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REPORT ON REPRESENTATION THEORY OF LIE SUPERALGEBRAS AND INFINITE-DIMENSIONAL LIE ALGEBRAS (1/3)

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The purpose of this note is to report on [CWZ] on certain connections between the representation theories of Lie superalgebras and Lie algebras funded by NSC-grant 93-2115-M-002-015 of the R.O.C. For more details the interested reader is referred to the paper. We are grateful to the NSC for supporting this work.

1. Introduction

In 1977 Kac classified finite-dimensional complex simple Lie superalgebras [K]. Since then the representation theory of these Lie superalgebras has been studied extensively. Especially the problem of finding a character formula for the finite-dimensional irreducible representations. It turned out that the problem of character for the most basic Lie superalgebras, e.g. general linear Lie superalgebras and ortho-sympletic Lie superalgebras, is the most challenging.

For the general linear Lie superalgebra many partial results have been obtained. In 1996 Serganova [S] was the first to obtain a complete solution. She defined Kazhdan-Lusztig polynomials [KL], and found an algorithm to compute them. But her method is difficult to implement in practice.

In 2003 Brundan [B] gave a satisfying solution to the problem by reformulating these polynomials as coefficients of a certain transition matrix on a Fock space.

In [CWZ] we offer another different solution to this problem.

2. Combinatorial Character Formula for $gl(m|n)$

Let $gl(m|n)$ be the general linear superalgebra over $\mathbb{C}$ and let $\mathbb{C}^{m|n}$ be the complex superspace of dimension $(m|n)$. Let $T^k = (\mathbb{C}^{m|n})^{\otimes k}$ be the k-th tensor power, on which $gl(m|n)$ acts and it is known that T^k is completely reducible. By [BR, Sv] the irreducible modules of $gl(m|n)$ that appear in T^k are precisely of the following form: Let λ be a partition lying in the $(m|n)$-hook, i.e. $\lambda_1 \leq n$, where

$$\lambda = (\lambda_{-m}, \cdots, \lambda_{-1}; \lambda_1, \lambda_2, \cdots),$$

We write $\lambda = (\lambda^{<0}; \lambda^{>0})$, where

$$\lambda^{<0} = (\lambda_{-m}, \cdots, \lambda_{-1}), \quad \lambda^{>0} = (\lambda_1, \lambda_2, \cdots).$$

Let $\lambda^\natural := (\lambda^{<0}; (\lambda^{>0})')$. Then the irreducible representations of $gl(m|n)$ that appear in T^k are precisely of highest weights $\lambda^\natural$ such that $|\lambda| = k$.

In what follows for μ a highest weight of $gl(m|n)$ we let $L_n(\mu)$ denote the irreducible module of highest weight μ.

The character of $L_n(\lambda^\natural)$ [BR, Sv] is given by the *Hook Schur polynomial* in $x_{-m},\cdots,x_{-1}$ and $x_1,\cdots,x_n$, which is as follows: Let $s_\lambda(x_{-m},\cdots,x_{-1};x_1,\cdots,x_n,\cdots)$ be the Schur function associated to the partition λ and regard it as a character of the infinite-dimensional Lie algebra $gl(m+\infty)$. We will write $\mathfrak{L}_\infty(\lambda)$ or simply just $\mathfrak{L}(\lambda)$ for the irreducible representation of $gl(m+\infty)$ of highest weight λ.

Let ω_+ be the involution defined by

$$\omega_+(\prod_{i=1}^\infty \frac{1}{1-tx_i}) = \prod_{i=1}^\infty (1+tx_i).$$

Here t is an indeterminate. We apply ω_+ to $s_\lambda(x_{-m},\cdots,x_{-1};x_1,\cdots)$ to obtain the so-called Hook Schur function associated to $\lambda^\natural$. Now we set the variables $x_{n+1}=x_{n+2}=\cdots=0$, and the resulting polynomial is precisely the Hook Schur polynomial associated to the partition $\lambda^\natural$. That is

$$\mathrm{ch}_{L_n(\lambda^\natural)} = \omega_+(\mathrm{ch}_{\mathfrak{L}(\lambda)})(x_{-m},\cdots,x_{-1};x_1,\cdots,x_n,0,\cdots). \tag{2.1}$$

In fact one can let $n\to\infty$ and one thus obtain

$$\mathrm{ch}_{L(\lambda^\natural)} = \omega_+(\mathrm{ch}_{\mathfrak{L}(\lambda)}), \tag{2.2}$$

where $L(\lambda^\natural)$ denotes the irreducible $gl(m|\infty)$-module of highest weight $\lambda^\natural$.

