$ (resp. $\pi^* \Lambda$). Set $d_{m_0} := \dim \Phi_{m_0}(X)$ (resp. $d_\Lambda := \dim \Gamma$). The Bertini Theorem implies that the general member of the moving part M_Λ of $\pi^*(\Lambda)$ is irreducible whenever $d_\Lambda \geq 2$ and, otherwise, $M_\Lambda \equiv a_\Lambda F$, where $a_\Lambda := \deg f_* \mathcal{O}_{X'}(M_\Lambda)$ and F is a general fiber of f . We set

$$\theta_\Lambda := \begin{cases} 1, & \text{if } d_\Lambda \geq 2; \\ a_\Lambda, & \text{if } d_\Lambda = 1. \end{cases}$$

Recall our definition in [7, Definition 2.4], the *generic irreducible element* Σ of $\pi^*(\Lambda)$ is defined as follows:

$$\Sigma_\Lambda := \begin{cases} \text{the general member of the moving part of } \pi^*(\Lambda), & \text{if } d_\Lambda \geq 2; \\ F, & \text{if } d_\Lambda = 1. \end{cases}$$

By the above setting, we always have

$$m_0 \pi^*(K_X) \sim_{\mathbb{Q}} \theta_\Lambda \Sigma_\Lambda + E'_\Lambda$$

for some effective $\mathbb{Q}$ -divisor E'_Λ on X' .

Convention. Whenever we are working on the complete linear system $|m_0 K_X|$, we will use parallel notations such as $d_{m_0}, \theta_{m_0}, \dots$ (or even just $d, \theta, \dots$, for simplicity).

We discuss the special case with $d_\Lambda = 1$. Clearly the general fiber F is nonsingular projective of dimension $\dim(X) - 1$. Replace X' by its birational model, we may assume that there is a birational contraction morphism $\sigma : F \rightarrow F_0$ onto a minimal model F_0 . We have the following “canonical restriction inequality”:

Lemma 2.1. *Keep the above settings. Suppose that $d_\Lambda = 1$. The following holds:*

- (i) if $b := g(\Gamma) > 0$, then $\pi^*(K_X)|_F \sim \sigma^*(K_{F_0})$;
- (ii) if $b = 0$, then

$$\pi^*(K_X)|_F \geq \frac{\theta_\Lambda}{m_0 + \theta_\Lambda} \sigma^*(K_{F_0}).$$

Proof. Statement (i) follows from Chen [14, Lemma 2.5].

Assume $\Gamma \cong \mathbb{P}^1$. Choose a sufficiently large and divisible integer m so that both $|m\pi^*(K_X)|$ and $|mK_{F_0}|$ are base point free. By Kawamata's extension theorem [20, Theorem A], we have the surjective map:

$$H^0(X', m\theta_\Lambda(K_{X'} + F)) \longrightarrow H^0(F, m\theta_\Lambda K_F).$$

Since $|m(\theta_\Lambda + m_0)K_{X'}| \supseteq |m\theta_\Lambda(K_{X'} + F)|$, $\text{Mov}|m\theta_\Lambda K_F| = |m\theta_\Lambda \sigma^*(K_{F_0})|$ and $|m(\theta_\Lambda + m_0)\pi^*(K_X)| = |M_{m(\theta_\Lambda + m_0)}|$, we obtain the following inequality:

$$m(\theta_\Lambda + m_0)\pi^*(K_X)|_F = M_{m(\theta_\Lambda + m_0)}|_F \geq m\theta_\Lambda \sigma^*(K_{F_0}),$$

which implies (ii). $\square$

2.2. Key inequalities on 3-folds. Let X be minimal 3-fold of general type. Assume that $\Lambda \subset |m_0 K_X|$ is a linear system of positive dimension. As in 2.1, we obtain an induced fibration $f : X' \rightarrow \Gamma$. Pick a generic irreducible element S of $|m_0 K_{X'}|$. Let $|G|$ be a given base point free linear system on S . Pick a generic irreducible element C of $|G|$. Since $\pi^*(K_X)|_S$ is nef and big, Kodaira's lemma implies that $\pi^*(K_X)|_S \geq \beta C$ for some rational number $\beta > 0$. Then, by [7, Inequality (2.1)], one has

$$K_X^3 \geq \frac{\theta\beta}{m_0} \xi \quad (2.1)$$

where $\xi := (\pi^*(K_X) \cdot C)_{X'}$. Besides, by [7, Remark 2.12], one has

$$\xi \geq \frac{\deg(K_C)}{1 + \frac{m_0}{\theta} + \frac{1}{\beta}}. \quad (2.2)$$

For any positive integer m so that $\alpha_m := (m - 1 - \frac{m_0}{\theta} - \frac{1}{\beta})\xi > 1$, by Chen–Zuo [13, Theorem 3.1], one has

$$\xi \geq \frac{\deg(K_C) + \lceil \alpha_m \rceil}{m}. \quad (2.3)$$

We have a stronger form of Inequality (2.3) when C is “even”:

Lemma 2.2. *Under the above situation, if C is an even divisor on S (i.e. $\frac{1}{2}C \in \text{Pic}(S)$), then, for any $m > 0$ so that $\alpha_m > 0$, one has*

$$\xi \geq \frac{\deg(K_C) + 2\lceil \frac{1}{2}\alpha_m \rceil}{m}. \quad (2.4)$$

Proof. We refer to the proof for Chen–Zuo [13, Theorem 3.1]. The key point is to estimate $\deg(D)$ where $D = \lceil Q \rceil|_C$ and Q is a $\mathbb{Q}$ -divisor on S with $(Q \cdot C) = \alpha_m$. Since $\deg(D) \geq \alpha_m > 0$ and $\deg(D)$ is even, we naturally have

$$\deg(D) = 2(\lceil Q \rceil \cdot \frac{1}{2}C) \geq 2\lceil \frac{1}{2}\alpha_m \rceil$$

where we note that $(\lceil Q \rceil \cdot \frac{1}{2}C)$ is a positive integer. Clearly the rest of the proof of Chen–Zuo [13, Theorem 3.1] implies Inequality (2.4). $\square$

When $d_\Lambda = 1$, Lemma 2.1(ii) implies the following:

$$\xi = (\pi^*(K_X) \cdot C)_{X'} \geq \frac{\theta}{m_0 + \theta} (\sigma^*(K_{F_0}) \cdot C)_F. \quad (2.5)$$

2.3. Other useful Lemmas.

Lemma 2.3. (see Mařek [25, Proposition 4] or [12, Lemma 2.6]) *Let S be a nonsingular projective surface. Let L be a nef and big $\mathbb{Q}$ -divisor on S satisfying the following conditions:*

- (1) $L^2 > 8$.
- (2) $(L \cdot C_x) \geq 4$ for all irreducible curves C_x passing through any very general point $x \in S$.

Then the linear system $|K_S + \lceil L \rceil|$ separates two distinct points in very general positions. Consequently, $|K_S + \lceil L \rceil|$ gives a birational map.

Lemma 2.4. *Let $\sigma : S \rightarrow S_0$ be a birational contraction from a nonsingular projective surface S of general type onto the minimal model S_0 . Assume that $(K_{S_0}^2, p_g(S_0)) \neq (1, 2)$ and that C is a moving curve on S . Then $(\sigma^*(K_{S_0}) \cdot C) \geq 2$.*

Proof. When $K_{S_0}^2 \geq 2$, this is due to Hodge index theorem. When $(K_{S_0}^2, p_g(S_0)) = (1, 0)$, this is due to Miyaoka [26, Lemma 5]. When $(K_{S_0}^2, p_g(S_0)) = (1, 1)$, $(\sigma^*(K_{S_0}) \cdot C) = 1$ implies $K_{S_0} \equiv \sigma_* C$ by Hodge index theorem. According to Bombieri [3], we know that S_0 is simply connected. Thus $K_{S_0} \sim \sigma_* C$, which is impossible since $|K_{S_0}|$ is not movable. $\square$

Lemma 2.5. *Let $\sigma : S \rightarrow S_0$ be the birational contraction onto the minimal model S_0 from a nonsingular projective surface S of general type. Assume that $(K_{S_0}^2, p_g(S_0)) \neq (1, 2)$ and that $\tilde{C}$ is a curve on S passing through very general points. Then $(\sigma^*(K_{S_0}) \cdot \tilde{C}) \geq 2$.*

Proof. In fact, by the projection formula, this is equivalent to see $(K_{S_0} \cdot C_0) \geq 2$ for any curve $C_0 \subset S_0$ passing through very general points of S_0 .

On the contrary, let us assume $(K_{S_0} \cdot C_0) \leq 1$. Then $g(C_0) \geq 2$ implies $C_0^2 \geq 1$. The Hodge index theorem says $K_{S_0}^2 = 1$ and $K_{S_0} \equiv C_0$. Recall that S_0 is not a $(1, 2)$ surface. So S_0 must be either a $(1, 0)$ surface or a $(1, 1)$ surface.

If $(K_{S_0}^2, p_g(S_0)) = (1, 0)$, then $q(S_0) = 0$ and the torsion element $\theta := K_{S_0} - C_0$ is of order ≤ 5 (see Reid [27]) and $h^0(S_0, C_0) = 1$. Thus there are at most finite number of such curves on S_0 since $\#\text{Tor}(S_0) \leq 5$, which is absurd by the choice of C_0 .

If $(K_{S_0}^2, p_g(S_0)) = (1, 1)$, then $q(S_0) = 0$ and $K_{S_0} \sim C_0$ since $\text{Tor}(S_0) = 0$ by Bombieri [3, Theorem 15] and thus C_0 is the unique canonical curve of S_0 , which is absurd as well. $\square$

2.4. The birationality principle.

Definition 2.6. Pick two different generic irreducible elements S', S'' (resp. C', C'') in $|M_{m_0}|$ (resp. in $|G|$).

- (1) We say that $|mK_{X'}|$ *distinguishes* S' and S'' if $\Phi_{|mK_{X'}|}(S') \neq \Phi_{|mK_{X'}|}(S'')$.
- (2) We say that $|mK_{X'}|$ *distinguishes* C' and C'' if $\Phi_{|mK_{X'}|}(C') \neq \Phi_{|mK_{X'}|}(C'')$.

We will apply the useful, but technical theorem in Chen-Zuo [13] for the birationality of Φ_m .

Theorem 2.7. (see Chen-Zuo [13, Theorem 3.1] or [7, Theorem 2.11, Part 2]) *Keep the same notations as above. Assume that, for some $m > 0$, $|mK_{X'}|$ distinguishes S' and S'' , C' and C'' for generic $S' \neq S''$, $C' \neq C''$. Then Φ_m is birational under one of the following conditions:*

- (i) $\alpha_m > 2$;
- (ii) $\alpha_m > 1$ and C is not hyper-elliptic.

3. The lower bound of K^3 in terms of m_0

In the study of 3-dimensional explicit birational geometry, a challenging problem is to determine whether a given weighted basket $\mathbb{B}$ is geometric, i.e. equal to $\mathbb{B}_X$ for some 3-fold X or not. By exploiting geometric properties, one might be able to have a better estimation of the lower bound of K_X^3 , and hence exclude some non-geometric formal baskets. In fact, in [7, 2.19~2.31], we already proved some effective inequalities for K_X^3 . We shall go further along this direction in this section

Let X be a minimal 3-fold of general type. Assume $P_{m_0}(X) \geq 2$. Mostly we will take $\Lambda = |m_0K_X|$. Keep the settings in 2.1 and 2.2.

3.1. The case $d_{m_0} = 3$.

If we take $|G|$ to be $|S|_S|$, then $\beta = \frac{1}{m_0}$. It is known, from [7, 2.19], that $\deg(K_C) \geq 6$, $\xi \geq \frac{10}{3m_0+2}$ and $K_X^3 \geq \frac{\xi}{m_0^2}$. Take $m = 5m_0 + 4, \dots, (2t+1)m_0 + 2t$, successively. Then, by (2.3), one has $\xi \geq \frac{17}{5m_0+4}, \frac{24}{7m_0+6}, \dots, \frac{7t+3}{(2t+1)m_0+2t}$ respectively. Taking the limit, we obtain $\xi \geq \frac{7}{2m_0+2}$. Therefore

$$K_X^3 \geq \frac{7}{2m_0^2(m_0+1)}. \quad (3.1)$$

In fact, for each small m_0 , the explicit lower bound of K^3 can be slightly improved by the same trick and here is the result:

Table A1

$m_0 =$	2	3	4	5	6	7	8
$\xi \geq$	4/3	1	3/4	5/8	1/2	6/13	2/5
$K^3 \geq$	1/3	1/9	3/64	1/40	1/72	6/637	1/160
$m_0 =$	9	10	11	12	13	14	15
$\xi \geq$	4/11	1/3	3/10	5/18	1/4	6/25	2/9
$K^3 \geq$	4/891	1/300	3/1210	5/2592	1/696	3/2450	2/2025

3.2. The case $d_{m_0} = 2$.

If we take $|G| = |S|_S$, then $\beta \geq \frac{P_{m_0}-2}{m_0}$. By Inequality (2.3), one has $\xi \geq \frac{2}{2m_0+1}$. Take $m = 3m_0+2, 5m_0+4, \dots, (2t+1)m_0+2t$ successively. One gets from Inequality (2.3) that $\xi \geq \frac{4}{3m_0+2}, \frac{7}{5m_0+4}, \dots, \frac{3t+1}{(2t+1)m_0+2t}$. Taking the limit, we have $\xi \geq \frac{3}{2m_0+2}$. By Inequality (2.1), we have

$$K_X^3 \geq \frac{3(P_{m_0}-2)}{2m_0^2(m_0+1)} \geq \frac{3}{2m_0^2(m_0+1)}. \quad (3.2)$$

In fact, we have the following estimation for each small m_0 , which slightly improves [7, Table A]:

Table A2

$m_0 =$	2	3	4	5	6	7	8
$\xi \geq$	1/2	2/5	1/3	1/4	2/9	1/5	1/6
$K^3 \geq$	1/8	2/45	1/48	1/100	1/162	1/245	1/384
$m_0 =$	9	10	11	12	13	14	15
$\xi \geq$	2/13	1/7	1/8	2/17	1/9	1/10	2/21
$K^3 \geq$	2/1053	1/700	1/968	1/1224	1/1521	1/1960	2/4725

Under the same situation, if there exists a number $m_1 > 0$ such that $d_{m_1} = 3$, then, since $(m_1\pi^*(K_X)|_F \cdot C) \geq 2$, we have $\xi \geq \frac{2}{m_1}$. Thus Inequality (2.1) reads:

$$K_X^3 \geq \frac{2(P_{m_0}-2)}{m_0^2 m_1} \geq \frac{2}{m_0^2 m_1}. \quad (3.3)$$

3.3. The case $d_{m_0} = 1, b = g(\Gamma) > 0$.

We have $S = F$ by definition. Pick a very large number $l > 0$. Take $|G| := |l\sigma^*(K_{F_0})|$ which is base point free by the surface theory. By definition, we have $\theta \geq P_{m_0} \geq 2$. Since $\pi^*(K_X)|_F \sim \sigma^*(K_{F_0})$ by Lemma 2.1(i), we see $\beta = \frac{1}{l}$ and thus Inequality (2.1) implies

$$K_X^3 \geq \frac{P_{m_0}}{m_0} \cdot \frac{1}{l} \cdot lK_{F_0}^2 \geq \frac{P_{m_0}}{m_0}. \quad (3.4)$$

3.4. The case $d_{m_0} = 1, b = 0$.

By Lemma 2.1(ii), we have

$$K_X^3 \geq \frac{\theta}{m_0} \pi^*(K_X)|_F^2 \geq \frac{\theta^3}{m_0(m_0+\theta)^2} \cdot K_{F_0}^2. \quad (3.5)$$

We will choose suitable linear system $|G|$ on F depending on the numerical type of F . From the surface theory, we know that either $K_{F_0}^2 \geq 2$ or $(K_{F_0}^2, p_g(F)) = (1, 2), (1, 1), (1, 0)$.

Subcase 3.4.1. $K_{F_0}^2 \geq 2$.

Inequality (3.5) implies

$$K_X^3 \geq \frac{2\theta^3}{m_0(m_0 + \theta)^2}. \quad (3.6)$$

Subcase 3.4.2. $(K_{F_0}^2, p_g(F_0)) = (1, 2)$.

Take $|G| := \text{Mov}|K_F|$. Then C , as a generic irreducible element of $|G|$, is a smooth curve of genus 2 (see [1]). By Lemma 2.1(ii), we have $\beta = \frac{\theta}{m_0 + \theta} \geq \frac{1}{m_0 + 1}$.

Inequality (2.2) implies $\xi \geq \frac{\theta}{m_0 + \theta}$. Take $m = \lfloor \frac{3m_0 + 3\theta}{\theta} \rfloor + 1 > \frac{3m_0 + 3\theta}{\theta}$. Then, since $\alpha_m \geq (m - 1 - \frac{m_0}{\theta} - \frac{1}{\beta})\xi > 1$, Inequality (2.3) gives $\xi \geq \frac{4}{\lfloor \frac{3m_0 + 3\theta}{\theta} \rfloor + 1} \geq \frac{4\theta}{3m_0 + 4\theta}$. Inductively, take $m = \lfloor \frac{(1 + \frac{2}{3}(4^t - 1))m_0 + 3 \cdot 4^{t-1}\theta}{4^{t-1}\theta} \rfloor + 1$, one gets $\xi \geq \frac{4^t\theta}{(1 + \frac{2}{3}(4^t - 1))m_0 + 4^t\theta}$ and hence $\xi \geq \frac{3\theta}{2m_0 + 3\theta}$ by taking the limit. Thus we have

$$K_X^3 \geq \frac{3\theta^3}{m_0(m_0 + \theta)(2m_0 + 3\theta)} \geq \frac{3}{m_0(m_0 + 1)(2m_0 + 3)}. \quad (3.7)$$

A similar calculation leads to the following better estimation for smaller m_0 :

Table A3

$m_0 =$	2	3	4	5	6	7	8
$\xi \geq$	1/2	1/3	2/7	1/4	1/5	2/11	1/6
$K^3 \geq$	1/12	1/36	1/70	1/120	1/210	1/308	1/432
$m_0 =$	9	10	11	12	13	14	15
$\xi \geq$	1/7	2/15	1/8	1/9	2/19	1/10	1/11
$K^3 \geq$	1/630	1/825	1/1056	1/1404	1/1729	1/2100	1/2640

Subcase 3.4.3. $(K_{F_0}^2, p_g(F_0)) = (1, 1)$.

Since $|\sigma^*(K_{F_0})|$ is not moving, we have to take $|G| := |2\sigma^*(K_{F_0})|$ which is base point free by the surface theory. Naturally the generic irreducible element C of $|G|$ is even and $\deg(K_C) = 6$.

By Lemma 2.1(ii), we have $\beta = \frac{\theta}{2m_0 + 2\theta}$. Take $m = \lfloor \frac{3m_0 + 3\theta}{\theta} \rfloor + 1$. Since $\xi > 0$, we have $\alpha_m > 0$. Thus Lemma 2.2 implies $\xi \geq \frac{8\theta}{3m_0 + 4\theta}$. Thus Inequality (2.1) reads

$$K_X^3 \geq \frac{4\theta^3}{m_0(m_0 + \theta)(3m_0 + 4\theta)}. \quad (3.8)$$

For each small m_0 , we have the following better estimation:

Table A4

$m_0 =$	2	3	4	5	6	7	8
$\xi \geq$	6/7	2/3	1/2	4/9	3/8	1/3	2/7
$K^3 \geq$	1/14	1/36	1/80	1/135	1/224	1/336	1/504
$m_0 =$	9	10	11	12	13	14	15
$\xi \geq$	4/15	6/25	2/9	1/5	4/21	14/79	1/6
$K^3 \geq$	1/675	3/2750	1/1188	1/1560	1/1911	1/2370	1/2880

Subcase 3.4.4. $(K_{F_0}^2, p_g(F_0)) = (1, 0)$.

Modulo further birational modification, we may assume that $\text{Mov}|2K_F|$ is base point free. Take $|G| = \text{Mov}|2K_F|$. By Catanese-Pignatelli [4], the generic irreducible element C of $|G|$ is a smooth curve of genus ≥ 3 . By Lemma 2.1(ii), we have $\beta = \frac{\theta}{2m_0+2\theta} \geq \frac{1}{2m_0+2}$. Lemma 2.4 implies $\xi \geq \frac{\theta}{m_0+\theta} \cdot (\sigma^*(K_{F_0}) \cdot C) \geq \frac{2\theta}{m_0+\theta}$. Thus we have

$$K_X^3 \geq \frac{\theta^3}{m_0(m_0 + \theta)^2}. \quad (3.9)$$

Of course, for each small m_0 , one might get slightly better estimation for ξ and K_X^3 .

Variant 3.4.5. If there exists a positive integer m_1 such that $P_{m_1} \geq 2$ and that $|m_0K_{X'}|$ and $|m_1K_{X'}|$ are not composed with the same pencil. We may take $|G| = |M_{m_1}|_F$ and then we have $\beta = \frac{1}{m_1}$. Thus Inequality (2.1) and Lemma 2.4 imply

$$K_X^3 \geq \frac{2\theta_{m_0}^2}{m_0m_1(m_0 + \theta_{m_0})}, \quad (3.10)$$

provided that $(K_{F_0}^2, p_g(F_0)) \neq (1, 2)$.

3.5. Some other inequalities.

Corollary 3.1. *Let X be a minimal 3-fold of general type. Assume $P_{m_0} = 2$. Keep the same notation as above. Suppose that the general fiber F of the induced fibration from Φ_{m_0} is not a $(1, 2)$ surface, and that $P_{m_1} \geq 2$ for some integer $m_1 > 0$. Then*

$$K_X^3 \geq \min\left\{\frac{(P_{m_1} - 1)^3}{m_1(m_1 + P_{m_1} - 1)^2}, \frac{2}{m_0m_1(m_0 + 1)}\right\}.$$

Proof. If $|m_0K_{X'}|$, $|m_1K_{X'}|$ are composed with the same pencil, then both $|m_0K_{X'}|$ and $|m_1K_{X'}|$ induce the same fibration $f : X' \rightarrow \Gamma$. Consider $\tilde{\Lambda} = |m_1K_{X'}|$. Then, $\theta_{m_1} \geq P_{m_1} - 1$. Since F is not a $(1, 2)$ surface and by comparing Inequality 3.4, 3.6, 3.8 and 3.9, we have

$$K_X^3 \geq \frac{(P_{m_1} - 1)^3}{m_1(m_1 + P_{m_1} - 1)^2}.$$

Suppose that $|m_0K_{X'}|$, $|m_1K_{X'}|$ are not composed with the same pencil. We have $\beta = \frac{1}{m_1}$. Then we have Inequality (3.10) as in Variant 3.4.5. $\square$

Now we are able to study the more restricted case:

Proposition 3.2. *Let X be a minimal 3-fold of general type. Assume that $P_{m_0}(X) \geq 4$ and $d_{m_0} = 2$, then*

$$K_X^3 \geq \min \left\{ \frac{8}{m_0(m_0 + 2)^2}, \frac{6}{m_0^2(m_0 + 2)} \right\}.$$

Proof. We need to study the image surface W' of X' through the morphism $\Phi_{|m_0 K_{X'}|}$. In fact, we have the Stein factorization

$$\Phi_{m_0} := \Phi_{|m_0 K_{X'}|} : X' \xrightarrow{f} \Gamma \xrightarrow{s} W' \subset \mathbb{P}^{P_{m_0}-1}.$$

Denote by H' a very ample divisor on W' such that $M_{m_0} \sim \Phi_{m_0}^*(H')$. Furthermore one has $M_{m_0}|_S \equiv \tilde{a}_{m_0}C$ for a general member $S \in |M_{m_0}|$ and the integer $\tilde{a}_{m_0} \geq \deg(s) \deg(W') \geq \deg(W') \geq P_{m_0} - 2$, where C is a general fiber of f . Set $|G| := |M_{m_0}|_S|$.

Case 1. $\tilde{a}_{m_0} \geq 3$.

We have $\beta \geq \frac{3}{m_0}$. Inequality (2.2) implies $\xi \geq \frac{6}{4m_0+3}$. Take $m = 2m_0 + 2$. Then Inequality (2.3) gives $\xi \geq \frac{2}{m_0+1}$. Take $m = \lfloor \frac{11m_0+9}{6} \rfloor + 1$. Since $\alpha_m > (\frac{11m_0+9}{6} - 1 - m_0 - \frac{1}{\beta})\xi \geq 1$, Inequality (2.3) implies $\xi \geq \frac{24}{11m_0+15}$. Thus, we have

$$K_X^3 \geq \frac{72}{m_0^2(11m_0 + 15)}. \quad (3.11)$$

Case 2. $\tilde{a}_{m_0} = 2$.

Automatically we have $P_{m_0} = 4$, which also implies that $\deg(W') = 2$ and $\deg(s) = 1$. Recall that an irreducible surface (in $\mathbb{P}^3$) of degree 2 is one of the following surfaces (see, for instance, Reid [28, p. 30, Ex. 19]):

- (a) W' is the cone $\overline{\mathbb{F}}_2$ obtained by blowing down the unique section with the self-intersection (-2) on the Hirzebruch ruled surface $\mathbb{F}_2$;
- (b) $W' \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Case 2.a. $W' = \overline{\mathbb{F}}_2$.

Replacing by its birational model, we may assume that Φ_{m_0} factors through the minimal resolution $\mathbb{F}_2$ of W' . So we have the factorization of $\Phi_{m_0} : X' \xrightarrow{h} \mathbb{F}_2 \xrightarrow{\nu} W'$ where h is a fibration and ν is the minimal resolution of W' . Set $\hat{H} = \nu^*(H')$. We know that $H'^2 = 2$ and hence $\hat{H}^2 = 2$. Noting that $\hat{H}$ is nef and big on $\mathbb{F}_2$, we can write

$$\hat{H} \sim \mu G_0 + nT$$

where μ and n are integers, G_0 denotes the unique section with $G_0^2 = -2$, and T is the general fiber of the ruling on $\mathbb{F}_2$. The property of $\hat{H}$ being nef and big implies that $\mu > 0$ and $n \geq 2\mu \geq 2$. Now let

$pr : \mathbb{F}_2 \longrightarrow \mathbb{P}^1$ be the ruling. Set $\tilde{f} := pr \circ h : X' \longrightarrow \mathbb{P}^1$, which is a fibration with connected fibers. Denote by F a general fiber of $\tilde{f}$. We have

$$M_{m_0} \sim \Phi_{m_0}^*(H') = h^*(\hat{H}) \geq 2F.$$

Let $\Lambda = |2F| \preceq |m_0 K_{X'}|$. Clearly we have $\theta_\Lambda = 2$, $d_\Lambda = 1$ and $b = 0$. By Inequalities (3.6), (3.7), (3.8) and (3.9), we have

$$K_X^3 \geq \frac{8}{m_0(m_0 + 2)^2}. \quad (3.12)$$

Case 2.b. $W' = \mathbb{P}^1 \times \mathbb{P}^1$.

We have an induced fibration $f : X' \longrightarrow W' = \mathbb{P}^1 \times \mathbb{P}^1$. Since a very ample divisor H' on W' with $H'^2 = 2$ is linearly equivalent to $L_1 + L_2 = q_1^*(\text{point}) + q_2^*(\text{point})$ where q_1, q_2 are projections from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^1$ respectively. Set $\tilde{f}_i := q_i \circ f : X' \longrightarrow \mathbb{P}^1$, $i = 1, 2$. Then $\tilde{f}_1$ and $\tilde{f}_2$ are two fibrations onto $\mathbb{P}^1$. Let F_1 and F_2 be general fibers of $\tilde{f}_1$ and $\tilde{f}_2$, respectively. Then $F_1 \cap F_2$ is simply a general fiber C of f . We will estimate ξ in an alternative way. In fact, the following argument is similar to the proof of [13, Theorem 3.1].

Since $\tilde{a}_{m_0} = 2$, we have $S|_S \sim 2C$. On the other hand, we have $S \geq F_1 + F_2$. Modulo further birational modifications, we may write $m_0 \pi^*(K_X) \equiv F_1 + F_2 + H'_{m_0}$ where H'_{m_0} is an effective $\mathbb{Q}$ -divisor with simple normal crossing supports. For any integer $m > m_0 + 1$, we consider the linear system

$$|K_{X'} + \lceil (m - m_0 - 1)\pi^*(K_X) \rceil + F_1 + F_2| \preceq |mK_{X'}|.$$

Since $(m - m_0 - 1)\pi^*(K_X) + F_2$ is nef and big, Kawamata-Viehweg vanishing ([19, 33]) gives the surjective map:

$$\begin{aligned} & H^0(K_{X'} + \lceil (m - m_0 - 1)\pi^*(K_X) \rceil + F_2 + F_1) \\ \longrightarrow & H^0(F_1, K_{F_1} + \lceil (m - m_0 - 1)\pi^*(K_X) \rceil|_{F_1} + C). \end{aligned}$$

Using the vanishing theorem again, one gets the surjective map:

$$H^0(F_1, K_{F_1} + \lceil (m - m_0 - 1)\pi^*(K_X) \rceil|_{F_1} + C) \longrightarrow H^0(C, K_C + \hat{D}_m)$$

where $\hat{D}_m := \lceil (m - m_0 - 1)\pi^*(K_X) \rceil|_C$ with

$$\deg(\hat{D}_m) \geq (m - m_0 - 1)\xi.$$

When m is large enough so that $\deg(\hat{D}_m) \geq 2$, the above two surjective maps directly implies

$$m\xi \geq \deg(K_C) + \deg(\hat{D}_m) \geq 2 + \lceil (m - m_0 - 1)\xi \rceil. \quad (3.13)$$

In particular, we have $\xi \geq \frac{2}{m_0+1}$.

Take $m = 2m_0 + 3$. Then $(m - m_0 - 1)\xi > 2$ and Inequality (3.13) gives $\xi \geq \frac{5}{2m_0+3}$.

Assume $m_0 > 1$ and take $m = 2m_0 + 2$. One gets $\xi \geq \frac{5}{2m_0+2}$. Take $m = \lfloor \frac{7m_0+12}{5} \rfloor = \lfloor \frac{7m_0+7}{5} \rfloor + 1 > \frac{7m_0+7}{5}$, one has $\xi \geq \frac{4}{m} \geq \frac{20}{7m_0+12}$.

Inductively, take $m = \lfloor \frac{(2+\frac{5}{3}(4^t-1))m_0+2+\frac{10}{3}(4^t-1)}{5 \cdot 4^{t-1}} \rfloor$ for $t \geq 1$, one has $\xi \geq \frac{5 \cdot 4^t}{(2+\frac{5}{3}(4^t-1))m_0+2+\frac{10}{3}(4^t-1)}$. We have $\xi \geq \frac{3}{m_0+2}$ by taking the limit and hence

$$K_X^3 \geq \frac{1}{m_0} \cdot (\pi^*(K_X)|_S)^2 \geq \frac{2}{m_0^2} \cdot \xi \geq \frac{6}{m_0^2(m_0+2)}. \quad (3.14)$$

We conclude the statement by comparing 3.11, 3.12 and 3.14. $\square$

Corollary 3.3. *Let X be a minimal 3-fold of general type. The following holds:*

$$K_X^3 \geq \begin{cases} \min \left\{ \frac{8}{m_0(m_0+2)^2}, \frac{7}{2m_0^2(m_0+1)} \right\}, & \text{when } P_{m_0} \geq 4; \\ \frac{3}{2m_0^2(m_0+1)}, & \text{when } P_{m_0} = 3. \end{cases}$$

Proof. When $P_{m_0} \geq 4$, $d_{m_0} = 3, 2, 1$ and the inequality follows from comparing Inequality (3.1), Proposition 3.2, Inequalities (3.4, 3.6, 3.7, 3.8, 3.9) (with $\theta_{m_0} = 3$), respectively.

When $P_{m_0} = 3$, $d_{m_0} = 2, 1$ and the inequality follows immediately by comparing Inequality (3.2) with Inequalities (3.4, 3.6, 3.7, 3.8, 3.9) (with $\theta_{m_0} = 2$). $\square$

4. Threefolds with $\delta(V) \leq 12$

The purpose of this section is to prove the following sharper bounds:

Theorem 4.1. *Let X be a minimal projective 3-fold of general type with $2 \leq \delta(X) \leq 12$. Then $K_X^3 \geq v(\delta(X))$, where the function $v(x)$ is defined as follows:*

x	2	3	4	5	6	7
$v(x)$	1/14	1/36	1/90	1/135	1/224	1/336
x	8	9	10	11	12	—
$v(x)$	1/504	1/675	3/2750	1/1188	1/1560	—

We are going to estimate the lower bound of the volume, case by case, for a given δ . The discussion here relies on those formulae in [6, (3.6)-(3.12)].

Proposition 4.2. *If $P_2(X) \geq 2$, then $K_X^3 \geq \frac{1}{14}$.*

Proof. Set $m_0 = 2$. By Table A1, Table A2, Inequalities (3.4) and (3.6), Table A3, Table A4 and Corollary 3.3, we have $K_X^3 \geq \frac{1}{14}$ unless $P_2 = 2$, $d_2 = 1$, $b = 0$ and F is of type $(1, 0)$.

In the remaining case, we have that $\chi(\mathcal{O}_X) = 1$ by [7, Lemma 2.32]. By [7, Lemma 3.2], one has $P_4 \geq 2P_2 \geq 4$. If $d_4 \geq 2$, then $K_X^3 \geq \frac{1}{12}$ by Inequality (3.10) (with $m_0 = 2$, $m_1 = 4$, $\theta_2 = 1$). If $d_4 = 1$, then $|2K_{X'}|$ and $|4K_{X'}|$ are composed with the same pencil. Thus we have $K_X^3 \geq \frac{27}{196} > \frac{1}{8}$ by Inequality (3.9) (with $m_0 = 4$, $\theta_4 = 3$). $\square$

Proposition 4.3. *If $P_3(X) \geq 2$, then $K_X^3 \geq \frac{1}{36}$.*

Proof. Take $m_0 = 3$ and $\Lambda = |3K_{X'}|$. One has $K_X^3 \geq \frac{1}{36}$ by Table A1, Table A2, Inequalities (3.4), (3.6), Table A3, Table A4 and Corollary 3.3 ($m_0 = 3$) unless we are in Subcase 3.4.4 with $P_3 = 2$. That is, $P_3 = 2, d_3 = 1, b = 0$ and F is of type $(1, 0)$. Again, $\chi(\mathcal{O}_X) = 1$. Thus, for any $m \geq 2$, [7, Lemma 3.2] implies $P_{m+2} \geq P_m + P_2$.

By Corollary 3.1, if $P_4 \geq 3$ (resp. $P_5 \geq 3$), then $K_X^3 \geq \frac{1}{24}$ (resp. $\frac{1}{30}$). Suppose that both $P_4 \leq 2$ and $P_5 \leq 2$, then $P_5 = 2$ and $P_2 = 0$. By [6, (3.6)], $n_{1,2}^0 = 5 - 8 + P_4 < 0$, which is a contradiction. Hence either P_4 or $P_5 \geq 3$ in this case and we are done. $\square$

Proposition 4.4. *If $P_4(X) \geq 2$, then $K_X^3 \geq \frac{1}{90}$.*

Proof. Similarly, we have $K_X^3 \geq \frac{1}{80}$ unless $P_4 = 2, b = 0$ and F is of $(1, 0)$ type. In fact, in this situation, we have at least $K_X^3 \geq \frac{1}{100}$ by Inequality (3.9). We will go a little bit further to investigate this situation.

0. We may and do assume that $P_2 \leq 1$ and $P_3 \leq 1$.

1. If $P_7 \geq 3$ (resp. $P_6 \geq 3, P_5 \geq 3$), then $K^3 \geq \frac{8}{567} > \frac{1}{80}$ (resp. $\frac{1}{60}, \frac{1}{50}$) by Corollary 3.1 (with $m_0 = 4$, and $m_1 = 7, 6, 5$ respectively). So we may assume $P_5, P_6, P_7 \leq 2$. Since $P_6 \geq P_4 + P_2$, we see that $P_2 = 0$ and $P_6 = P_4 = 2$.

2. If $P_3 = 0$, then $n_{1,3}^0 = P_5 - 2 \geq 0$ implies $P_5 = 2$. Now $n_{1,4}^5 = 3 - \sigma_5 \geq 0$ gives $\sigma_5 \leq 3$. However $n_{1,3}^5 \geq 0$ implies $\sigma_5 \geq 4$, a contradiction. We thus assume that $P_3 = 1$ from now on.

3. We thus can make the following complete table for $B^{(5)}$ depending on P_5, σ_5 :

No.	P_5	σ_5	$B^{(5)}$	K^3	$\epsilon + P_7$
1	1	0	$\{2 \times (1, 2), (2, 5), 5 \times (1, 4)\}$	$1/20$	4
2	1	1	$\{3 \times (1, 2), (1, 3), 4 \times (1, 4), (1, r)\}$	$1/r - 1/6$	4
3	2	1	$\{(1, 2), 2 \times (2, 5), 3 \times (1, 4), (1, r)\}$	$1/r - 3/20$	5
4	2	2	$\{2 \times (1, 2), (2, 5), (1, 3), 2 \times (1, 4), (1, r_1), (1, r_2)\}$	$1/r_1 + 1/r_2 - 11/30$	5
5	2	3	$\{3 \times (1, 2), 2 \times (1, 3), (1, 4), (1, r_1), (1, r_2), (1, r_3)\}$	$1/r_1 + r_2 + r_3 - 7/12$	5

4. By definition, one has $\sigma_5 \leq \epsilon \leq 2\sigma_5$. Note that No. 1 is impossible because $\epsilon = 0$ but $P_7 \leq 2$ implies that $\epsilon \geq 2$, a contradiction. In No. 3, $P_5 = 2$ implies $P_7 = 2$ and hence $\epsilon = 3 > 2\sigma_5$, a contradiction.

In No. 2, one must have $P_7 = 2$ and $\epsilon = 2 = 2\sigma_5$. Hence $r \geq 6$. Then it follows that $K^3 \leq K^3(B^{(5)}) \leq 0$, a contradiction. Similarly, in No. 4, $K^3(B^{(5)}) > 0$ only when $r_1 = r_2 = 5$. But then $\epsilon = 2$, a contradiction.

5. It remains to consider No. 5. Note that $K^3(B^{(5)}) > 0$ only when $r_1 = r_2 = r_3 = 5$ and $K^3(B^{(5)}) = \frac{1}{60}$. There are only finitely many possible packings. Among them, we search for baskets with $K^3 \geq \frac{1}{100}$. It turns out there is only one new baskets

$$B_{90} = \{3 \times (1, 2), 2 \times (1, 3), (2, 9), 2 \times (1, 5)\}$$

with $K^3(B_{90}) = \frac{1}{90}$. $\square$

Proposition 4.5. *If $P_5 \geq 2$, then $K_X^3 \geq \frac{1}{135}$.*

Proof. Similarly, we have $K_X^3 \geq \frac{1}{135}$ unless $P_5 = 2$, $b = 0$ and F a $(1, 0)$ surface, for which we have $K_X^3 \geq \frac{1}{180}$. Furthermore, we may assume that $P_m \leq 2$ for $m = 6, 7, 8$ by Corollary 3.1. It suffices to consider: $\chi(\mathcal{O}_X) = 1$, $P_2 = 0$, $P_3 = 0, 1$, $P_4 = 0, 1$, $P_5 = P_7 = 2$ and $P_4 \leq P_6 \leq P_8 \leq 2$.