The derivation of (2.1) and (2.2) relies heavily the complete reducibility structure of T^k and also not all the finite-dimensional highest weights of $gl(m|n)$ are of the form $\lambda^\natural$, where λ lies in the $(m|n)$-hook.

In [CWZ] it is shown that (2.1) and (2.2) indeed hold for general finite-dimensional highest weight modules of $gl(m|n)$. But we need to be careful, as one cannot simply apply ω_+ to any character of $L(\mu)$, for any μ a $gl(m|n)$-highest weight. First note that if μ is a finite-dimensional $gl(m|n)$-highest weight, then writing

$$\mu = (\mu_{-m},\cdots,\mu_{-1};\mu_1,\cdots,\mu_n),$$

one has $\mu_i-\mu_{i+1}\in\mathbb{Z}_+$, for $i=-m,\cdots,-2,1,\cdots,n-1$. Let $\mathfrak{h}$ be the standard Cartan subalgebra and let $x_j=e^{\delta_j}$. Here δ_j is the j-th fundamental weight.

Recall that $gl(m|n)$ has a standard consistent $\mathbb{Z}$-gradation

$$gl(m|n) = gl(m|n)_{-1}\oplus gl(m|n)_0\oplus gl(m|n)_{+1},$$

where $gl(m|n)_0 = gl(m)\oplus gl(n)$. Let $L^0_n(\mu)$ be the irreducible $gl(m)\oplus gl(n)$-module of highest weight μ and extend it trivially to a $gl(m|n)_{+1}\oplus gl(m|n)_0$-module. The Kac module is

$$K_n(\mu) = \mathrm{Ind}^{gl(m|n)}_{gl(m|n)_{\geq 0}} L^0_n(\mu).$$

By Kac's results one knows that $K_n(\mu)$ is irreducible if and only if μ is a typical weight. Since the character of $K_n(\mu)$ is easy, the typical case can be ignored.

Now by tensoring with the one-dimensional representation of $gl(m|n)$ of highest weight $(\alpha,\cdots,\alpha;-\alpha,\cdots,-\alpha)$, $\alpha\in\mathbb{C}$, if necessary, we may restrict ourselves to the case when μ is integral. Now tensoring with $(-p,\cdots,-p;p,\cdots,p)$, $p\in\mathbb{Z}_+$, if necessary, we may restrict ourselves to the case when μ has the form that $(\mu_1,\cdots,\mu_n)$ is a partition. Now for such a μ it is easy to see that the character of $L_n(\mu)$ is a symmetric function in the variables $x_1,\cdots,x_n$, and so we may apply the involution ω_+ in the limiting case $n \to \infty$. We now can state the following theorem.

Theorem 2.1. [CWZ] *For μ as above we have*

$$ch_{L_n(\mu)}(x_{-m}, \cdots, x_{-1}; x_1, \cdots, x_n) = \omega_+(ch_{\mathcal{L}(\mu^\natural)})(x_{-m}, \cdots, x_{-1}; x_1, \cdots, x_n, 0, 0, \cdots),$$

where we regard $\mu^\natural$ as a $gl(m+\infty)$-highest weight.

We will explain below how to prove this theorem.

3. Brundan's Kazhdan-Lusztig Theory of $gl(m|n)$

Motivated by [LLT] Brundan [B] shows that the Kazhdan-Lusztig polynomials for the finite-dimensional representations of $gl(m|n)$ can be realized as coefficients of a certain transition matrix between the standard monomial basis and the dual canonical basis on certain Fock space. We will recall some of his results that we will need in the sequel.

Let $U_q(gl(\infty))$ be the quantum group of $gl(\infty)$ acting on its standard module V, whose standard basis we parameterize by integers. Let $W = V^*$ be the corresponding restricted dual, on which the quantum group also acts naturally. Letting v_a, $a \in \mathbb{Z}$, be the standard basis for V we let $w_a \in V^*$ be defined by

$$w_a(v_b) = (-q)^{-a}\delta_{ab}.$$

We can form the module $\Lambda^m(V) \otimes \Lambda^n(W)$, on which $U_q(gl(\infty))$ acts (via a comultiplication).

Let $f : I(m|n) = \{-m, \cdots, -1; 1, \cdots, n\} \to \mathbb{Z}$ with $f(-m) > \cdots > f(-1)$ and $f(1) < f(2) < \cdots < f(n)$ and denote the set of such f by $\mathbb{Z}_+^{m|n}$. Let $X_+^{m|n}$ be the set of finite-dimensional integral highest weights for $gl(m|n)$ and let ρ be the half (super) sum of positive roots of $gl(m|n)$. We have a bijection between $X_+^{m|n} \to \mathbb{Z}_+^{m|n}$ given by

$$\lambda \to f_\lambda,$$

where $f_\lambda(i) = (\lambda + \rho|\delta_i)$. The bilinear form here is the usual super bilinear form and we take ρ to be

$$\rho = (m, m-1, \cdots, 1| -1, -2, -3, \cdots, -n).$$

Thus the super Bruhat ordering on $X_+^{m|n}$ induces a partial ordering $\succcurlyeq$ on $\mathbb{Z}_+^{m|n}$ and we can import the notion of degree of atypicality to $\mathbb{Z}_+^{m|n}$.