We look at $B^{(5)}$ with $K^3 > 0$ according to (P_3, P_4, P_6) and σ_5 . It turns out that there is only one,

$$B^{(5)} = \{2 \times (2, 5), 3 \times (1, 3), (1, 4), (1, 6)\}$$

with $K^3(B^{(5)}) = \frac{1}{60}$, given by $(P_3, P_4, P_6) = (1, 1, 2)$ and $\sigma_5 = 2$. Now $P_8 = 2$ and hence

$$B^{(7)} = \{2 \times (2, 5), 2 \times (1, 3), (2, 7), (1, 6)\}.$$

However $K^3(B^{(7)}) = \frac{1}{210} < \frac{1}{180}$, which is impossible. $\square$

Proposition 4.6. *If $P_6 \geq 2$, then $K_X^3 \geq \frac{1}{224}$.*

Proof. Similarly, we have $K_X^3 \geq \frac{1}{224}$ unless $P_6 = 2$, $b = 0$ and F a $(1, 0)$ surface, for which we have $K_X^3 \geq \frac{1}{294}$. Again, we may assume that $P_m \leq 2$ for $m = 7, 8, 9, 10$. Therefore, it remains to consider such a situation that $\chi(\mathcal{O}_X) = 1$, $P_2 = 0$, $P_4 \leq 1$, $P_3 \leq P_5 \leq 1$, $P_7 \leq P_9 \leq 2$ and $P_8 = P_{10} = 2$. According to the value of (P_3, P_4, P_5) and σ_5 , we have the following table.

No.	(P_3, P_4, P_5)	σ_5	$B^{(5)}$	K^3	$\epsilon + P_7$
1	(0,0,0)	0	$\{5 \times (1, 2), 4 \times (1, 3), (1, 4)\}$	$1/12$	2
2	(0,0,1)	0	$\{3 \times (1, 2), 2 * (2, 5), 3 * (1, 3)\}$	$1/10$	3
3	(0,1,0)	0	$\{6 * (1, 2), (1, 3), 3 * (1, 4)\}$	$1/12$	3
4	(0,1,1)	0	$\{4 * (1, 2), 2 * (2, 5), 2 * (1, 4)\}$	$1/10$	4
5	(0,1,1)	1	$\{5 * (1, 2), 1 * (2, 5), (1, 3), (1, 4), (1, r)\}$	$1/r - 7/60$	4
6	(0,1,1)	2	$\{6 * (1, 2), 2 * (1, 3), (1, r_1), (1, r_2)\}$	$1/r_1 + 1/r_2 - 1/3$	4
7	(1,0,1)	0	$\{(2, 5), 6 * (1, 3), (1, 4)\}$	$1/20$	2
8	(1,0,1)	1	$\{(1, 2), 7 * (1, 3), (1, r)\}$	$1/r - 1/6$	2
9	(1,1,1)	0	$\{(1, 2), (2, 5), 3 * (1, 3), 3 * (1, 4)\}$	$1/20$	3
10	(1,1,1)	1	$\{2 * (1, 2), 4 * (1, 3), 2 * (1, 4), (1, r)\}$	$1/r - 1/6$	3

1. It is clear that No. 2, 3, 4, 9 are not allowed for $\epsilon = 0$ and hence $P_7 \geq 3$.

2. In No. 1, 7, the baskets allow at most one packing at level 7, i.e., $\epsilon_7 \leq 1$. However, $P_7 = 2$ and $P_8 = 2$ yield $\epsilon_7 \geq 2$, a contradiction.

3. Consider No. 10. Since $K^3 = \frac{1}{r} - \frac{1}{6} > 0$, it follows that $r = 5$. So $\epsilon = 1$ and $P_7 = 2$. Then $\epsilon_7 = 2$ and

$$B^{(7)} = \{2 \times (1, 2), 2 \times (1, 3), 2 \times (2, 7), (1, 5)\}.$$

This already implies $\epsilon_8 = 0$ and so we get $P_9 = 3$, a contradiction.

4. Consider No. 8. Since $K^3 > 0$, thus we get

$$B^{(5)} = \{(1, 2), 7 \times (1, 3), (1, 5)\}.$$

Since $B^{(5)}$ allows no further packing, hence $K_X^3 = \frac{1}{30}$ in this case.

5. Consider No. 5. Since $K^3 > 0$, $r = 6, 7, 8$. It is easy to see that the basket with the smallest volume and dominated by $B^{(5)}$ is

$$B_{210} = \{(7, 15), (2, 7), (1, 6)\}$$

with $K^3 = \frac{1}{210}$. Thus $K_X^3 \geq \frac{1}{210}$.

6. Finally Consider No. 6. Since $K^3 > 0$, $(r_1, r_2) = (5, 5), (5, 6), (5, 7)$. It is easy to see that the basket with the smallest volume and dominated by $B^{(5)}$ is

$$B_{105} = \{6 \times (1, 2), 2 \times (1, 3), (1, 5), (1, 7)\}$$

with $K^3 = \frac{1}{105}$. Thus $K_X^3 \geq \frac{1}{105}$. $\square$

Note that, when $\delta(X) \geq 7$, we can utilize our explicit classification in [7, Section 3]. We shall omit some details to avoid unnecessary redundancy.

Proposition 4.7. *If $P_7 \geq 2$, then $K_X^3 \geq \frac{1}{336}$.*

Proof. Similarly, we have $K_X^3 \geq \frac{1}{336}$ unless $P_7 = 2$, $b = 0$, F a $(1, 0)$ surface and $\chi(\mathcal{O}_X) = 1$. Again, we may assume that $P_m \leq 2$ for $m = 8, 9$. Hence $P_9 = 2$ and $P_2 = 0$.

By $\epsilon_6 = 0$, we have $P_4 + P_5 + P_6 = P_3 + 2 + \epsilon$. Hence $(P_3, P_4, P_5, P_6) = (0, 0, 1, 1), (0, 1, 0, 1), (0, 1, 1, 1)$ or $(1, 1, 1, 1)$ which corresponds to Cases IV, V, VI, and VIII in [7, Section 3] respectively. The classification implies that, if $K_X^3 < \frac{1}{336}$, then $B_X \succeq B_{\min}$, where $B_{\min}$ is a minimal positive basket and belongs to one of the following:

- (b1) $B_{6,4} = \{(1, 2), (6, 13), (1, 3), 2 \times (1, 5)\}$ with $K^3(B_{6,4}) = \frac{1}{390}$ and $P_9(B_{6,4}) = 3$;
- (b2) $B_{6,6} = \{3 \times (1, 2), (3, 7), (2, 5), (1, 4), (1, 6)\}$ with $K^3(B_{6,6}) = \frac{1}{420}$ and $P_9(B_{6,4}) = 3$;
- (b3) $B_{8,3} = \{2 \times (2, 5), (1, 3), (3, 11), (1, 4)\}$ with $K^3(B_{8,3}) = \frac{1}{660}$.

Clearly, Case b1 can not happen because $P_9(B_X) \geq P_9(B_{\min}) = 3$.

In the Case b2, for the similar reason, $B_X \neq B_{6,6}$. Thus $B_X \succeq B_{60} := \{4 \times (1, 2), 2 \times (2, 5), (1, 4), (1, 6)\}$ and so $K_X^3 \geq K^3(B_{60}) = \frac{1}{60}$.

Finally, in Case b3, the proof of [7, Theorem 3.11] implies that $B_X \neq B_{8,3}$ and $B_X \succeq B_{210} = \{2 \times (2, 5), (1, 3), (2, 7), 2 \times (1, 4)\}$ with $K_X^3 \geq K^3(B_{210}) = \frac{1}{210}$. We have proved the statement. $\square$

It is now immediately to see the following consequences:

Corollary 4.8. *(=Corollary 1.5) Let X be a minimal projective 3-fold of general type with $K_X^3 < \frac{1}{336}$. Then $\delta(X) \geq 8$.*

Proposition 4.9. *Let X be a minimal projective 3-fold of general type.*

- (1) *If $P_8 \geq 2$, then $K_X^3 \geq \frac{1}{504}$.*
- (2) *If $P_9 \geq 2$, then $K_X^3 \geq \frac{1}{675}$.*
- (3) *If $P_{10} \geq 2$, then $K_X^3 \geq \frac{3}{2750}$.*

- (4) If $P_{11} \geq 2$, then $K_X^3 \geq \frac{1}{1188}$.
 (5) If $P_{12} \geq 2$, then $K_X^3 \geq \frac{1}{1560}$.

Proof. We only prove (1). Other statements can be proved similarly.

When $P_8 \geq 2$, Table A1, Table A2, Inequalities (3.4), (3.6), Table A3 and Table A4 imply $K_X^3 \geq \frac{1}{504}$ unless we are in Subcase 3.4.4, for which one has $K_X^3 \geq \frac{1}{420}$ by [7, Theorem 1.2(2)] since $\chi(\mathcal{O}_X) = 1$. $\square$

Propositions 4.2, 4.3, 4.4, 4.5, 4.6, 4.7 and 4.9 imply Theorem 4.1.

An interesting by-product is the following:

Corollary 4.10. (=Corollary 1.7(1)) *Let X be a minimal projective 3-fold of general type with $p_g(X) = 1$. Then $K_X^3 \geq \frac{1}{75}$.*

Proof. We distinguish the following cases.

Case 1. $P_4 \geq 3$.

By Corollary 3.3, $K_X^3 \geq \frac{3}{160}$.

Case 2. $P_4 = 2$.

We have $K_X^3 \geq \frac{1}{70}$ by Inequalities (3.4), (3.6) and Table A3 unless $b = 0$ and F is either a $(1, 1)$ or a $(1, 0)$ surface, for which we necessarily have $h^2(\mathcal{O}_X) = 0$ and thus $\chi(\mathcal{O}_X) = 0$. Reid's Riemann-Roch formula implies $P_5 > P_4 = 2$. Now Corollary 3.1 (with $m_0 = 4$, $m_1 = 5$) yields $K_X^3 \geq \frac{1}{50}$.

Case 3. $P_4 = 1$.

Since $p_g(X) = 1$, one has $P_m > 0$ for all $m > 1$. By [6, (3.10)], we have

$$P_4 + P_5 + P_6 = 3P_2 + P_3 + P_7 + \epsilon \geq 3P_2 + P_3 + P_7.$$

If $P_4 = 1$ (which implies $P_3 = P_2 = 1$), then we have

$$P_5 \geq (P_7 - P_6) + 3 \geq 3.$$

Then, from [6, (3.6)], $n_{1,4}^0 \geq 0$ implies $\chi(\mathcal{O}_X) \geq 3$. Due to our previous result [5, Corollary 1.2] for irregular 3-folds, we may assume $q(X) = 0$. Thus we have $h^2(\mathcal{O}_X) = \chi(\mathcal{O}_X) \geq 3$. Take a sub-pencil Λ of $|5K_X|$. Then Λ induces a fibration $f : X' \rightarrow \Gamma$ after Stein factorization. Let F be the general fiber and F_0 be the minimal model of F .

Claim. $K_{F_0}^2 \geq 2$.

Proof. Clearly we may write

$$f_*\omega_{X'} = \mathcal{O}_\Gamma \oplus \mathcal{O}_\Gamma(e_2) \oplus \cdots \oplus \mathcal{O}_\Gamma(e_{p_g(F)-1})$$

with $-2 \leq e_j \leq -1$ for all j , since $p_g(X') = 1$. Note that we have

$$\begin{aligned} h^2(\mathcal{O}_X) &= h^1(f_*\omega_{X'}) + h^0(R^1f_*\omega_{X'}) \\ &\leq (p_g(F) - 1) + h^0(R^1f_*\omega_{X'}). \end{aligned}$$

If $q(F) > 0$, we have $K_{F_0}^2 \geq 2$ by the surface theory. If $q(F) = 0$, we have $R^1f_*\omega_{X'} = 0$ and thus $p_g(F) \geq h^2(\mathcal{O}_X) + 1 \geq 4$. Hence we have $K_{F_0}^2 \geq 4$ by the Noether inequality. $\square$

If $d_5 \geq 2$, then we may set $m_1 = 5$ and apply Inequality (3.10), which gives $K_X^3 \geq \frac{1}{75}$.

If $d_5 = 1$, then $|5K_{X'}|$ and Λ are composed with the same pencil. Thus we have $\theta_5 \geq 2$ and Inequality (3.6) gives $K_X^3 \geq \frac{16}{245}$. $\square$

5. Threefolds with $\delta(V) \geq 13$

Let X be a minimal projective 3-fold of general type with $\delta(X) \geq 13$. Now we are in the natural position to classify baskets $\mathbb{B}(X)$ with $\delta(X) \geq 13$. In fact, we have $\mathbb{B}^{12} \succeq \mathbb{B}(X) \succeq \mathbb{B}_{\min}$ for certain minimal positive basket $\mathbb{B}_{\min}$ listed in [7, Table C], where $\mathbb{B}^{12}$ is also listed there. However, as pointed out in [7, Proposition 4.5], our earlier classification in [7, Table C] is not clean since some minimal baskets in Table C are actually known to be “non-geometric”.

Recall that, by definition, a geometric weighted basket is a basket of a projective threefold of general type. Hence the following properties hold:

- A. $P_m P_n \leq P_{m+n}$ if $P_m = 1$ and $n > 0$.
- B. $P_m \geq 0$ for all $m > 0$.
- C. $K^3 \geq f(m_0)$ for some explicit function $f(x)$ given in Sections 3 and 4 provided that $P_{m_0} \geq 2$.

Indeed, if $\mathbb{B}^{12}$ violates one of A, B, C , then so does $\mathbb{B}(X)$. Therefore $\mathbb{B}(X)$ is non-geometric. If $\mathbb{B}_{\min}$ is non-geometric (e.g. cases No. 3a, 5b, 10a, $\dots$, etc.), then we need to check all baskets between $\mathbb{B}^{12}$ and $\mathbb{B}_{\min}$. The following Table H consists of non-geometric baskets with $\delta \geq 13$. We keep the same notation as in Table C.

Table H

No.	$(P_{12}, \dots, P_{24})$	$(n_{1,2}, n_{4,9}, \dots, n_{1,5})$ or $B_{\min}$	K^3	Offending
3a	(1, 0, 0, 1, 0, 0, 2, 0, 3, 1, 1, 1, 3)	$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{17}{30030}$	$P_8 P_8 > P_{16}$
5b	(1, 0, 1, 2, 0, 0, 3, 0, 2, 1, 2, 2, 3)	$\{(5, 13), (4, 15), *\}$	$\frac{1}{1170}$	$P_8 P_8 > P_{16}$
8	(1, 0, 2, 1, 0, 1, 3, 1, 4, 3, 2, 2, 5)	(7, 1, 0, 1, 0, 2, 0, 0, 6, 0, 2, 0, 0, 1)	$\frac{1}{770}$	$P_6 P_{10} > P_{16}$
9	(1, 0, 2, -1, 1, 0, 2, 0, 1, 2, 1, 0, 2)	(9, 0, 0, 2, 0, 0, 1, 1, 4, 0, 1, 0, 0, 1, 0)	$\frac{1}{5544}$	$P_{15} = -1$

10a	(1, 0, 2, 1, 2, -1, 2, 0, 2, 2, 1, 2, 4)	$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{1680}$	$P_{17} = -1$
11a	(1, 0, 2, 0, 2, 0, 2, 2, 2, 1, 1, 1, 3)	$\{(3, 8), (4, 11), *\} \succ \{(7, 19), *\}$	$\frac{1}{2660}$	$P_8 P_{14} > P_{22}$
13	(1, 0, 3, -1, 1, 1, 3, 1, 3, 3, 3, 1, 4)	(12, 0, 0, 2, 0, 2, 0, 2, 4, 0, 2, 0, 0, 1, 0)	$\frac{1}{3465}$	$P_{15} = -1$
15a	(1, 0, 3, 0, 1, 0, 2, 0, 3, 1, 1, 1, 4)	$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{1}{2520}$	$P_8 P_{14} > P_{22}$
15b	(1, 0, 2, 0, 1, 0, 3, 0, 3, 2, 1, 1, 4)	$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{36036}$	$P_8 P_{14} > P_{22}$
15c	(1, 0, 3, 1, 2, 0, 3, 1, 3, 2, 2, 2, 5)	$\{(7, 16), (7, 19), *\}$	$\frac{1}{31920}$	$P_8 P_{14} > P_{22}$
16c	(1, 0, 2, 1, 1, -1, 3, -1, 2, 2, 1, 1, 3)	$\{(5, 13), (7, 16), *\}$	$\frac{1}{16016}$	$P_{17} = -1$
18a	(1, 0, 3, 0, 1, 0, 2, 1, 2, 2, 2, 1, 3)	$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{1}{3080}$	$P_6 P_{11} > P_{17}$
19	(1, 0, 2, 0, 1, 1, 3, 0, 2, 2, 2, 1, 3)	(8, 0, 1, 1, 0, 1, 0, 1, 5, 0, 1, 0, 0, 1, 0)	$\frac{1}{3465}$	$P_9 P_{14} > P_{23}$
20a	(1, 0, 1, 1, 1, 0, 3, -1, 2, 1, 0, 1, 3)	$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{16380}$	$P_{19} = -1$
21a	(1, 1, 1, 1, 2, 0, 2, 1, 2, 1, 2, 2, 3)	$\{(1, 3), (3, 10), *\} \succ \{(4, 13), *\}$	$\frac{1}{4680}$	$P_8 P_9 > P_{17}$
22	(1, 0, 1, 1, 1, 0, 2, 1, 3, 1, 1, 1, 3)	(7, 1, 0, 1, 0, 1, 1, 0, 5, 1, 0, 0, 1, 0, 1)	$\frac{1}{9240}$	$P_8 P_9 > P_{17}$
23a	(1, 0, 2, 1, 2, 0, 2, 1, 3, 1, 2, 2, 3)	$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{2640}$	$P_8 P_9 > P_{17}$
24	(1, 0, 2, 0, 0, 1, 3, 0, 3, 2, 2, 0, 3)	(10, 1, 0, 1, 0, 3, 0, 1, 6, 0, 2, 0, 0, 1, 0)	$\frac{1}{3465}$	$P_8 P_8 > P_{16}$
26a	(1, 0, 3, 1, 1, 1, 3, 0, 4, 1, 2, 2, 5)	$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{1}{1260}$	$P_9 P_{10} > P_{19}$
27.1	(1, 0, 2, 2, 1, 1, 5, 0, 4, 3, 3, 3, 6)	$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{45045}$	$P_9 P_{10} > P_{19}$
27.2	(1, 0, 2, 2, 1, 1, 5, -1, 3, 2, 2, 2, 4)	$\{(2, 5), (5, 13), *\} \succ \{(7, 18), *\}$	$\frac{1}{1386}$	$P_{19} = -1$
27a	(1, 0, 2, 2, 1, 1, 5, -1, 3, 2, 2, 2, 3)	$\{(2, 5), (7, 18), *\} \succ \{(9, 23), *\}$	$\frac{1}{1386}$	$P_{19} = -1$
27b	(1, 0, 2, 2, 1, 1, 5, -1, 3, 2, 2, 2, 5)	$\{(5, 13), (5, 18), *\}$	$\frac{1}{1170}$	$P_{19} = -1$
29a	(1, 1, 3, 1, 2, 2, 2, 1, 3, 1, 2, 2, 3)	$\{(5, 14), (1, 3), *\} \succ \{(6, 17), *\}$	$\frac{1}{5335}$	$P_9 P_{14} > P_{23}$
32b	(1, 0, 3, 1, 1, 1, 3, 1, 3, 2, 3, 2, 4)	$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{1}{1386}$	$P_9 P_{14} > P_{23}$
33a	(1, 1, 2, 0, 2, 1, 1, 1, 2, 2, 1, 2, 3)	$\{(3, 10), (2, 7), *\} \succ \{(5, 17), *\}$	$\frac{1}{2856}$	$P_6 P_{16} > P_{22}$
34b	(1, 1, 2, 0, 1, 1, 3, 0, 3, 3, 1, 2, 4)	$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{1170}$	$P_6 P_{13} > P_{19}$
39a	(1, 1, 2, 1, 3, 0, 2, 1, 3, 2, 2, 3, 4)	$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{1680}$	$P_6 P_{16} > P_{22}$
39b	(1, 1, 2, 1, 3, 1, 2, 1, 3, 2, 2, 3, 5)	$\{(3, 10), (2, 7), *\} \succ \{(5, 17), *\}$	$\frac{1}{5355}$	$P_6 P_{16} > P_{22}$
40.1	(1, 1, 2, 1, 2, 1, 4, 0, 4, 3, 2, 3, 6)	$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{32760}$	$P_6 P_{13} > P_{19}$
40a	(1, 1, 2, 1, 2, 1, 4, -1, 3, 2, 1, 2, 4)	$\{(4, 10), (3, 8), *\} \succ \{(7, 18), *\}$	$\frac{1}{2520}$	$P_6 P_{13} > P_{19}$
40b	(1, 1, 2, 1, 2, 1, 4, 0, 4, 3, 1, 2, 5)	$\{(2, 5), (6, 16), *\} \succ \{(8, 21), *\}$	$\frac{1}{1260}$	$P_6 P_{13} > P_{19}$
43a	(1, 1, 3, 0, 2, 1, 2, 1, 3, 2, 2, 2, 4)	$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{1}{2520}$	$P_7 P_8 > P_{15}$
43b	(1, 1, 2, 0, 2, 1, 3, 1, 3, 3, 2, 2, 4)	$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{36036}$	$P_7 P_8 > P_{15}$
44a	(1, 1, 2, 1, 2, 1, 4, 1, 3, 4, 2, 2, 4)	$\{(2, 5), (6, 16), *\} \succ \{(8, 21), *\}$	$\frac{1}{1386}$	$P_7 P_{18} > P_{25} = 3$
44b	(1, 1, 2, 1, 2, 0, 3, 0, 2, 3, 2, 2, 3)	$\{(7, 16), (5, 13), *\}$	$\frac{1}{16016}$	$P_7 P_{10} > P_{17}$
46a	(1, 1, 1, 1, 2, 1, 3, 0, 3, 1, 1, 2, 3)	$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{16380}$	$P_9 P_{10} > P_{19}$
50a	(1, 1, 3, 1, 2, 2, 3, 1, 4, 2, 3, 3, 5)	$\{(4, 11), (1, 3), *\} \succ \{(5, 14), *\}$	$\frac{1}{1260}$	$P_7 P_{14} > P_{21}$
51a	(1, 1, 2, 2, 2, 2, 5, 0, 3, 3, 3, 3, 4)	$\{(4, 10), (3, 8), *\} \succ \{(7, 18), *\}$	$\frac{1}{1386}$	$P_6 P_{13} > P_{19}$
51b	(1, 1, 2, 2, 2, 2, 5, 0, 3, 3, 3, 3, 5)	$\{(5, 13), (5, 18), *\}$	$\frac{1}{1170}$	$P_6 P_{13} > P_{19}$
52a	(1, 1, 2, 1, 1, 0, 2, 1, 2, 2, 1, 2, 3)	$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{2184}$	$P_5 P_{12} > P_{17}$
56a	(1, 1, 2, 2, 1, 1, 2, 1, 3, 2, 2, 3, 3)	$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{1680}$	$P_5 P_{14} > P_{19}$
57	(1, 0, 2, 2, 0, 1, 3, 1, 3, 2, 2, 2, 3)	(3, 0, 1, 2, 0, 5, 0, 0, 4, 0, 0, 1, 0, 0, 0)	$\frac{1}{1386}$	$P_7 P_9 > P_{16}$
58a	(1, 1, 2, 2, 2, 0, 2, 1, 3, 2, 2, 3, 4)	$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{1680}$	$P_5 P_{12} > P_{17}$
59a	(1, 1, 2, 1, 2, 1, 2, 3, 2, 2, 2, 2, 3)	$\{(3, 8), (4, 11), *\} \succ \{(7, 19), *\}$	$\frac{1}{2660}$	Item C
60a	(1, 1, 1, 2, 1, 1, 3, 0, 3, 1, 1, 2, 3)	$\{(2, 5), (3, 8), *\} \succ \{(5, 13), *\}$	$\frac{1}{16380}$	$P_9 P_{10} > P_{19}$
61	(1, 1, 1, 2, 1, 1, 2, 2, 3, 2, 2, 2, 3)	(0, 1, 0, 1, 0, 3, 1, 0, 2, 0, 0, 0, 1, 0, 0)	$\frac{1}{9240}$	Item C
62a	(1, 1, 2, 2, 2, 1, 2, 2, 3, 2, 3, 3, 3)	$\{(4, 9), (3, 7), *\} \succ \{(7, 16), *\}$	$\frac{1}{2640}$	Item C
63	(1, 1, 3, 1, 2, 1, 3, 2, 3, 3, 2, 2, 4)	(5, 0, 1, 2, 0, 1, 1, 1, 3, 0, 1, 0, 0, 0, 1)	$\frac{1}{5544}$	Item C

By eliminating non-geometric baskets, we obtain a shorter list of baskets, listed in Table F-0, F-1, F-2 in the Appendix. We summarize some observations from the Tables.

Theorem 5.1. (=Theorem 1.4) *Let X be a minimal projective 3-fold of general type with the weighted basket $\mathbb{B}(X) := \{B_X, P_2, \chi(\mathcal{O}_X)\}$. If $\delta(X) \geq 13$, then $P_2 = 0$ and $\mathbb{B}(X)$ belongs to one of the types listed in Tables F-0~ F-2 in Appendix. Furthermore, the following holds:*

- (1) $\delta(X) = 18$ if and only if $\mathbb{B}(X) = \{B_{2a}, 0, 2\}$ (see Table F-0 for B_{2a}) with $K_X^3 = \frac{1}{1170}$.
- (2) $\delta(X) \neq 16, 17$.

- (3) $\delta(X) = 15$ if and only if $\mathbb{B}(X)$ is among one of the cases in Table F-1. One has $K_X^3 \geq \frac{1}{1386}$.
- (4) $\delta(X) = 14$ if and only if $\mathbb{B}(X)$ is among one of the cases in Table F-2. One has $K_X^3 \geq \frac{1}{1680}$.
- (5) $\delta(X) = 13$ if and only if $\mathbb{B}(X) = \{B_{41}, 0, 2\}$ (see Table F-0 for B_{41}) with $K_X^3 = \frac{1}{252}$.

Theorem 4.1, Theorem 5.1 and [11, Theorem 1.4] imply the following:

Corollary 5.2. (=Theorem 1.6(2)) *Let X be a minimal projective 3-fold of general type. Then $K_X^3 \geq \frac{1}{1680}$, and equality holds if and only if $\chi(\mathcal{O}_X) = 2$, $P_2 = 0$ and $B_X = B_{7a}$ or $B_X = B_{36a}$ (cf. Table F-2).*

Theorem 5.1, together with the explicit calculation, also implies the following:

Corollary 5.3. *Let X be a minimal projective 3-fold of general type. Then,*

- (1) if $\delta(X) = 13$, $P_m > 0$ for all $m \geq 10$;
- (2) if $\delta(X) = 14, 15, 18$, $P_m > 0$ for all $m \geq 20$.

6. Birationality

Theorem 6.1. *Let X be a minimal projective 3-fold of general type. If $\delta(X) = 18$, then Φ_m is birational for all $m \geq 61$.*

Proof. Set $m_0 = 18$. By Theorem 5.1, we know that $B_X = B_{2a}$, $P_2 = 0$, $\chi(\mathcal{O}_X) = 2$, $P_{19} = 0$, $P_{24} = 3$ and $K_X^3 = \frac{1}{1170}$. By [5, Corollary 1.2], we see $q(X) = 0$. Thus $|18K_X|$ induces a fibration $f : X' \rightarrow \Gamma \cong \mathbb{P}^1$. We have $h^2(\mathcal{O}_{X'}) = h^2(\mathcal{O}_X) = 1$. Pick a general fiber F . Since $P_{19}(X) = P_{19}(\mathbb{B}_{2a}) = 0$, we have $H^0(X', K_{X'} + F) = 0$.

Claim 6.1.1. $p_g(F) = 1$.

Proof. Since $\chi(\mathcal{O}_{X'}) > 1$, we have $p_g(F) > 0$ by [7, Lemma 2.32]. On the other hand, we have the long exact sequence:

$$H^0(X', K_{X'} + F) \rightarrow H^0(F, K_F) \rightarrow H^1(X', K_{X'}) \rightarrow H^1(X', K_{X'} + F)$$

which implies $h^0(K_F) \leq h^1(X', K_{X'}) = h^2(\mathcal{O}_{X'}) = 1$. Thus we get $p_g(F) = 1$. $\square$

We have $P_m > 0$ for all $m \geq 20$ by Corollary 5.3 (2). Consider the linear systems

$$|K_{X'} + \lceil n\pi^*(K_X) \rceil + F| \leq |(n+19)K_{X'}|.$$

Clearly $|(n+19)K_{X'}|$ distinguish different general fibers F as long as $n \geq 19$. By Kawamata-Viehweg vanishing,

$$\begin{aligned} |K_{X'} + \lceil n\pi^*(K_X) \rceil + F|_F &= |K_F + \lceil n\pi^*(K_X) \rceil|_F \\ &\succeq |K_F + \lceil L_n \rceil| \end{aligned}$$

where we set $L_n := n\pi^*(K_X)|_F$.

Claim 6.1.2. $L_n^2 > 8$ whenever $n \geq 42$.

Proof. Since $p_g(F) = 1$, we are in Subcase 3.4.1 or Subcase 3.4.3.

Let us consider Subcase 3.4.1 (i.e. $K_{F_0}^2 \geq 2$) first. We have

$$(\pi^*(K_X)|_F)^2 \geq \frac{1}{19^2} K_{F_0}^2 \geq \frac{2}{19^2}$$

by Lemma 2.1(ii). Thus $L_n^2 > 8$ whenever $n > 38$.

If $K_{F_0}^2 = 1$, we shall estimate L_n^2 in an alternative way. Suppose that $|24K_{X'}|$ and $|18K_{X'}|$ are not composed with the same pencil. Take $|G| := |M_{24}|_F$. Pick a generic irreducible element C of $|G|$. Then we have $\xi = (\pi^*(K_X)|_F \cdot C) \geq \frac{2}{19}$ by Lemma 2.4. Thus $(\pi^*(K_X)|_F)^2 \geq \frac{1}{24}\xi \geq \frac{1}{12 \cdot 19}$. Since $r(X) = 2340$ and $r(X)(\pi^*(K_X)|_F)^2$ is an integer, we see $(\pi^*(K_X)|_F)^2 \geq \frac{11}{2340}$. So we have $L_n^2 > 8$ whenever $n \geq 42$.

Assume that $|24K_{X'}|$ and $|18K_{X'}|$ are composed with the same pencil. Since $P_{24} = 3$, we may set $m_0 = 24$ and $\Lambda = |24K_{X'}|$. We have $\theta = 2$. The argument in Subcase 3.4.3 implies that

$$(\pi^*(K_X)|_F)^2 \geq \frac{4\theta^2}{(\tilde{m}_0 + \theta)(3m_0 + 4\theta)} = \frac{1}{130}.$$

We have $L_n^2 > 8$ whenever $n \geq 33$. □

For very general curves $\tilde{C}$ on F , one has

$$(L_n \cdot \tilde{C}) \geq \frac{n}{19} (\sigma^*(K_{F_0}) \cdot \tilde{C}) \geq \frac{2n}{19}$$

by Lemma 2.5. Therefore, $(L_n \cdot \tilde{C}) \geq 4$ for $n \geq 38$. Lemma 2.3 implies that $|K_F + \lceil L_n \rceil|$ gives a birational map for $n \geq 42$. Thus Φ_m is birational for all $m \geq 61$. □

Theorem 6.2. *Let X be a minimal projective 3-fold of general type. If $\delta(X) \leq 15$, then Φ_m is birational for all $m \geq 56$.*

Proof. Set $m_0 = \delta(X)$. By considering a sub-pencil Λ of $|m_0 K_X|$, we may always assume that we have an induced fibration $f : X' \rightarrow \Gamma$ onto a curve Γ . By Chen-Hacon [9], we may assume $q(X) = 0$. Thus $\Gamma \cong \mathbb{P}^1$. By [7, Corollary 3.13] and [7, Lemma 2.32], we know that $\delta(X) \leq 10$ as long as F is a $(1, 0)$ surface. Therefore it suffices to consider the following 3 cases:

1. $\delta(X) \leq 15$ and F is a $(1, 2)$ surface.
2. $\delta(X) \leq 15$ and F is neither a $(1, 2)$ surface nor a $(1, 0)$ surface.
3. $\delta(X) \leq 10$ and F is a $(1, 0)$ surface.

Case 1. Without losing of generality, let us assume $\delta(X) = 15$. Take $|G|$ to be the moving part of $|K_F|$. Then, by Table A3, we have $\xi \geq \frac{1}{11}$. We have $m_0 = 15$ and $\beta \mapsto \frac{1}{16}$. So $\alpha_m > 2$ whenever $m \geq 55$. By Corollary 5.3, $|mK_{X'}|$ separates different general fibers F as long as

$m \geq 35$. On the other hand, Kawamata-Viehweg vanishing and Lemma 2.1 imply the following, whenever $m \geq 49$,

$$\begin{aligned} |mK_{X'}|_F &\succeq |K_{X'} + \lceil (m-16)\pi^*(K_X) \rceil + F|_F \\ &\succeq |K_F + \lceil (m-16)\pi^*(K_X)|_F \rceil \\ &\succeq |(K_F + \lceil Q_m \rceil + C) + C| \end{aligned}$$

where Q_m is a nef and big $\mathbb{Q}$ -divisor. Thus, by [7, Lemma 2.17], Φ_m distinguishes different generic curves C for $m \geq 49$. Finally Theorem 2.7 implies that Φ_m is birational for all $m \geq 55$.

Case 2. Still assume $\delta(X) = 15$. Parallel to the respective argument in the proof of Theorem 6.1, one knows that $|mK_{X'}|$ distinguishes different general fibers F for $m \geq 35$. By the surface theory, we see that F is either a surface with $K_{F_0}^2 \geq 2$ or a $(1, 1)$ surface. We want to study the linear system $|K_F + \lceil L_n \rceil|$. In fact, by the estimation in Subcase 3.4.1 and Table A4, we have $L_n^2 \geq \frac{n^2}{32.6} > 8$ whenever $n \geq 40$. Similarly we have $(L_n \cdot \tilde{C}) \geq 4$ for all $n \geq 32$ and for all curves $\tilde{C}$ on F passing through very general points. By Lemma 2.3, we see that $|K_F + \lceil L_n \rceil|$ gives a birational map for all $n \geq 40$. Similar to what discussed in the proof of Theorem 6.1, we have proved that Φ_m is birational for all $m \geq n + 16 \geq 56$.

Case 3. When $\delta(X) \leq 10$, we have much better birationality result even though F is a $(1, 0)$ surface. In fact, parallel argument shows that Φ_m is birational for all $m \geq 39$. The proof is more or less similar to above ones. We leave it as an exercise to interested readers. $\square$

Theorems 5.1, 6.1, and 6.2 imply Theorem 1.6 (2).

7. Threefolds with $\delta(V) = 2$

This section is devoted to classifying minimal projective threefolds of general type with $\delta(X) = 2$, that is, $p_g(X) \leq 1$ and $P_2(X) \geq 2$.

Assume that $P_2 \geq 2$. We first recall the following known results:

- (a) If $d_2 = 3$, then Φ_m is birational for all $m \geq 7$ by [7, Theorem 2.20].
- (b) If $d_2 = 2$, Φ_m is birational for all $m \geq 10$ by [7, Theorem 2.22].
- (c) If $q(X) > 0$, then Φ_m is birational for all $m \geq 7$ by Chen-Hacon [9] and for $m = 6$ by Chen-Chen-Jiang [8].

The purpose of this section is to prove that Φ_m is birational for $m \geq 11$ and classify threefolds such that Φ_{10} is not birational. Therefore, we may and do assume that $q(X) = 0$, $d_2 = 1$ and $b = g(\Gamma) = 0$. Let F be the general fiber of the induced fibration $f : X' \rightarrow \mathbb{P}^1$ from Φ_2 .

7.1. Birationality of Φ_m for $m \geq 11$.

Lemma 7.1. $|mK_{X'}|$ distinguishes different general fibers of f for all $m \geq 9$.

Proof. When $p_g(F) > 0$, by [7, Proposition 2.15 (i)], one has $P_k > 0$ for $k \geq 7$. Thus, for all $m \geq 9$, $mK_{X'} \geq F$, hence $|mK_{X'}|$ distinguishes different general fibers of f .

When $p_g(F) = 0$, one has $\chi(\mathcal{O}_X) \leq 1$ (cf. [7, Lemma 2.32]). By [7, Lemma 3.2], one has $P_5 \geq P_2 > 0$. Then clearly $P_k > 0$ for all $k \geq 5$. Thus, for all $m \geq 7$, $mK_{X'} \geq F$ and hence $|mK_{X'}|$ distinguishes different general fibers of f . $\square$

Proposition 7.2. Assume $P_2(X) \geq 2$, $q(X) = 0$, $d_2 = 1$ and F is not a $(1, 2)$ surface. Then Φ_m is birational for all $m \geq 10$.

Proof. Set $L_n := n\pi^*(K_X)|_F$ which is a nef and big $\mathbb{Q}$ -divisor on F . Kawamata-Viehweg vanishing gives the following surjective map:

$$H^0(X', K_{X'} + \lceil n\pi^*(K_X) \rceil + F) \longrightarrow H^0(F, K_F + \lceil L_n \rceil|_F).$$

Together with Lemma 7.1, it is sufficient to prove that $|K_F + \lceil L_n \rceil|$ gives a birational map for $n \geq 7$ because

$$|(n+3)K_{X'}| \succeq |K_{X'} + \lceil n\pi^*(K_X) \rceil + F|.$$

Claim 7.2.1. If $K_{F_0}^2 \geq 2$ or F_0 is of type $(1, 0)$, then $|K_F + \lceil L_n \rceil|$ is birational for $n \geq 7$.

First of all, for any curve $\tilde{C} \subset F$ passing through very general points of F , we estimate $(L_n \cdot \tilde{C})$ for $n \geq 7$. Clearly we have $g(\tilde{C}) \geq 2$. Set $m_0 = 2$ and $\Lambda = |2K_{X'}|$. By Lemma 2.1 and Lemma 2.5, we have

$$(L_n \cdot \tilde{C}) \geq 7(\pi^*(K_X)|_F \cdot \tilde{C}) \geq \frac{7}{3}(\sigma^*(K_{F_0}) \cdot \tilde{C}) > 4.$$

If $K_{F_0}^2 \geq 2$, then we have

$$L_n^2 \geq 49(\pi^*(K_X)|_F)^2 \geq 49\left(\frac{1}{3}\sigma^*(K_{F_0})\right)^2 \geq \frac{98}{9} > 8.$$

If F_0 is a $(1, 0)$ surface, we have $P_4 \geq 2P_2 \geq 4$ since $\chi(\mathcal{O}_X) \leq 1$. When $d_4 \geq 2$, we set $m_0 = 2$, $\Lambda = |2K_{X'}|$ and $|G| = |M_4|_F$. Then $\beta = \frac{1}{4}$, $\xi \geq \frac{1}{3}(\sigma^*(K_{F_0}) \cdot C) \geq \frac{2}{3}$ and so $L_n^2 \geq \frac{49}{6} > 8$.

When $d_4 = 1$, we set $m_0 = 4$ and $\Lambda = |4K_{X'}|$. Clearly $|2K_{X'}|$ and $|4K_{X'}|$ induce the same fibration f . Take $|G| = |2\sigma^*(K_{F_0})|$. Since $\theta \geq 3$, we have $\beta \geq \frac{3}{14}$ by Lemma 2.1. Thus $\xi \geq \frac{6}{7}$ and so $L_n^2 \geq 49 \cdot \frac{3}{14} \cdot \frac{6}{7} > 8$. By Lemma 2.3, the Claim follows.

Claim 7.2.2. If F_0 is a $(1, 1)$ surface, then $|K_F + \lceil L_n \rceil|$ is birational for $n \geq 7$.