One has the standard monomial basis for $\Lambda^m(V) \otimes \Lambda^n(W)$ given by

$$K_f = v_{f(-m)} \wedge \cdots \wedge v_{f(-1)} \otimes w_{f(1)} \wedge \cdots \wedge w_{f(n)}.$$

The space $\Lambda^m(V) \otimes \Lambda^n(W)$ (actually a topological completion of it) admits a bar-involution compatible with $\succcurlyeq$ and hence by the usual arguments going back to Kazhdan and Lusztig the space $\Lambda^m(V) \otimes \Lambda^n(W)$ has two sets of distinguished bar-invariant bases called the canonical and the dual canonical basis, also parameterized by $\mathbb{Z}_+^{m|n}$. We will denote the canonical basis element corresponding to f by U_f and the dual canonical basis element by L_f. We have the following standard theorem.

Theorem 3.1.
$$L_f = \sum_{g \preccurlyeq f} \ell_{gf} K_f, \quad U_f = \sum_{g \preccurlyeq f} u_{gf} K_f,$$
where $u_{ff} = 1$ *and* $u_{gf} \in q\mathbb{Z}[q]$*, and where* $\ell_{ff} = 1$ *and* $\ell_{gf} \in q^{-1}\mathbb{Z}[q^{-1}]$*, for* $g \neq f$*.*

Now let $\lambda \in X_+^{m|n}$ and consider the category $O^{m|n}$ of finite-dimensional $gl(m|n)$-modules. Let $U_n(\lambda)$ denote the tilting module (with $K_n(\lambda)$ at the bottom of its Kac flag). For an element M in $O^{m|n}$ let us denote by $[M]$ the corresponding element in the Grothendieck group.

Theorem 3.2. [B]
$$[L_n(\lambda)] = \sum_{\mu \preccurlyeq \lambda} \ell_{\mu\lambda}(1)[K_n(\mu)], \quad [U_n(\lambda)] = \sum_{\mu \preccurlyeq \lambda} u_{\mu\lambda}(1)[K_n(\mu)],$$
where here $u_{\mu\lambda}$ *and* $\ell_{\mu\lambda}$ *are defined via the bijection.*

In fact there is a natural correspondence between the combinatorial picture and the representation theoretical picture given by
$$K_{f_\lambda} \longleftrightarrow K_n(\lambda), \quad U_{f_\lambda} \longleftrightarrow U_n(\lambda), \quad L_{f_\lambda} \longleftrightarrow L_n(\lambda).$$
Brundan has provided formulas to compute these polynomials, and thus obtains a solution to finding the characters of $gl(m|n)$.

Let $X_{++}^{m|n}$ be the subset of $X_+^{m|n}$ consisting of λ with $(\lambda_1, \cdots, \lambda_n)$ a partition. Now let us restrict to a smaller category $O_+^{m|n}$, which consists of modules whose composition factors are of the form $L_n(\lambda)$, where $\lambda \in X_{++}^{m|n}$. This subcategory carries all the information of the category $O^{m|n}$. One observes from Brundan's formulas for u_{gf} and ℓ_{gf} that these polynomials satisfy certain stability.

4. Kazhdan-Lusztig Theory of $gl(m+n)$

We can form the $U_q(gl(\infty))$-module $\Lambda^m(V) \otimes \Lambda^n(V)$.

Let $f : I(m|n) \to \mathbb{Z}$ with $f(-m) > \cdots > f(-1)$ and $f(1) > f(2) > \cdots > f(n)$ and denote the set of such f by $\mathbb{Z}_+^{m+n}$.