Following the similar argument as above, it is easy to see that $L_n^2 \geq \frac{64}{7} > 8$ and $(L_n \cdot \tilde{C}) \geq 4$ for all $n \geq 8$. We consider the linear system $|K_F + \lceil 7\pi^*(K_X)|_F \rceil|$ in an alternative way. Note that $|2\sigma^*(K_{F_0})|$ is

base point free. Pick a generic irreducible element $C \in |2\sigma^*(K_{F_0})|$. Since $\mathcal{O}_\Gamma(1) \hookrightarrow f_*\omega_{X'}$, we have $f_*\omega_{X'/\Gamma}^2 \hookrightarrow f_*\omega_{X'}^{10}$. The semi-positivity implies that $f_*\omega_{X'/\Gamma}^2$ is generated by global sections, which directly implies $10K_{X'}|_F \geq C$. Thus Φ_{10} distinguishes different C . By Lemma 2.1, we have $6\pi^*(K_X)|_F \equiv C + H_6$ for an effective $\mathbb{Q}$ -divisor H_6 on F . Thus the vanishing theorem implies

$$|K_F + \lceil 7\pi^*(K_X)|_F - H_6 \rceil|_C = |K_C + D|$$

with $\deg(D) \geq 2(\lceil 7\pi^*(K_X)|_F - C - H_6 \rceil \cdot \sigma^*(K_{F_0})) \geq 2$. Since C is non-hyperelliptic, $|K_C + D|$ gives a birational map. Thus $|K_F + \lceil 7\pi^*(K_X)|_F|$ is birational. $\square$

Proposition 7.3. *Assume $P_2(X) \geq 2$, $q(X) = 0$, $d_2 = 1$ and F a $(1, 2)$ surface. Then Φ_m is birational for all $m \geq 11$.*

Proof. Take $|G|$ to be the moving part of $|\sigma^*(K_{F_0})|$. Modulo birational modifications, we may assume that $|G|$ is base point free. Pick a generic irreducible element C of $|G|$. It is also known that $g = 2$.

Claim 7.3.1 The linear system $|mK_{X'}|$ distinguishes different general members of $|G|$ for $m \geq 9$.

Proof. Clearly $|G|$ is composed with a rational pencil since $q(F) = 0$. We shall prove $|mK_{X'}|_F \succeq |G|$ and thus the statement follows. In fact, by Lemma 2.1, we have

$$3\pi^*(K_X) \equiv \sigma^*(K_{F_0}) + H_3$$

for an effective $\mathbb{Q}$ -divisor H_3 on F . Thus, for $m \geq 10$,

$$Q_m := (m - 3)\pi^*(K_X)|_F - 2H_3 - 2\sigma^*(K_{F_0}) \equiv (m - 9)\pi^*(K_X)|_F$$

is nef and big. It follows that $K_F + \lceil Q_m \rceil + \sigma^*(K_{F_0}) > 0$ by [7, Lemma 2.14]. We thus have the following:

$$\begin{aligned} |mK_{X'}|_F &\succeq |K_{X'} + F + \lceil (m - 3)\pi^*(K_X) \rceil|_F \\ &= |K_F + \lceil (m - 3)\pi^*(K_X) \rceil|_F \\ &\succeq |K_F + \lceil (m - 3)\pi^*(K_X)|_F - 2H_3 \rceil| \\ &= |(K_F + \lceil Q_m \rceil + \sigma^*(K_{F_0})) + \sigma^*(K_{F_0})| \\ &\succeq |\sigma^*(K_{F_0})| \succeq |G| \end{aligned}$$

where the first equality follows from the Kawamata-Viehweg vanishing ([19, 33]). Therefore, $|mK_{X'}|$ distinguishes general members of $|G|$ for $m \geq 10$. Moreover, for $m = 9$,

$$\begin{aligned} |9K_{X'}|_F &\succeq |5K_{X'}|_F \succeq |K_{X'} + \lceil 2\pi^*(K_X) \rceil + F|_F \\ &= |K_F + \lceil 2\pi^*(K_X) \rceil|_F \succeq |G| \end{aligned}$$

where the equality is again due to Kawamata-Viehweg vanishing. Hence $|9K_{X'}|$ distinguishes general members of $|G|$ as well, which asserts the claim. $\square$

From Table A3, one has $\xi \geq \frac{1}{2}$. Take $m \geq 11$, then $\alpha_m \geq \frac{5}{2} > 2$. This means that $|mK_{X'}|_C$ distinguishes points on C . Thus, by Theorem 2.7 and Claim 7.3.1, Φ_m is birational for all $m \geq 11$. $\square$

Now Theorem 1.8.1 follows from Proposition 7.2 and Proposition 7.3. That is, if $P_2 \geq 2$, then Φ_m is birational for $m \geq 11$.

If either $\xi > \frac{1}{2}$ or $\beta > \frac{1}{3}$ then $\alpha_{10} > 2$. Hence the following consequence is immediate.

Corollary 7.4. *Let X be a minimal projective 3-fold of general type. Assume $P_2(X) \geq 2$, $q(X) = 0$, $d_2 = 1$ and F_0 a $(1, 2)$ surface. If either $\xi > \frac{1}{2}$ or $\beta > \frac{1}{3}$ or $P_2 > 2$, then Φ_{10} is birational.*

Proposition 7.2, Proposition 7.3 and Corollary 7.4 also imply the following:

Corollary 7.5. *Let X be a minimal projective 3-fold of general type. Assume $P_2 \geq 2$ and Φ_{10} is not birational. Then $P_2 = 2$, $q(X) = 0$ and $|2K_{X'}|$ is composed with a rational pencil of $(1, 2)$ surfaces.*

7.2. Classification. In the rest of this section, we classify minimal 3-folds X of general type which satisfy the following assumptions:

($\sharp$) $P_2(X) = 2$ and Φ_{10} is not birational.

Note that Corollary 7.5 implies that $|2K_X|$ induces a fibration $f : X' \rightarrow \mathbb{P}^1$ with the general fiber F a $(1, 2)$ surface.

Lemma 7.6. *If X satisfies ($\sharp$), then $0 \leq \chi(\mathcal{O}_X) \leq 3$.*

Proof. Note that the general fiber F of f is a $(1, 2)$ surface. Since $q(F) = 0$, we have $q(X) = 0$, $h^2(\mathcal{O}_X) = h^1(\mathbb{P}^1, f_*\omega_{X'})$ and $p_g(X) = h^0(f_*\omega_{X'})$. Since $P_2(X) = 2$ implies $p_g(X) \leq 1$, we see $\chi(\mathcal{O}_X) \geq 0$. By Fujita's semi-positivity ([16]), we have $\chi(\mathcal{O}_X) \leq 3$. $\square$

Theorem 7.7. *Let X be a minimal projective 3-fold of general type. Assume $P_2 = 2$, $q(X) = 0$ and F a $(1, 2)$ surface. Then Φ_{10} is birational under one of the following conditions:*

- (1) $P_3 \geq 4$;
- (2) $P_4 \geq 6$;
- (3) $P_5 \geq 8$;
- (4) $P_6 \geq 14$.

Proof. We set $m_0 = 2$. Pick a general fiber F of $f : X' \rightarrow \Gamma$ and a generic irreducible element C of $|G| := \text{Mov}|\sigma^*(K_{F_0})|$ on F . For $m_1 = 3, 4, 5$ and 6 , we have $P_{m_1} \geq 4$. Modulo further birational modifications to π , we may assume that the moving part $|M_{m_1}|$ of $|m_1K_{X'}|$ is base point free. We consider the following natural maps:

$$H^0(X', S_{m_1}) \xrightarrow{\mu_{m_1}} H^0(F, S_{m_1}|_F) \xrightarrow{\nu_{m_1}} H^0(C, S_{m_1}|_C)$$

where $S_{m_1} \in |M_{m_1}|$ denotes the general member.

Let $\text{Mov}|S_{m_1}|_F|$ be the moving part of $|S_{m_1}|_F|$ and let T_{m_1} be a general element in $\text{Mov}|S_{m_1}|_F|$ when $h^0(F, S_{m_1}|_F) > 1$. Clearly

$$(S_{m_1} \cdot C)_{X'} \geq (T_{m_1} \cdot C)_F \geq 0.$$

Since F and C are general, both μ_{m_1} and ν_{m_1} are non-zero maps. In particular, $h^0(F, S_{m_1}|_F) > 0$ and $h^0(C, S_{m_1}|_C) > 0$.

Let $F_{(r)}$ be a general element in $\text{Mov}|S_{m_1} - rF|$ if $h^0(S_{m_1} - rF) \geq 2$. Let $C_{(r)}$ be a general element in $\text{Mov}|T_{m_1} - rC|$ if $h^0(T_{m_1} - rC) \geq 2$. Replace X' by its birational modification, we may and do assume that $\text{Mov}|S_{m_1} - rF|$ is free.

Clearly, for $0 < r \leq \frac{h^0(X', S_{m_1})}{h^0(F, S_{m_1}|_F)}$, we have

$$h^0(X', S_{m_1} - rF) \geq h^0(X', S_{m_1}) - r \cdot h^0(F, S_{m_1}|_F). \quad (7.1)$$

Claim 7.7.1. *If $(T_{m_1} \cdot C) \leq 1$, then $(T_{m_1} \cdot C) = 0$.*

Proof. In fact, if $|T_{m_1}| \neq \emptyset$ and $|T_{m_1}|$ is not composed of the same pencil as that of $|C|$, then $\Phi_{|T_{m_1}|}(C)$ is a curve and so $h^0(C, T_{m_1}|_C) \geq 2$. Note that $g(C) = 2$. The Riemann-Roch theorem and the Clifford theorem imply that $(T_{m_1} \cdot C) = \deg(T_{m_1}|_C) \geq 2$, a contradiction. Hence either $|T_{m_1}|$ is composed of the same pencil as that of $|C|$ on F or $|T_{m_1}| = \emptyset$. Claim 7.7.1 now follows. $\square$

Claim 7.7.2. *Keep the same notation as above. Then Φ_{10} is birational under one of the following conditions:*

- (1) $(T_{m_1} \cdot C) > \frac{m_1}{2}$;
- (2) $T_{m_1} \cdot C = 0$ and $h^0(F, T_{m_1}) > 1 + \frac{m_1}{3}$;
- (3) $T_{m_1} \geq tC$ for some rational number $t > \frac{m_1}{3}$;
- (4) either $|T_{m_1}| = \emptyset$ and $P_{m_1} > 1 + \frac{m_1}{2}$ or $|T_{m_1}| \neq \emptyset$ and $\lfloor \frac{P_{m_1}-1}{h^0(F, T_{m_1})} \rfloor > \frac{m_1}{2}$.
- (5) $F_{(r)}$ (resp. $C_{(r)}$) is algebraically equivalent to F (resp. C) and $\frac{r+1}{m_1} > \frac{1}{2}$ (resp. $\frac{r+1}{m_1} > \frac{1}{3}$).

Proof. If $(T_{m_1} \cdot C) > \frac{m_1}{2}$, then $\xi \geq \frac{1}{m_1}(S_{m_1} \cdot C) \geq \frac{1}{m_1}(T_{m_1} \cdot C) > \frac{1}{2}$. Then Corollary 7.4 implies that Φ_{10} is birational, which proves (1).

Now we prove (4). We claim that we have

$$m_1 \pi^*(K_X) \geq S_{m_1} \geq rF$$

for an integer $r > \frac{m_1}{2}$. In fact, when $|T_{m_1}| = \emptyset$, $|S_{m_1}|$ is composed of the same pencil as that of $|F|$ and we may take $r := P_{m_1} - 1$. When $|T_{m_1}| \neq \emptyset$, we may take $r = \lfloor \frac{P_{m_1}-1}{h^0(F, T_{m_1})} \rfloor$ and then $S_{m_1} \geq rF$ since $\dim(\mu_{m_1}) \leq h^0(F, T_{m_1})$. Then Lemma 2.1 implies $\beta \geq \frac{r}{m_1+r} > \frac{1}{3}$. So Φ_{10} is birational by Corollary 7.4, which asserts (4).

Since $m_1 \pi^*(K_X)|_F \geq T_{m_1} \geq tC$, we have $\beta > \frac{1}{3}$ and Φ_{10} is birational by Corollary 7.4, which proves (3).

If $(T_{m_1} \cdot C) = 0$ and $h^0(F, T_{m_1}) > 1 + \frac{m_1}{3}$, then $|T_{m_1}|$ is composed of the same pencil as that of $|C|$ and $T_{m_1} \geq tC$ where $t \geq h^0(T_{m_1}) - 1$. Hence Φ_{10} is birational by (3), which proves (2).

Finally, if $F_{(r)}$ is algebraically equivalent to F , then $S_{m_1} \geq F_{(r)} + F \sim (r+1)F$. Hence $\beta \geq \frac{r+1}{m_1+r+1} > \frac{1}{3}$. Thus Φ_{10} is birational by Corollary 7.4. If $C_{(r)}$ is algebraically equivalent to C , then we have $\beta \geq \frac{r+1}{m_1} > \frac{1}{3}$ as well. Hence Φ_{10} is birational, which verifies (5). $\square$

Return to the proof of Theorem 7.7.

Part I. $P_3 \geq 4$. Set $m_1 = 3$. By Claim 7.7.2 (1), (2) and Claim 7.7.1, we may assume $(T_3 \cdot C) = 0$ and $h^0(F, T_3) \leq 2$. Also by Claim 7.7.2 (4), we may assume $|T_3| \neq \emptyset$ and $h^0(F, T_3) = 2$.

By Inequality (7.1), one gets $h^0(S_3 - F) \geq 2$. Clearly we have that $S_3 \geq F + F_{(1)}$ and that, by assumption, $F_{(1)}$ is nef. Since $r = 1$ and $\frac{r+1}{m_1} = \frac{2}{3} > \frac{1}{2}$, we may assume that $F_{(1)}$ is not algebraically equivalent to F by Claim 7.7.2 (5).

Now clearly we have $h^0(F, F_{(1)}|_F) \geq 2$. Note that we have

$$|10K_{X'}| \succeq |K_{X'} + [6\pi^*(K_X)] + F_{(1)} + F|.$$

Kawamata-Viehweg vanishing gives the surjective map:

$$\begin{aligned} & H^0(X', K_{X'} + [6\pi^*(K_X)] + F_{(1)} + F) \\ \longrightarrow & H^0(F, K_F + [6\pi^*(K_X)]|_F + F_{(1)}|_F). \end{aligned}$$

It is sufficient to verify the birationality of the rational map defined by $|K_F + [6\pi^*(K_X)]|_F + \Gamma_{(1)}|$ where $\Gamma_{(1)}$ is a generic irreducible element in $\text{Mov}|F_{(1)}|_F|$.

We claim that $(\pi^*(K_X) \cdot \Gamma_{(1)}) \geq \frac{1}{2}$. In fact, if $\Gamma_{(1)}$ is algebraically equivalent to C , then $(\pi^*(K_X) \cdot \Gamma_{(1)}) = \xi \geq \frac{1}{2}$ by Table A3. On the other hand, if $\Gamma_{(1)}$ is not algebraically equivalent to C , then we should have $(\Gamma_{(1)} \cdot C) \geq 2$. By Lemma 2.1, $(\pi^*(K_X)|_F \cdot \Gamma_{(1)}) \geq \frac{1}{3}(C \cdot \Gamma_{(1)}) \geq \frac{2}{3}$.

Clearly $|K_F + [6\pi^*(K_X)]|_F + \Gamma_{(1)}|$ distinguishes different generic $\Gamma_{(1)}$'s since $K_F + [6\pi^*(K_X)]|_F > 0$. Now by the vanishing theorem again we have the following surjective map:

$$H^0(F, K_F + [6\pi^*(K_X)]|_F + \Gamma_{(1)}) \longrightarrow H^0(\Gamma_{(1)}, K_{\Gamma_{(1)}} + D)$$

where $D := [6\pi^*(K_X)]|_{\Gamma_{(1)}}$ with $\deg(D) \geq 6(\pi^*(K_X) \cdot \Gamma_{(1)}) > 2$. So Φ_{10} is birational by the ordinary birationality principle.

Part II. $P_4 \geq 6$. We set $m_1 = 4$. By Claim 7.7.2 (1) and (4), we may assume $(T_4 \cdot C) \leq 2$ and $h^0(F, T_4) \geq 2$. Claim 7.7.1 implies either $(T_4 \cdot C) = 0$ or $(T_4 \cdot C) = 2$.

(II-1). If $h^0(F, T_4) = 2$, we have $h^0(X', S_4 - 2F) \geq 2$ by Inequality (7.1). We consider $F_{(2)}$ and may assume that $F_{(2)}$ is not algebraically equivalent to F by Claim 7.7.2 (5). Now $h^0(F, F_{(2)}|_F) \geq 2$ and pick a

generic irreducible element $\Gamma_{(2)}$ of $\text{Mov}|F_{(2)}|_F|$. By Kawamata-Viehweg vanishing, we have

$$\begin{aligned} |10K_{X'}|_F &\succeq |K_{X'} + \lceil 5\pi^*(K_X) \rceil + F_{(2)} + 2F|_F \\ &= |K_F + \lceil 5\pi^*(K_X) \rceil|_F + F_{(2)}|_F| \\ &\succeq |K_F + \lceil 5\pi^*(K_X) \rceil|_F + \Gamma_{(2)}|. \end{aligned}$$

When C is algebraically equivalent to $\Gamma_{(2)}$ (in particular, $C \sim \Gamma_{(2)}$) due to the fact that $q(F) = 0$, since

$$\deg(5\pi^*(K_X)|_C) = 5\xi \geq \frac{5}{2}$$

and

$$|K_F + \lceil 5\pi^*(K_X) \rceil|_F + \Gamma_{(2)}|_C = |K_C + \lceil 5\pi^*(K_X) \rceil|_F|_C|$$

with $\deg(\lceil 5\pi^*(K_X) \rceil|_C) > 2$, we see that $\Phi_{10}|_C$ is birational by Lemma 7.1 and Claim 7.3.1.

When C is not algebraically equivalent to $\Gamma_{(2)}$, we have $(\Gamma_{(2)} \cdot C) \geq 2$ and

$$K_F + \lceil 5\pi^*(K_X) \rceil|_F + \Gamma_{(2)} \geq K_F + \lceil Q_1 + C \rceil + \Gamma_{(2)}$$

for certain nef and big $\mathbb{Q}$ -divisor Q_1 on F by Lemma 2.1. The vanishing theorem also shows that

$$|K_F + \lceil Q_1 \rceil + \Gamma_{(2)} + C|_C = |K_C + (Q_1 + \Gamma_{(2)})|_C|$$

gives a birational map since $\deg((Q_1 + \Gamma_{(2)})|_C) > 2$. Thus we have shown that Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

(II-2). If $(T_4 \cdot C) = 0$ and $h^0(F, T_4) \geq 3$, Φ_{10} is birational by Claim 7.7.2 (2).

(II-3). If $(T_4 \cdot C) = 2$ and $h^0(F, T_4) \geq 3$, then $|T_4|$ is not composed of the same pencil as that of $|C|$ and $h^0(C, T_4|_C) \geq 2$. By the Riemann-Roch and the Clifford theorem, we see $\deg(T_4|_C) = h^0(C, T_4|_C) = 2$. Thus $\dim \text{im}(\nu_4) = 2$.

(II-3-1). If $h^0(F, T_4) \geq 4$, we have $h^0(F, T_4 - C) \geq 2$. Denote by $C_{(1)}$ a generic irreducible element of $\text{Mov}|T_4 - C|$. Then we have $T_4 \geq C + C_{(1)}$ and we may assume that C is not algebraically equivalent to $C_{(1)}$ by Claim 7.7.2 (5), which implies $(C_{(1)} \cdot C) \geq 2$. By the Kawamata-Viehweg vanishing and properties of the roundup operator, we have

$$\begin{aligned} |10K_{X'}|_F &\succeq |K_{X'} + \lceil 3\pi^*(K_X) \rceil + S_4 + F|_F \\ &= |K_F + \lceil 3\pi^*(K_X) \rceil|_F + S_4|_F| \\ &\succeq |K_F + \lceil 3\pi^*(K_X) \rceil|_F + C_{(1)} + C| \end{aligned}$$

and

$$|K_F + \lceil 3\pi^*(K_X) \rceil|_F + C_{(1)} + C|_C = |K_C + D|,$$

where $D := (\lceil 3\pi^*(K_X) \rceil|_F + C_{(1)})|_C$ with $\deg(D) > (C_{(1)} \cdot C) \geq 2$. Thus Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

(II-3-2). If $h^0(F, T_4) = 3$, we have $h^0(S_4 - F) \geq 3$. Again, we pick a general member $F_{(1)} \in \text{Mov}|S_4 - F|$. Consider the natural map:

$$H^0(X', F_{(1)}) \xrightarrow{\mu'_4} H^0(F, F_{(1)}|_F) \subset H^0(F, S_4|_F).$$

When $\dim \text{im}(\mu'_4) = 3$, we see $\dim \nu_4(\text{im}(\mu'_4)) = \dim \nu_4(\text{im}(\mu_4)) = 2$; when $\dim \text{im}(\mu'_4) = 2$, we consider the situation $\dim \nu_4(\text{im}(\mu'_4)) \leq 1$ at first. In both cases, $h^0(F, F_{(1)}|_F - C) > 0$ and thus $F_{(1)}|_F - C \geq 0$. By the vanishing theorem once more, we have

$$\begin{aligned} |10K_{X'}|_F &\succeq |K_{X'} + [5\pi^*(K_X)] + F_{(1)} + F|_F \\ &= |K_F + [5\pi^*(K_X)]|_F + F_{(1)}|_F| \\ &\succeq |K_F + [5\pi^*(K_X)]_F| + C|. \end{aligned}$$

Applying the vanishing theorem again, we see

$$|K_F + [5\pi^*(K_X)]_F| + C|_C = |K_C + D|,$$

where $D := ([5\pi^*(K_X)]_F)|_C$ with $\deg(D) \geq 5\xi > 2$. Thus Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

When $\dim \text{im}(\mu'_4) = \dim \nu_4(\text{im}(\mu'_4)) = 2$, then $|F_{(1)}|_F$ is not composed with the same pencil as that of $|C|$. In particular, $(F_{(1)} \cdot C) \geq 2$. By Lemma 2.1, we have

$$K_F + [5\pi^*(K_X)]_F + F_{(1)}|_F \geq K_F + [Q_2 + C] + F_{(1)}|_F$$

for certain nef and big $\mathbb{Q}$ -divisor Q_2 . Since the vanishing theorem gives

$$|K_F + [Q_2] + F_{(1)}|_F + C|_C = |K_C + D'|$$

with $\deg(D') > (F_{(1)} \cdot C) \geq 2$, we see Φ_{10} is birational too by Lemma 7.1 and Claim 7.3.1.

Consider the last case $\dim \text{im}(\mu'_4) = 1$. We see that $|F_{(1)}|$ is composed of the same pencil as that of $|F|$ and $F_{(1)} \geq 2F$. Thus $S_4 \geq 3F$ and, since $\frac{3}{m_1} > \frac{1}{2}$, Φ_{10} is birational by Claim 7.7.2 (5).

Part III. $P_5 \geq 8$. We set $m_1 = 5$. By Claim 7.7.1 and Claim 7.7.2 (1), (2) and (4), we may assume $(T_5 \cdot C) = 2$ and $h^0(F, T_5) \geq 3$. Clearly $|T_5|$ is not composed of the same pencil as that of $|C|$ and so that $h^0(C, T_5|_C) \geq 2$. By the Riemann-Roch and the Clifford theorem, we see $\deg(T_5|_C) = h^0(C, T_5|_C) = 2$. Thus $\dim \text{im}(\nu_5) = 2$.

(III-1). If $h^0(F, T_5) \geq 4$, we have $h^0(F, T_5 - C) \geq 2$. Let $C_{(1)}$ be a generic irreducible element in $\text{Mov}|T_5 - C|$. Thus we have $T_5 \geq C + C_{(1)}$ and we may assume that $C_{(1)}$ is not algebraically equivalent to C by Claim 7.7.2 (5). Hence $(C_{(1)} \cdot C) \geq 2$. By the Kawamata-Viehweg vanishing and properties of the roundup operator, we have the following:

$$\begin{aligned} |10K_{X'}|_F &\succeq |K_{X'} + [2\pi^*(K_X)] + S_5 + F|_F \\ &= |K_F + [2\pi^*(K_X)]|_F + S_5|_F| \\ &\succeq |K_F + [2\pi^*(K_X)]_F| + C_{(1)} + C| \end{aligned}$$

and $|K_F + \lceil 2\pi^*(K_X)|_F \rceil + C_{(1)} + C|_C = |K_C + D|$, with
 $\deg(D) > (C_{(1)} \cdot C) \geq 2$.

Thus Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

(III-2). If $h^0(F, T_5) = 3$, we have $h^0(S_5 - F) \geq 5$. Let $F_{(1)} \in \text{Mov}|S_5 - F|$ be a general member. We consider the natural map:

$$H^0(X', F_{(1)}) \xrightarrow{\mu'_5} H^0(F, F_{(1)}|_F) \subset H^0(F, S_5|_F).$$

Clearly we have $\dim \text{im}(\mu'_5) \leq h^0(F, T_5) = 3$.

When $\dim \text{im}(\mu'_5) = 3$, we see $\dim \nu_5(\text{im}(\mu'_5)) = \dim \nu_5(\text{im}(\mu_5)) = 2$. Thus $|F_{(1)}|_F$ is not composed of the same pencil as that of $|C|$. Pick a generic irreducible element $\Gamma_{(1)}$ in the moving part of $|F_{(1)}|_F$. Then $(\Gamma_{(1)} \cdot C) \geq 2$. By the vanishing theorem, we have

$$\begin{aligned} |10K_{X'}|_F &\succeq |K_{X'} + \lceil 4\pi^*(K_X) \rceil + F_{(1)} + F|_F \\ &= |K_F + \lceil 4\pi^*(K_X) \rceil|_F + F_{(1)}|_F \\ &\succeq |K_F + \lceil 4\pi^*(K_X)|_F \rceil + \Gamma_{(1)}|. \end{aligned}$$

Applying Lemma 2.1, we have

$$|K_F + \lceil 4\pi^*(K_X)|_F \rceil + \Gamma_{(1)}| \succeq |K_F + \lceil Q_3 + C \rceil + \Gamma_{(1)}|$$

where Q_3 is certain nef and big $\mathbb{Q}$ -divisor on F . Applying the vanishing once more, we have

$$|K_F + \lceil Q_3 \rceil + \Gamma_{(1)} + C|_C = |K_C + D|$$

with $\deg(D) > (\Gamma_{(1)} \cdot C) \geq 2$. Thus Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

When $\dim \text{im}(\mu'_5) \leq 2$, we have $h^0(X', F_{(1)} - 2F) \geq 1$ and hence $S_5 - 3F \geq 0$. Therefore Φ_{10} is birational by Claim 7.7.2 (5).

Part IV. $P_6 \geq 14$. We set $m_1 = 6$. By Claim 7.7.1 and Claim 7.7.2 (1), (2) and (4), we may assume $2 \leq (T_6 \cdot C) \leq 3$ and $h^0(F, T_6) \geq 4$. Clearly $|T_6|$ is not composed of the same pencil as that of $|C|$. Thus, by the Riemann-Roch theorem and the Clifford theorem, $\dim \text{im}(\nu_6) = h^0(C, T_6|_C) = 2$.

(IV-1). If $h^0(F, T_6) \geq 5$, then we see $h^0(F, T_6 - C) \geq 3$. We pick a general member $C_{(1)}$ in $\text{Mov}|T_6 - C|$. By Claim 7.7.2 (5), we may assume that $|C_{(1)}|$ is not composed of the same pencil as that of $|C|$. We shall analyze the natural map $\nu'_6 : H^0(F, C_{(1)}) \mapsto H^0(C, C_{(1)}|_C)$. Clearly $2 \leq \dim \text{im}(\nu'_6) \leq h^0(C, T_6|_C) = 2$.

Since $C_{(1)}$ is not algebraically equivalent to C , one has $(C_{(1)} \cdot C) \geq 2$. By the vanishing theorem, we have

$$\begin{aligned} |10K_{X'}|_F &\succeq |K_{X'} + \lceil \pi^*(K_X) \rceil + S_6 + F|_F \\ &\succeq |K_F + \lceil \pi^*(K_X)|_F \rceil + C_{(1)} + C|_C \end{aligned}$$

and $|K_F + \lceil \pi^*(K_X)|_F \rceil + C_{(1)} + C|_C = |K_C + D|$ with $\deg(D) > (C_{(1)} \cdot C) = 2$. Thus Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1.

(IV-2). If $h^0(F, T_6) = 4$, we have $h^0(S_6 - F) \geq 10$. We pick a general member $F_{(1)} \in \text{Mov}|S_6 - F|$ and consider the natural map:

$$H^0(X', F_{(1)}) \xrightarrow{\mu'_6} H^0(F, F_{(1)}|_F) \subset H^0(F, S_6|_F).$$

Clearly we have $\dim \text{im}(\mu'_6) \leq h^0(F, T_6) = 4$.

When $\dim \text{im}(\mu'_6) \leq 3$, we have $F_{(1)} - 3F \geq 0$ and then $S_6 \geq 4F$. By Claim 7.7.2 (5), Φ_{10} is birational.

When $\dim \text{im}(\mu'_6) = 4$, we see $\dim \nu_6(\text{im}(\mu'_6)) = \dim \nu_6(\text{im}(\mu_6)) = 2$. Thus $h^0(F, F_{(1)}|_F - C) = 2$. Furthermore $|F_{(1)}|_F$ is not composed of the same pencil as that of $|C|$. Noting that a divisor of degree 1 can not move on C , we see $(F_{(1)} \cdot C) \geq 2$. Denote by $\Gamma_{(1)}$ a general irreducible element of $\text{Mov}|F_{(1)}|_F - C|$. Noting that $S_6 \geq F_{(1)} + F$ and applying the vanishing theorem, we have

$$\begin{aligned} |10K_{X'}| &\succeq |K_{X'} + \lceil 3\pi^*(K_X) \rceil + F_{(1)} + F| \\ &\succeq |K_F + \lceil 3\pi^*(K_X)|_F \rceil + F_{(1)}|_F|. \end{aligned}$$

If $\Gamma_{(1)}$ is not algebraically equivalent to C , we have $(\Gamma_{(1)} \cdot C) \geq 2$. The vanishing theorem gives

$$|K_F + \lceil 3\pi^*(K_X)|_F \rceil + \Gamma_{(1)} + C|_C = |K_C + D|$$

with $\deg(D) > (\Gamma_{(1)} \cdot C) \geq 2$. Thus Φ_{10} is birational by Lemma 7.1 and Claim 7.3.1. If $\Gamma_{(1)}$ is algebraically equivalent to C , we have $F_{(1)}|_F \geq 2C$ and write

$$F_{(1)}|_F = 2C + H_6$$

where H_6 is an effective divisor on F . Since $3\pi^*(K_X)|_F + F_{(1)}|_F - C - \frac{1}{2}H_6$ is nef and big, the Kawamata-Viehweg vanishing theorem implies the following surjective map

$$H^0(F, K_F + \lceil 3\pi^*(K_X)|_F \rceil + F_{(1)}|_F - \frac{1}{2}H_6) \longrightarrow H^0(C, D')$$

where $D' := \lceil 3\pi^*(K_X)|_F + F_{(1)}|_F - \frac{1}{2}H_6 - C \rceil|_C$ with $\deg(D') \geq 3\xi + \frac{1}{2}(F_{(1)} \cdot C) > 2$. Thus we see that Φ_{10} is birational again by Lemma 7.1 and Claim 7.3.1. So we conclude the theorem. $\square$

Corollary 7.8. (=Theorem 1.8(2)) *Let X be a minimal projective 3-fold of general type with $\delta(X) = 2$. If Φ_{10} is not birational, then the weighted basket $\mathbb{B}(X) = (B_X, P_2, \chi(\mathcal{O}_X))$ are dominated by an initial basket listed in Tables II-1, II-2, II-3 in Appendix.*

Proof. By Lemma 7.6 and Theorem 7.7, we see $0 \leq \chi(\mathcal{O}_X) \leq 3$, $P_2(X) = 2$, $P_3(X) \leq 3$, $P_4(X) \leq 5$, $P_5(X) \leq 7$ and $P_6(X) \leq 13$. According to [6, Section 3], the total number of numerical types of $\mathbb{B}(X)$ is finite. We give a list of $\mathbb{B}^0(X)$ in Tables II-1, II-2 and II-3. $\square$

8. Projective 4-folds of general type with positive geometric genus

In order to study 4-folds of general type, we need to prove a slightly general statement on 3-folds.

Theorem 8.1. *Let $\nu : \tilde{X} \rightarrow X$ be a birational morphism from a nonsingular projective 3-fold $\tilde{X}$ of general type onto a minimal model X with $p_g(X) > 0$. Let Q_λ be any $\mathbb{Q}$ -divisor on $\tilde{X}$ satisfying $Q_\lambda \equiv \lambda \nu^*(K_X)$ for some rational number $\lambda > 16$. Then $|K_{\tilde{X}} + \lceil Q_\lambda \rceil|$ gives a birational map onto its image. In particular, Φ_m is birational for all $m \geq 18$.*

Proof. From the proof of Corollary 4.10, we only need to consider the following two cases:

Case 1. $P_4 \geq 2$;

Case 2. $P_4 = 1$ and $P_5 \geq 3$.

Set $m_0 = 4$ (resp. 5) in Case 1 (resp. Case 2). Take a sub-pencil $\Lambda \subset |m_0 K_X|$. We use the same set up as in 2.1. We may and do assume that π factors through ν , i.e. there is a birational morphism $\mu : X' \rightarrow \tilde{X}$ so that $\pi = \nu \circ \mu$ and that $\mu^*(\{Q_\lambda\}) \cup \{\text{exc. divisors of } \mu\}$ has simple normal crossing supports.

Since

$$\mu_* \mathcal{O}_{X'}(K_{X'} + \lceil \mu^*(Q_\lambda) \rceil) \subseteq \mu_* \mathcal{O}_{X'}(K_{X'} + \mu^* \lceil Q_\lambda \rceil) \subseteq \mathcal{O}_{\tilde{X}}(K_{\tilde{X}} + \lceil Q_\lambda \rceil),$$

it is sufficient to prove the birationality of $\Phi_{|K_{X'} + \lceil \mu^*(Q_\lambda) \rceil|}$. We write $Q'_\lambda := \mu^*(Q_\lambda) \equiv \lambda \pi^*(K_X)$.

We have an induced fibration $f : X' \rightarrow \Gamma$ onto a smooth projective curve. Let F be a general fiber of f . Recall that we have $m_0 \pi^*(K_X) \sim_{\mathbb{Q}} \theta F + E'_\Lambda$ for a positive integer θ and an effective $\mathbb{Q}$ -divisor E'_Λ on X' .

Without lose of generality, we may assume $p_g(X) = 1$ (the case with $p_g(X) > 1$ is much easier). Clearly one has $p_g(F) > 0$.

Claim 8.1.1. One has $h^0(X', K_{X'} + \lceil Q'_\lambda \rceil) > 0$ for $\lambda > 2m_0 + 1$.

By Lemma 2.1,

$$\pi^*(K_X)|_F \equiv \frac{1}{m_0 + 1} \sigma^*(K_{F_0}) + H_{m_0}$$

for certain effective $\mathbb{Q}$ -divisor H_{m_0} on F . Since $Q'_\lambda - F - \frac{1}{\theta} E'_\Lambda \equiv (\lambda - \frac{m_0}{\theta}) \pi^*(K_X)$ is nef and big, Kawamata-Viehweg vanishing implies the surjective map:

$$H^0(X', K_{X'} + \lceil Q'_\lambda - \frac{1}{\theta} E'_\Lambda \rceil) \rightarrow H^0(F, K_F + \lceil Q'_\lambda - \frac{1}{\theta} E'_\Lambda \rceil|_F). \quad (8.1)$$

Let

$$\begin{aligned} Q_{\lambda, F} : &= (Q'_\lambda - \frac{1}{\theta} E'_\Lambda)|_F - (m_0 + 1)H_{m_0} - \sigma^*(K_{F_0}) \\ &\equiv (\lambda - \frac{m_0}{\theta} - m_0 - 1)\pi^*(K_X)|_F, \end{aligned}$$

which is nef and big. Since $p_g(F) > 0$, we have

$$\begin{aligned} & h^0(F, K_F + \lceil Q'_\lambda - \frac{1}{\theta} E'_\Lambda \rceil|_F) \\ & \geq h^0(F, K_F + \lceil (Q'_\lambda - \frac{1}{\theta} E'_\Lambda)|_F - (m_0 + 1)H_{m_0} \rceil) \\ & = h^0(F, K_F + \lceil Q_{\lambda, F} \rceil + \sigma^*(K_{F_0})) \geq 2 \end{aligned}$$

by [7, Lemma 2.14]. This verifies the Claim.

Claim 8.1.2. *The linear system $|K_{X'} + \lceil Q'_\lambda \rceil|$ distinguishes different general fibers of f for any $\lambda > 3m_0 + 1$.*

Proof. When $g(\Gamma) = 0$, we consider $Q'_\zeta := Q'_\lambda - F - \frac{1}{\theta} E'_\Lambda \equiv \zeta \pi^*(K_X)$ with $\zeta = \lambda - \frac{m_0}{\theta}$. It is clear that $K_{X'} + \lceil Q'_\lambda \rceil \geq (K_{X'} + \lceil Q'_\zeta \rceil) + F$ and hence $|K_{X'} + \lceil Q'_\lambda \rceil|$ distinguishes different general fibers by Claim 8.1.1 since $\zeta > 2m_0 + 1$.

When $g(\Gamma) > 0$, we have $\theta \geq 2$. Pick two different general fibers F_1 and F_2 of f . The vanishing theorem gives the surjective map:

$$\begin{aligned} & H^0(X', K_{X'} + \lceil Q'_\lambda - \frac{2}{\theta} E'_\Lambda \rceil) \\ & \longrightarrow \bigoplus_{i=1}^2 H^0(F_i, (K_{X'} + \lceil Q'_\lambda - F_1 - F_2 - \frac{2}{\theta} E'_\Lambda \rceil + F_1 + F_2)|_{F_i}) \end{aligned}$$

where we note that $(K_{X'} + \lceil Q'_\lambda - F_1 - F_2 - \frac{2}{\theta} E'_\Lambda \rceil)|_{F_i} \geq 0$ due to Claim 8.1.1 and the fact $(F_1 + F_2)|_{F_i} = 0$. Hence $|K_{X'} + \lceil Q'_\lambda \rceil|$ distinguishes F_1 and F_2 . $\square$

Now we discuss two cases independently.

Case 1. $P_4 \geq 2$.

If F is a $(1, 2)$ surface, we take $|G| := \text{Mov}|\sigma^*(K_{F_0})|$ and a general member $C \in |G|$. By the surjection map in (8.1) and Claim 8.1.2, it is sufficient to study the linear system $|K_F + \lceil (Q'_\lambda - \frac{1}{\theta} E'_\Lambda)|_F \rceil|$. For any t , let

$$L_{\lambda, t} := (Q'_\lambda - \frac{1}{\theta} E'_\Lambda)|_F - t\sigma^*(K_{F_0}) - 5tH_4 \equiv (\lambda - \frac{4}{\theta} - 5t)\pi^*(K_X)|_F,$$

which is nef and big as long as $\lambda - \frac{4}{\theta} - 5t > 0$. Notice also that $(Q'_\lambda - \frac{1}{\theta} E'_\Lambda)|_F \geq L_{\lambda, t} + t\sigma^*(K_{F_0})$. For simplicity, $L_{\lambda, 0}$ is denoted by L_λ . In fact, for $\lambda > 14$ and by [7, Lemma 2.14], one has

$$K_F + \lceil Q'_\lambda - \frac{1}{\theta} E'_\Lambda \rceil|_F \geq (K_F + \lceil L_{\lambda, 2} \rceil + \sigma^*(K_{F_0})) + C \geq C.$$

Thus $|K_F + \lceil (Q'_\lambda - \frac{1}{\theta} E'_\Lambda)|_F \rceil|$ separates different general curves C when $\lambda > 14$. Restricting to the curve C , one sees by the vanishing theorem that

$$|K_F + \lceil (Q'_\lambda - \frac{1}{\theta} E'_\Lambda)|_F \rceil|_C \geq |K_F + \lceil L_{\lambda, 1} \rceil + C|_C = |K_C + \lceil L_{\lambda, 1} \rceil|_C|.$$

Since $\deg([L_{\lambda,1}]|_C) \geq (\lambda - \frac{4}{\theta} - 5)\xi > 2$ for $\xi \geq 2/7$ (cf. Table A3 with $m_0 = 4$). Thus $\Phi_{|K_{X'} + [Q'_\lambda]|}$ separates points on the general curve C and hence is birational when $\lambda > 16$.

Assume that F is not a $(1,2)$ surface. We would like to study $|K_F + [L_\lambda]|$ where $L_\lambda := (Q'_\lambda - \frac{1}{\theta}E'_\lambda)|_F$, making use of the relation (8.1). If $K_{F_0}^2 \geq 2$, Inequalities (3.4), (3.6) imply

$$L_\lambda^2 \geq \frac{2(\lambda - 4)^2}{25} > 8$$

whenever $\lambda > 14$. If F is a $(1,1)$ surface, then we have $q(X) = g(\Gamma) \geq 0$ and $h^2(\mathcal{O}_X) = 0$ as seen in the proof of Case 2 of Corollary 4.10. Hence we have $\chi(\mathcal{O}_X) \leq 0$ and Reid's Riemann-Roch formula gives $P_5 > P_4 \geq 2$. In particular, we have $P_5 \geq 3$. We omit the discussion for the situation when $|5K_{X'}|$ and $|4K_{X'}|$ are composed with the same pencil since that is a comparatively much better case. So may assume that $|5K_{X'}|_F$ is moving on F . If we take $|G_1| := \text{Mov}[5\pi^*(K_X)]|_F|$, we have $\beta_{G_1} = \frac{1}{5}$. Then, by Lemma 2.1 and Lemma 2.4, we have

$$L_\lambda^2 \geq \frac{(\lambda - 4)^2}{25}(\sigma^*(K_{F_0}) \cdot G_1) \geq \frac{2(\lambda - 4)^2}{25} > 8$$

whenever $\lambda > 14$. Finally, for both cases, $(L_\lambda \cdot \tilde{C}) \geq \frac{2(\lambda-4)}{5} \geq 4$ for $\lambda \geq 14$ and for any very general curve $\tilde{C}$ on F . Therefore, by Lemma 2.3, $|K_F + [L_\lambda]|$ gives a birational map when $\lambda \geq 14$.

Hence, when $P_4 \geq 2$, $\Phi_{|K_{X'} + [Q'_\lambda]|}$ is birational for $\lambda > 16$.

Case 2. $P_4 = 1$ and $P_5 \geq 3$.

We set $m_0 = 5$. If $d_5 = 1$, we set $\Lambda = |5K_X|$. Then we are in much better situation than that of $P_3 = 2$ since we have $\theta \geq 2$ (and noting that $\frac{\theta}{m_0} = \frac{2}{5} > \frac{1}{3}$). We omit the details and leave this as an exercise to interested readers.

If $d_5 \geq 2$, we take a sub-pencil $\Lambda \subset |5K_X|$ and Λ induces a fibration $f : X' \rightarrow \Gamma$ onto a smooth complete curve Γ . As we have seen in Case 3 of Corollary 4.10, the general fiber F satisfies $K_{F_0}^2 \geq 2$. For the similar reason, we can take $m_1 = 5$ and $|G| := \text{Mov}[m_1 K_{X'}]|_F|$. Pick a generic irreducible element C in $|G|$. Lemma 2.1 implies $\xi = (\pi^*(K_X) \cdot C) \geq \frac{1}{6}(\sigma^*(K_{F_0}) \cdot C) \geq \frac{1}{3}$. We may write $5\pi^*(K_X)|_F \equiv C + N_5$ for an effective $\mathbb{Q}$ -divisor N_5 on F . For two different generic irreducible curves C_1 and C_2 in $|G|$, we set

$$L_{\lambda,2} := (Q'_\lambda - \frac{1}{\theta}E'_\lambda)|_F - C_1 - C_2 - 2N_5,$$

and

$$L_{\lambda,1} := (Q'_\lambda - \frac{1}{\theta}E'_\lambda)|_F - C - N_5$$

respectively. It is clear that they are both nef and big whenever $\lambda > 15$.

Thanks to the vanishing theorem, we have the surjective map:

$$\begin{aligned} H^0(F, K_F + \lceil L_\lambda - 2N_5 \rceil) &\longrightarrow H^0(C_1, K_{C_1} + \lceil L_{\lambda,2} \rceil|_{C_1} + C_2|_{C_1}) \\ &\oplus H^0(C_2, K_{C_2} + \lceil L_{\lambda,2} \rceil|_{C_2} + C_1|_{C_2}) \end{aligned}$$

if $\lambda > 15$. It is clear that

$$H^0(C_i, K_{C_i} + \lceil L_{\lambda,2} \rceil|_{C_i} + C_{2-i}|_{C_i}) \neq 0$$

since $L_{\lambda,2}$ is nef and big. Hence $|K_F + \lceil (Q'_\lambda - \frac{1}{\theta}E'_\Lambda)|_F - 2N_5 \rceil| = |K_F + \lceil L_\lambda - 2N_5 \rceil|$ separates different general curves C in $|G|$. This also implies that $|K_F + \lceil (Q'_\lambda - \frac{1}{\theta}E'_\Lambda)||$ can distinguish C_1 and C_2 . Now applying the vanishing theorem once more, we get the surjective map:

$$H^0(F, K_F + \lceil L_\lambda - N_5 \rceil) \longrightarrow H^0(C, K_C + \lceil L_{\lambda,1} \rceil|_C)$$

with

$$\deg(\lceil L_{\lambda,1} \rceil|_C) \geq (\lambda - \frac{5}{\theta} - 5)\xi > 2$$

whenever $\lambda > 16$ for $\xi \geq 1/3$. Thus, by Theorem 2.7, $|K_{X'} + \lceil Q'_\lambda \rceil|$ gives a birational map for $\lambda > 16$. So we conclude the statement of the theorem. $\square$

Theorem 8.2. (*=Theorem 1.11*) *Let V be a nonsingular projective 4-fold of general type. Then,*

- (1) *when $p_g(V) \geq 2$, $\Phi_{m,V}$ is birational for all $m \geq 35$;*
- (2) *when $p_g(V) \geq 19$, $\Phi_{m,V}$ is birational for all $m \geq 18$.*

Proof. Let Z be the minimal model of V . We set $m_0 = 1$, $\Lambda = |K_Z|$ and use the set up in 2.1. Thus we have an induced fibration $f : Z' \rightarrow \Gamma$.

First we consider the case $\dim \Gamma = 1$. Recall that we have $M_\Lambda \equiv \theta F$ for a general fiber F of f , where $\theta \geq p_g(Z) - 1$. It is clear that, when $m \geq 3$, $|mK_{Z'}|$ distinguishes different general fibers of f . Pick a general fiber $F = X'$, which is a nonsingular projective 3-fold of general type with $p_g(X') > 0$. Replace by its birational model, we may assume that there is a birational morphism $\nu : X' \rightarrow X$ onto a minimal model. By Lemma 2.1, we have

$$\pi^*(K_Z)|_{X'} \equiv \frac{\theta}{\theta + 1} \nu^*(K_X) + J_1$$

for an effective $\mathbb{Q}$ -divisor J_1 on X' . When m is large, since $(m - 1)\pi^*(K_Z) - X' - \frac{1}{\theta}E'_\Lambda$ is nef and big, Kawamata-Viehweg vanishing implies:

$$\begin{aligned} &|K_{Z'} + \lceil (m - 1)\pi^*(K_Z) - \frac{1}{\theta}E'_\Lambda \rceil|_{X'} \\ &= |K_{X'} + \lceil (m - 1)\pi^*(K_Z) - \frac{1}{\theta}E'_\Lambda \rceil_{X'}| \\ &\succeq |K_{X'} + \lceil R_m \rceil| \end{aligned}$$

where $R_m := ((m-1)\pi^*(K_Z) - X' - \frac{1}{\theta}E'_\Lambda)|_{X'}$. In fact, we have

$$\begin{aligned} R_m &\equiv (m-1-\frac{1}{\theta})\pi^*(K_Z)|_{X'} \\ &\equiv (\frac{m\theta}{\theta+1}-1)\nu^*(K_X) + (m-1-\frac{1}{\theta})J_1 \end{aligned}$$

Since $\frac{m\theta}{\theta+1}-1 > 16$ whenever either $m \geq 18$ and $p_g(Z) \geq 19$ or $m \geq 35$ and $p_g(Z) \geq 2$, Theorem 8.1 implies that $|K_{X'} + \lceil R_m - (m-1-\frac{1}{\theta})J_1 \rceil|$ gives a birational map. Thus statements of the theorem follow in this case.

Next we consider the case $\dim \Gamma \geq 2$. By definition, $\theta = 1$. Clearly it is sufficient to consider $\Phi_{|mK_{Z'}|}|_{X'}$ for a general member $X' \in |M_\Lambda|$. We consider a general X' and, similarly, we may assume that there is a birational morphism $\nu : X' \rightarrow X$ onto a minimal model X . Then Kawamata's extension theorem [20, Theorem A] still implies

$$\pi^*(K_Z)|_{X'} \geq \frac{1}{2}\nu^*(K_X). \quad (8.2)$$

We consider the linear system $|M_\Lambda|_{X'}|$, which may be assumed to be base point free modulo further birational modifications. Pick a generic irreducible element S of this linear system. We clearly have

$$\pi^*(K_Z)|_{X'} \geq M_\Lambda|_{X'} \geq S.$$

Modulo $\mathbb{Q}$ -linear equivalence, one has

$$2S \leq (\pi^*(K_Z) + X')|_{X'} \leq K_{X'}.$$

Thus Kawamata's extension theorem gives

$$\nu^*(K_X)|_S \geq \frac{2}{3}\sigma^*(K_{S_0}) \quad (8.3)$$

where $\sigma : S \rightarrow S_0$ is the contraction onto the minimal model S_0 of S . Both (8.2) and (8.3) imply

$$\pi^*(K_Z)|_S \geq \frac{1}{3}\sigma^*(K_{S_0}).$$

Write $\pi^*(K_Z)|_{X'} \equiv S + H_\Lambda$ where H_Λ is an effective $\mathbb{Q}$ -divisor on X' . Since $R_m - S - H_\Lambda \equiv (m-3)\pi^*(K_Z)|_{X'}$ is nef and big, the vanishing theorem implies

$$\begin{aligned} |K_{X'} + \lceil R_m - H_\Lambda \rceil|_S &= |K_S + \lceil R_m - S - H_\Lambda \rceil|_S \\ &\supseteq |K_S + \lceil R_{m,S} \rceil| \end{aligned}$$

where $R_{m,S} := (R_m - S - H_\Lambda)|_S$. Note that

$$\begin{aligned} R_{m,S} &\equiv (m-3)\pi^*(K_Z)|_S \\ &\equiv \frac{m-3}{3}\sigma^*(K_{S_0}) + E_{m,S} \end{aligned}$$

where $E_{m,S}$ is an effective $\mathbb{Q}$ -divisor on S . Now it is clear by Lemma 2.3 that $|K_S + \lceil R_{m,S} - E_{m,S} \rceil|$ gives a birational map whenever $m \geq 15$.

Again Kawamata-Viehweg vanishing shows that $|K_{X'} + \lceil R_m \rceil|$ distinguishes different elements S . Thus we have shown that $\Phi_{m,Z}$ is birational for all $m \geq 15$ in this case. We are done. $\square$

G. Brown and M. Reid kindly informed us the following interesting canonical 4-folds:

Example 8.3. The general hypersurfaces $W_{36} \subset \mathbb{P}(1, 1, 3, 5, 7, 18)$ and $Y_{36} \subset \mathbb{P}(1, 1, 4, 5, 6, 18)$ have canonical singularities, $p_g = 2$. It is clear that the 17-canonical maps of these two 4-folds are not birational.

Problem 8.4. It is a very interesting problem to find more examples of 4-folds of general type so that Φ_m is not birational for large m .

9. Appendix. Tables

Table F-0

Types	B_X	χ	K_X^3	$\delta(X)$
2a	$\{4 \times (1, 2), (4, 9), (2, 5), (5, 13), 3 \times (1, 3), 2 \times (1, 4)\}$	2	1/1170	18
41	$\{5 \times (1, 2), (4, 9), 2 \times (3, 8), (1, 3), 2 \times (2, 7)\}$	2	1/252	13

Table F-1

Types	B_X	χ	K_X^3	$\delta(X)$
2	$\{4 \times (1, 2), (4, 9), 2 \times (2, 5), (3, 8), 3 \times (1, 3), 2 \times (1, 4)\}$	2	1/360	15
3	$\{6 \times (1, 2), (5, 11), 4 \times (2, 5), (3, 8), 4 \times (1, 3), (2, 7), 2 \times (1, 4)\}$	3	23/9240	15
5.1	$\{7 \times (1, 2), (4, 9), 3 \times (2, 5), (5, 13), 4 \times (1, 3), (3, 11), (1, 4)\}$	3	61/25740	15
5.2	$\{7 \times (1, 2), (4, 9), 2 \times (2, 5), (7, 18), 4 \times (1, 3), (3, 11), (1, 4)\}$	3	1/660	15
5.3	$\{7 \times (1, 2), (4, 9), (2, 5), (9, 23), 4 \times (1, 3), (3, 11), (1, 4)\}$	3	47/45540	15
5a	$\{7 \times (1, 2), (4, 9), (11, 28), 4 \times (1, 3), (3, 11), (1, 4)\}$	3	1/1386	15
5b	$\{7 \times (1, 2), (4, 9), 3 \times (2, 5), (5, 13), 4 \times (1, 3), (4, 15)\}$	3	1/1170	15
53a	$\{3 \times (1, 2), (4, 9), 2 \times (2, 5), (5, 13), 3 \times (1, 3), (1, 5)\}$	2	1/1170	15

Table F-2

Types	B_X	χ	K_X^3	$\delta(X)$
1	$\{5 \times (1, 2), (3, 7), 3 \times (2, 5), 3 \times (1, 3), (3, 11)\}$	2	3/770	14
4	$\{7 \times (1, 2), (4, 9), 4 \times (2, 5), (4, 11), 3 \times (1, 3), (2, 7), 2 \times (1, 4)\}$	3	13/3465	14
4.5	$\{7 \times (1, 2), (4, 9), 4 \times (2, 5), (5, 14), 2 \times (1, 3), (2, 7), 2 \times (1, 4)\}$	3	1/630	14
5	$\{7 \times (1, 2), (4, 9), 4 \times (2, 5), (3, 8), 4 \times (1, 3), (3, 11), (1, 4)\}$	3	17/3960	14
5.4	$\{7 \times (1, 2), (4, 9), 4 \times (2, 5), (3, 8), 4 \times (1, 3), (4, 15)\}$	3	1/360	14
6	$\{9 \times (1, 2), 2 \times (3, 7), (2, 5), (4, 11), 4 \times (1, 3), 2 \times (2, 7), (1, 5)\}$	3	1/462	14
7	$\{5 \times (1, 2), (4, 9), (3, 7), 5 \times (1, 3), (2, 7), (1, 5)\}$	2	1/630	14
7a	$\{5 \times (1, 2), (7, 16), 5 \times (1, 3), (2, 7), (1, 5)\}$	2	1/1680	14
10	$\{8 \times (1, 2), (4, 9), (3, 7), 2 \times (3, 8), 5 \times (1, 3), (2, 7), (1, 4), (1, 5)\}$	3	1/630	14
11	$\{9 \times (1, 2), 2 \times (3, 7), (3, 8), (4, 11), 3 \times (1, 3), (3, 10), (1, 4), (1, 5)\}$	3	3/3080	14
12	$\{9 \times (1, 2), (4, 9), (2, 5), 2 \times (3, 8), 4 \times (1, 3), 2 \times (2, 7), (1, 5)\}$	3	1/252	14
12.1	$\{9 \times (1, 2), (4, 9), (5, 13), (3, 8), 4 \times (1, 3), 2 \times (2, 7), (1, 5)\}$	3	67/32760	14
12a	$\{9 \times (1, 2), (4, 9), (8, 21), 4 \times (1, 3), 2 \times (2, 7), (1, 5)\}$	3	1/630	14
14	$\{10 \times (1, 2), (3, 7), 2 \times (2, 5), 2 \times (3, 8), 6 \times (1, 3), 2 \times (2, 7), (1, 4), (1, 5)\}$	4	1/770	14
15	$\{11 \times (1, 2), (4, 9), (3, 7), 2 \times (2, 5), (3, 8), (4, 11), 5 \times (1, 3), 2 \times (2, 7), (1, 4), (1, 5)\}$	4	71/27720	14
15.1	$\{11 \times (1, 2), (4, 9), (3, 7), 2 \times (2, 5), (7, 19), 5 \times (1, 3), 2 \times (2, 7), (1, 4), (1, 5)\}$	4	47/23940	14
15.2	$\{11 \times (1, 2), (7, 16), 2 \times (2, 5), (3, 8), (4, 11), 5 \times (1, 3), 2 \times (2, 7), (1, 4), (1, 5)\}$	4	29/18480	14
16	$\{11 \times (1, 2), (4, 9), (3, 7), 2 \times (2, 5), 2 \times (3, 8), 6 \times (1, 3), (2, 7), (3, 11), (1, 5)\}$	4	43/13860	14

16.1	$\{11 \times (1, 2), (4, 9), (3, 7), (2, 5), (5, 13), (3, 8), 6 \times (1, 3), (2, 7), (3, 11), (1, 5)\}$	4	85/72072	14
16.2	$\{11 \times (1, 2), (7, 16), 2 \times (2, 5), 2 \times (3, 8), 6 \times (1, 3), (2, 7), (3, 11), (1, 5)\}$	4	13/6160	14
16.4	$\{11 \times (1, 2), (7, 16), 2 \times (2, 5), 2 \times (3, 8), 6 \times (1, 3), (5, 18), (1, 5)\}$	4	1/720	14
16.5	$\{11 \times (1, 2), (4, 9), (3, 7), 2 \times (2, 5), 2 \times (3, 8), 6 \times (1, 3), (5, 18), (1, 5)\}$	4	1/420	14
17	$\{9 \times (1, 2), 2 \times (3, 7), 2 \times (4, 11), 3 \times (1, 3), (2, 7), (1, 4), (1, 5)\}$	3	3/1540	14
18	$\{9 \times (1, 2), 2 \times (3, 7), (3, 8), (4, 11), 4 \times (1, 3), (3, 11), (1, 5)\}$	3	23/9240	14
18b	$\{9 \times (1, 2), 2 \times (3, 7), (7, 19), 4 \times (1, 3), (3, 11), (1, 5)\}$	3	83/43890	14
20	$\{7 \times (1, 2), 2 \times (4, 9), (2, 5), (3, 8), 6 \times (1, 3), (2, 7), (1, 4), (1, 5)\}$	3	1/504	14
21	$\{6 \times (1, 2), (4, 9), (3, 8), 3 \times (1, 3), (3, 10), (1, 5)\}$	2	1/360	14
23	$\{8 \times (1, 2), (4, 9), (3, 7), (2, 5), (4, 11), 4 \times (1, 3), (3, 10), (1, 4), (1, 5)\}$	3	19/13860	14
25	$\{9 \times (1, 2), (5, 11), (4, 9), 3 \times (2, 5), (3, 8), 7 \times (1, 3), 2 \times (2, 7), (1, 4), (1, 5)\}$	4	47/27720	14
25a	$\{9 \times (1, 2), (9, 20), 3 \times (2, 5), (3, 8), 7 \times (1, 3), 2 \times (2, 7), (1, 4), (1, 5)\}$	4	1/840	14
26	$\{10 \times (1, 2), 2 \times (4, 9), 3 \times (2, 5), (4, 11), 6 \times (1, 3), 2 \times (2, 7), (1, 4), (1, 5)\}$	4	41/13860	14
27	$\{10 \times (1, 2), 2 \times (4, 9), 3 \times (2, 5), (3, 8), 7 \times (1, 3), (2, 7), (3, 11), (1, 5)\}$	4	97/27720	14
27.3	$\{10 \times (1, 2), 2 \times (4, 9), 3 \times (2, 5), (3, 8), 7 \times (1, 3), (5, 18), (1, 5)\}$	4	1/360	14
28	$\{5 \times (1, 2), (5, 11), (3, 8), 4 \times (1, 3), (2, 7), (1, 5)\}$	2	23/9240	14
29	$\{6 \times (1, 2), (4, 9), (4, 11), 3 \times (1, 3), (2, 7), (1, 5)\}$	2	13/3465	14
29.1	$\{6 \times (1, 2), (4, 9), (5, 14), 2 \times (1, 3), (2, 7), (1, 5)\}$	2	1/630	14
30	$\{7 \times (1, 2), (5, 11), (3, 7), (2, 5), (4, 11), 5 \times (1, 3), (2, 7), (1, 4), (1, 5)\}$	3	1/924	14
31	$\{7 \times (1, 2), (5, 11), (3, 7), (2, 5), (3, 8), 6 \times (1, 3), (3, 11), (1, 5)\}$	3	1/616	14
32	$\{8 \times (1, 2), (4, 9), (3, 7), (2, 5), (4, 11), 5 \times (1, 3), (3, 11), (1, 5)\}$	3	2/693	14
32a	$\{8 \times (1, 2), (7, 16), (2, 5), (4, 11), 5 \times (1, 3), (3, 11), (1, 5)\}$	3	1/528	14
33	$\{5 \times (1, 2), 2 \times (3, 7), (3, 8), (1, 3), (3, 10), (2, 7)\}$	2	1/840	14
34	$\{7 \times (1, 2), (4, 9), (3, 7), 2 \times (2, 5), (3, 8), 3 \times (1, 3), 3 \times (2, 7)\}$	3	1/360	14
34a	$\{7 \times (1, 2), (7, 16), 2 \times (2, 5), (3, 8), 3 \times (1, 3), 3 \times (2, 7)\}$	3	1/560	14
35	$\{5 \times (1, 2), 2 \times (3, 7), (4, 11), (1, 3), 2 \times (2, 7)\}$	2	1/462	14
36	$\{4 \times (1, 2), (4, 9), (3, 7), (2, 5), 2 \times (1, 3), (3, 10), (2, 7)\}$	2	1/630	14
36a	$\{4 \times (1, 2), (7, 16), (2, 5), 2 \times (1, 3), (3, 10), (2, 7)\}$	2	1/1680	14
36b	$\{4 \times (1, 2), (4, 9), (3, 7), (2, 5), 2 \times (1, 3), (5, 17)\}$	2	4/5355	14
37	$\{6 \times (1, 2), 2 \times (4, 9), 3 \times (2, 5), 4 \times (1, 3), 3 \times (2, 7)\}$	3	1/315	14
38	$\{3 \times (1, 2), (5, 11), (3, 7), (2, 5), 3 \times (1, 3), 2 \times (2, 7)\}$	2	1/770	14
39	$\{7 \times (1, 2), (4, 9), (3, 7), (2, 5), 2 \times (3, 8), 2 \times (1, 3), (3, 10), (2, 7), (1, 4)\}$	3	1/630	14
40	$\{9 \times (1, 2), 2 \times (4, 9), 3 \times (2, 5), 2 \times (3, 8), 4 \times (1, 3), 3 \times (2, 7), (1, 4)\}$	4	1/315	14
42	$\{6 \times (1, 2), (5, 11), (3, 7), (2, 5), 2 \times (3, 8), 3 \times (1, 3), 2 \times (2, 7), (1, 4)\}$	3	1/770	14
43	$\{7 \times (1, 2), (4, 9), (3, 7), (2, 5), (3, 8), (4, 11), 2 \times (1, 3), 2 \times (2, 7), (1, 4)\}$	3	71/27720	14
43.1	$\{7 \times (1, 2), (7, 16), (2, 5), (3, 8), (4, 11), 2 \times (1, 3), 2 \times (2, 7), (1, 4)\}$	3	29/18480	14
43c	$\{7 \times (1, 2), (7, 16), (2, 5), (7, 19), 2 \times (1, 3), 2 \times (2, 7), (1, 4)\}$	3	31/31920	14
43.2	$\{7 \times (1, 2), (4, 9), (3, 7), (2, 5), (7, 19), 2 \times (1, 3), 2 \times (2, 7), (1, 4)\}$	3	47/23940	14
44	$\{7 \times (1, 2), (4, 9), (3, 7), (2, 5), 2 \times (3, 8), 3 \times (1, 3), (2, 7), (3, 11)\}$	3	43/13860	14
44.1	$\{7 \times (1, 2), (4, 9), (3, 7), (5, 13), (3, 8), 3 \times (1, 3), (2, 7), (3, 11)\}$	3	85/72072	14
44.2	$\{7 \times (1, 2), (4, 9), (3, 7), (2, 5), 2 \times (3, 8), 3 \times (1, 3), (5, 18)\}$	3	1/420	14
44.3	$\{7 \times (1, 2), (7, 16), (2, 5), 2 \times (3, 8), 3 \times (1, 3), (2, 7), (3, 11)\}$	3	13/6160	14
44c	$\{7 \times (1, 2), (7, 16), (2, 5), 2 \times (3, 8), 3 \times (1, 3), (5, 18)\}$	3	1/720	14
45	$\{3 \times (1, 2), 2 \times (4, 9), (3, 8), 3 \times (1, 3), (2, 7), (1, 4)\}$	2	1/504	14
46	$\{6 \times (1, 2), 2 \times (4, 9), 2 \times (2, 5), (3, 8), 3 \times (1, 3), (3, 10), (2, 7), (1, 4)\}$	3	1/504	14
46b	$\{6 \times (1, 2), 2 \times (4, 9), 2 \times (2, 5), (3, 8), 3 \times (1, 3), (5, 17), (1, 4)\}$	3	7/6120	14
48	$\{4 \times (1, 2), (4, 9), (3, 7), (4, 11), (1, 3), (3, 10), (1, 4)\}$	2	19/13860	14
49	$\{5 \times (1, 2), (5, 11), (4, 9), 2 \times (2, 5), (3, 8), 4 \times (1, 3), 2 \times (2, 7), (1, 4)\}$	3	47/27720	14
49a	$\{5 \times (1, 2), (9, 20), 2 \times (2, 5), (3, 8), 4 \times (1, 3), 2 \times (2, 7), (1, 4)\}$	3	1/840	14
50	$\{6 \times (1, 2), 2 \times (2, 9), 2 \times (2, 5), (4, 11), 3 \times (1, 3), 2 \times (2, 7), (1, 4)\}$	3	41/13860	14
51	$\{6 \times (1, 2), 2 \times (4, 9), 2 \times (2, 5), (3, 8), 4 \times (1, 3), (2, 7), (3, 11)\}$	3	97/27720	14
51.1	$\{6 \times (1, 2), 2 \times (4, 9), (2, 5), (5, 13), 4 \times (1, 3), (2, 7), (3, 11)\}$	3	71/45045	14
52	$\{4 \times (1, 2), (3, 7), 2 \times (2, 5), 2 \times (3, 8), 2 \times (1, 3), (1, 5)\}$	2	1/420	14
53	$\{3 \times (1, 2), (4, 9), 3 \times (2, 5), (3, 8), 3 \times (1, 3), (1, 5)\}$	2	1/360	14
54	$\{2 \times (1, 2), 2 \times (3, 7), 3 \times (2, 5), (3, 8), (1, 3), (2, 7)\}$	2	1/840	14

56	$\{(1, 2), (4, 9), (3, 7), 4 \times (2, 5), 2 \times (1, 3), (2, 7)\}$	2	1/630	14
58	$\{4 \times (1, 2), (4, 9), (3, 7), 4 \times (2, 5), 2 \times (3, 8), 2 \times (1, 3), (2, 7), (1, 4)\}$	3	1/630	14
59	$\{2 \times (1, 2), 2 \times (3, 7), 2 \times (2, 5), (3, 8), (4, 11), (1, 4)\}$	2	3/3080	14
60	$3 \times (1, 2), 2 \times (4, 9), 5(2, 5), (3, 8), 3 \times (1, 3), (2, 7), (1, 4)\}$	3	1/504	14
62	$\{(1, 2), (4, 9), (3, 7), 3 \times (2, 5), (4, 11), (1, 3), (1, 4)\}$	2	19/13860	14

Table II-1

No.	$B^0(X)$	K_X^3	χ	(P_3, P_4, P_5, P_6)
1	$\{5 * (1, 2), 2 * (1, 3)\}$	$1/6$	0	$(3, 5, 7, 11)$
2	$\{5 * (1, 2), (1, 3), (1, 4)\}$	$1/12$	0	$(3, 5, 6, 9)$
3	$\{18 * (1, 2), (1, 3), \}$	$1/3$	1	$(1, 5, 6, 13)$
4	$\{(18 - 4t) * (1, 2), 3t * (1, 3), (1, 4)\}, t = 0, 1, 2$	$1/4$	1	$(1 + t, 5, 5 + t, 11 + t)$
5	$\{(18 - 4t) * (1, 2), 3t * (1, 3), (1, 5)\}, 5 \leq r \leq 12; t = 0, 1, 2$	$1/r$	1	$(1 + t, 5, 5 + t, 10 + t)$
6	$\{(17 - 4t) * (1, 2), (2 + 3t) * (1, 3)\}, t = 0, 1, 2$	$1/6$	1	$(1 + t, 4, 4 + t, 9 + t)$
7	$\{(14 - 4t) * (1, 2), (2 + 3t) * (1, 3), 2 * (1, 4)\}, t = 0, 1$	$1/6$	1	$(2 + t, 5, 5 + t, 10 + t)$
8	$\{(14 - 4t) * (1, 2), (2 + 3t) * (1, 3), (1, 4), (1, 5)\}, t = 0, 1$	$7/60$	1	$(2 + t, 5, 5 + t, 9 + t)$
9	$\{(14 - 4t) * (1, 2), (2 + 3t) * (1, 3), (1, 4), (1, 6)\}, t = 0, 1$	$1/12$	1	$(2 + t, 5, 5 + t, 9 + t)$
10	$\{(14 - 4t) * (1, 2), (1 + 3t) * (1, 3), 3 * (1, 4)\}, t = 0, 1$	$1/12$	1	$(2 + t, 5, 4 + t, 8 + t)$
11	$\{(17 - 4t) * (1, 2), (1 + 3t) * (1, 3), (1, 4)\}, t = 0, 1, 2$	$1/12$	1	$(1 + t, 4, 3 + t, 7 + t)$

Table II-2

No.	$B^0(X)$	K_X^3	χ	(P_3, P_4, P_5, P_6)
1	$\{27 * (1, 2), 2 * (1, 3), (1, r)\}$	$\frac{1}{6} + \frac{1}{r}$	2	$(0, 5, 5, 13)$
2	$\{(27 - 4t) * (1, 2), (1 + 3t) * (1, 3), 2 * (1, 4)\}, t = 0, 1.$	$1/3$	2	$(t, 5, 4 + t, 12 + t)$
3	$\{(27 - 4t) * (1, 2), (1 + 3t) * (1, 3), (1, 4), (1, r)\}, 5 \leq r; t = 0, 1, 2.$	$\frac{1}{12} + \frac{1}{r}$	2	$(t, 5, 4 + t, 11 + t)$
4	$\{(27 - 4t) * (1, 2), (1 + 3t) * (1, 3), (1, r_1), (1, r_2)\}, (r_1, r_2) \in I_4; t = 0, 1, 2, 3.$	$\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{6}$	2	$(t, 5, 4 + t, 10 + t)$
5	$\{(26 - 4t) * (1, 2), (4 + 3t) * (1, 3)\}, t = 0, 1.$	$1/3$	2	$(t, 4, 4 + t, 12 + t)$
6	$\{(27 - 4t) * (1, 2), 3t * (1, 3), 3 * (1, 4)\}, t = 0, 1, 2, 3.$	$1/4$	2	$(t, 5, 3 + t, 10 + t)$
7	$\{(27 - 4t) * (1, 2), 3t * (1, 3), 2 * (1, 4), (1, r)\}, 5 \leq r \leq 12; t = 0, 1, 2, 3.$	$1/r$	2	$(t, 5, 3 + t, 9 + t)$
8	$\{(27 - 4t) * (1, 2), 3t * (1, 3), (1, 4), (1, r_1), (1, r_2)\}, (r_1, r_2) \in I_3; t = 0, 1, 2, 3.$	$\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{4}$	2	$(t, 5, 3 + t, 8 + t)$
9	$\{(27 - 4t) * (1, 2), 3t * (1, 3), 3 * (1, 5)\}, t = 0, 1, 2, 3.$	$1/10$	2	$(t, 5, 3 + t, 7 + t)$
10	$\{(26 - 4t) * (1, 2), (3 + 3t) * (1, 3), (1, 4)\}, t = 0, 1, 2, 3.$	$1/4$	2	$(0, 4, 3 + t, 10 + t)$
11	$\{(26 - 4t) * (1, 2), (3 + 3t) * (1, 3), (1, r)\}, 5 \leq r \leq 12; t = 0, 1, 2, 3.$	$1/r$	2	$(0, 4, 3 + t, 9 + t)$
12	$\{(25 - 4t) * (1, 2), (5 + 3t) * (1, 3)\}, t = 0, 1, 2, 3.$	$1/6$	2	$(t, 3, 2 + t, 8 + t)$
13	$\{(26 - 4t) * (1, 2), (2 + 3t) * (1, 3), 2 * (1, 4)\}, t = 0, 1, 2, 3.$	$1/6$	2	$(t, 4, 2 + t, 8 + t)$
14	$\{(26 - 4t) * (1, 2), (2 + 3t) * (1, 3), (1, 4), (1, 5)\}, t = 0, 1, 2, 3.$	$7/60$	2	$(t, 4, 2 + t, 7 + t)$
15	$\{(26 - 4t) * (1, 2), (2 + 3t) * (1, 3), (1, 4), (1, 6)\}, t = 0, 1, 2, 3.$	$1/12$	2	$(t, 4, 2 + t, 7 + t)$
16	$\{(25 - 4t) * (1, 2), (4 + 3t) * (1, 3), (1, 4)\}, t = 0, 1, 2, 3.$	$1/12$	2	$(t, 3, 1 + t, 6 + t)$
17	$\{(26 - 4t) * (1, 2), (1 + 3t) * (1, 3), 3 * (1, 4)\}, t = 0, 1, 2, 3.$	$1/12$	2	$(t, 4, 1 + t, 6 + t)$

where

$$\begin{aligned}
I_4 &= \{(r_1, r_2) | 1/r_1 + 1/r_2 \geq 1/4, r_i \geq 5\} \\
&= \{(5, 5), \dots, (5, 20), (6, 6), \dots, (6, 12), (7, 7), (7, 8), (7, 9), (8, 8)\} \\
I_3 &= \{(r_1, r_2) | 1/r_1 + 1/r_2 \geq 1/3, r_i \geq 5\} \\
&= \{(5, 5), (5, 6), (5, 7), (6, 6)\}.
\end{aligned}$$

Table II-3

	$B^0(X)$	K_X^3	χ	(P_3, P_4, P_5, P_6)
1	$\{32 * (1, 2), 5 * (1, 3), 2 * (1, 4), (1, r)\}, 5 \leq r.$	$\frac{1}{6} + \frac{1}{r}$	3	$(0, 5, 4, 13)$
2	$\{(32 - 4t) * (1, 2), (5 + 3t) * (1, 3), (1, 4), (1, r_1), (1, r_2)\}, (r_1, r_2) \in I_6, t \leq 1.$	$\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{12}$	3	$(t, 5, 4 + t, 12 + t)$
3	$\{(32 - 4t) * (1, 2), (5 + 3t) * (1, 3), (1, r_1), (1, r_2), (1, r_3)\}, (r_1, r_2, r_3) \in J, t \leq 2.$	$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{3}$	3	$(t, 5, 4 + t, 11 + t)$
4	$\{(31 - 4t) * (1, 2), (7 + 3t) * (1, 3), 2 * (1, 4)\}, t \leq 1.$	1/3	3	$(t, 4, 3 + t, 12 + t)$
5	$\{(31 - 4t) * (1, 2), (7 + 3t) * (1, 3), (1, 4), (1, r)\}, 5 \leq r; t \leq 2.$	$\frac{1}{12} + \frac{1}{r}$	3	$(t, 4, 3 + t, 11 + t)$
6	$\{(31 - 4t) * (1, 2), (7 + 3t) * (1, 3), (1, r_1), (1, r_2)\}, (r_1, r_2) \in I_4; t \leq 3.$	$\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{6}$	3	$(t, 4, 3 + t, 10 + t)$
7	$\{(30 - 4t) * (1, 2), (10 + 3t) * (1, 3)\}, t = 0, 1.$	1/3	3	$(t, 3, 3 + t, 12 + t)$
8	$\{(31 - 4t) * (1, 2), (6 + 3t) * (1, 3), 3 * (1, 4)\}, t = 0, 1, 2, 3.$	1/4	3	$(t, 4, 2 + t, 10 + t)$
9	$\{(31 - 4t) * (1, 2), (6 + 3t) * (1, 3), 2 * (1, 4), (1, r)\}, 5 \leq r \leq 12; t = 0, 1, 2, 3.$	1/r	3	$(t, 4, 2 + t, 9 + t)$
10	$\{(31 - 4t) * (1, 2), (6 + 3t) * (1, 3), (1, 4), (1, r_1), (1, r_2)\}, (r_1, r_2) \in I_3; t \leq 3.$	$\frac{1}{r_1} + \frac{1}{r_2} - \frac{1}{4}$	3	$(t, 4, 2 + t, 8 + t)$
11	$\{(31 - 4t) * (1, 2), (6 + 3t) * (1, 3), 3 * (1, 5)\}, t = 0, 1, 2, 3.$	1/10	3	$(t, 4, 2 + t, 7 + t)$
12	$\{(30 - 4t) * (1, 2), (9 + 3t) * (1, 3), (1, 4)\}, t = 0, 1, 2, 3.$	1/4	3	$(0, 3, 2 + t, 10 + t)$
13	$\{(30 - 4t) * (1, 2), (9 + 3t) * (1, 3), (1, r)\}, 5 \leq r \leq 12; t = 0, 1, 2, 3.$	1/r	3	$(0, 3, 2 + t, 9 + t)$
14	$\{(30 - 4t) * (1, 2), (8 + 3t) * (1, 3), 2 * (1, 4)\}, t = 0, 1, 2, 3.$	1/6	3	$(t, 3, 1 + t, 8 + t)$
15	$\{(30 - 4t) * (1, 2), (8 + 3t) * (1, 3), (1, 4), (1, 5)\}, t = 0, 1, 2, 3.$	7/60	3	$(t, 3, 1 + t, 7 + t)$
16	$\{(30 - 4t) * (1, 2), (8 + 3t) * (1, 3), (1, 4), (1, 6)\}, t = 0, 1, 2, 3.$	1/12	3	$(t, 3, 1 + t, 7 + t)$
17	$\{(30 - 4t) * (1, 2), (7 + 3t) * (1, 3), 3 * (1, 4)\}, t = 0, 1, 2, 3.$	1/12	3	$(t, 3, t, 6 + t)$

where

$$\begin{aligned}
I_4 &= \{(r_1, r_2) | 1/r_1 + 1/r_2 \geq 1/4, r_i \geq 5\} \\
&= \{(5, 5), \dots, (5, 20), (6, 6), \dots, (6, 12), (7, 7), (7, 8), (7, 9), (8, 8)\} \\
I_3 &= \{(r_1, r_2) | 1/r_1 + 1/r_2 \geq 1/3, r_i \geq 5\} \\
&= \{(5, 5), (5, 6), (5, 7), (6, 6)\}. \\
I_6 &= \{(r_1, r_2) | 1/r_1 + 1/r_2 \geq 1/6, r_i \geq 5\} \\
&= \{(5, s_5), (6, s_6), (7, s_7), (8, s_8), (9, s_9), (10, s_{10}), (11, 11), (11, 12), (11, 13), (12, 12)\}, \\
&\quad 5 \leq s_1, 6 \leq s_2, 7 \leq s_7 \leq 42, 8 \leq s_8 \leq 24, 9 \leq s_9 \leq 18, 10 \leq s_{10} \leq 15. \\
J &= \{(r_1, r_2, r_3) | 1/r_1 + 1/r_2 + 1/r_3 \geq 5/12, r_i \geq 5\} \\
&= \{(5, 5, s_1), (5, 6, s_2), (5, 7, s_3), (5, 8, 8), (5, 8, 9), (5, 8, 10), (5, 9, 9), (6, 6, s_4), (6, 7, 7), (6, 7, 8), \\
&\quad (6, 7, 9), (6, 8, 8), (7, 7, 7)\}, 5 \leq s_1 \leq 60, 6 \leq s_2 \leq 20, 7 \leq s_3 \leq 13, 6 \leq s_4 \leq 12.
\end{aligned}$$

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