Consider the Lie algebra $gl(m+n)$, which we equip with a $\mathbb{Z}$-gradation, similar to $gl(m|n)$ before:
$$gl(m+n) = gl(m+n)_{-1} \oplus gl(m+n)_0 \oplus gl(m+n)_{+1}.$$
Let $gl(m+n)_{\geq 0} = gl(m+n)_0 \oplus gl(m+n)_{+1}$ and let X_+^{m+n} be the set of $gl(m+n)_{\geq 0}$-locally finite integral highest weights for $gl(m+n)$ and let ρ_c be the half sum of positive roots of $gl(m+n)$. We have a bijection between $X_+^{m+n} \to \mathbb{Z}_+^{m+n}$ given by
$$\lambda \to f_\lambda,$$
where $f_\lambda(i) = (\lambda + \rho_c|\delta_i)_c$. Here we take ρ_c to be
$$\rho_c = (m, m-1, \cdots, 1|0, -1, -2, \cdots, -n+1).$$
Thus the Bruhat ordering on X_+^{m+n} induces a partial ordering $\geq$ on $\mathbb{Z}_+^{m+n}$

One has the standard monomial basis for $\Lambda^m(V) \otimes \Lambda^n(V)$ given by

$$\mathcal{K}_f = v_{f(-m)} \wedge \cdots \wedge v_{f(-1)} \otimes v_{f(1)} \wedge \cdots \wedge v_{f(n)}.$$

The space $\Lambda^m(V) \otimes \Lambda^n(V)$ also admits a bar-involution compatible with $\geq$ and hence $\Lambda^m(V) \otimes \Lambda^n(V)$ has the canonical and the dual canonical basis, parameterized by $\mathbb{Z}_+^{m+n}$. We will denote the canonical basis element corresponding to f by $\mathcal{U}_f$ and the dual canonical basis element by $\mathcal{L}_f$. We have the following standard theorem.

Theorem 4.1.

$$\mathcal{L}_f = \sum_{g \preccurlyeq f} \mathfrak{l}_{gf} \mathcal{K}_f, \quad \mathcal{U}_f = \sum_{g \preccurlyeq f} \mathfrak{u}_{gf} \mathcal{K}_f,$$

where $\mathfrak{u}_{ff} = 1$ *and* $\mathfrak{u}_{gf} \in q\mathbb{Z}[q]$, *and where* $\mathfrak{l}_{ff} = 1$ *and* $\mathfrak{l}_{gf} \in q^{-1}\mathbb{Z}[q^{-1}]$, *for* $g \neq f$.

Now let $\lambda \in X_+^{m+n}$ and consider the category O^{m+n} of finitely generated $gl(m+n)$-modules which are locally finite over $gl(m+n)_{\geq 0}$ and semisimple over $gl(m+n)_0$ such that the weight spaces are integral. Let $\mathcal{K}_n(\lambda)$ denote the generalized Verma module of highest weight λ and let $\mathcal{U}_n(\lambda)$ denote the tilting module (with $\mathcal{K}_n(\lambda)$ at the bottom of its generalized Verma flag). For an element M in O^{m+n} let us denote by $[M]$ the corresponding element in the Grothendieck group.

Theorem 4.2. [CWZ]

$$[\mathcal{L}_n(\lambda)] = \sum_{\mu \leq \lambda} \mathfrak{l}_{\mu\lambda}(1)[\mathcal{K}_n(\mu)], \quad [\mathcal{U}_n(\lambda)] = \sum_{\mu \leq \lambda} \mathfrak{u}_{\mu\lambda}(1)[\mathcal{K}_n(\mu)],$$

where here again $\mathfrak{u}_{\mu\lambda}$ *and* $\mathfrak{l}_{\mu\lambda}$ *are defined via the above bijection.*

Similarly to the super picture there is a natural correspondence between the combinatorial picture and the representation theoretical picture given by

$$\mathcal{K}_{f_\lambda} \longleftrightarrow \mathcal{K}_n(\lambda), \quad \mathcal{U}_{f_\lambda} \longleftrightarrow \mathcal{U}_n(\lambda), \quad \mathcal{L}_{f_\lambda} \longleftrightarrow \mathcal{L}_n(\lambda).$$

One can obtain quite explicit formulas for these polynomials quite similar to Brundan's formulas for $gl(m|n)$.

Let X_{++}^{m+n} be the subset of X_+^{m+n} consisting of λ with $(\lambda_1, \cdots, \lambda_n)$ a partition. Now let us restrict to a smaller category O_+^{m+n}, which consists of modules whose composition factors are of the form $\mathcal{L}_n(\lambda)$, where $\lambda \in X_{++}^{m+n}$. Again this subcategory carries all the information of the category O^{m+n}. From our explicit formulas for $\mathfrak{u}_{gf}$ and $\mathfrak{l}_{gf}$ one observes that these polynomials again satisfy certain stability.

5. An isomorphism between $\Lambda^m(V) \otimes \Lambda^\infty(V)$ and $\Lambda^m(V) \otimes \Lambda^\infty(V^*)$

Using the stability of the polynomials u_{gf} and ℓ_{gf} one can generalize Brundan's results to the $n = \infty$ situation.

Also using the stability of the polynomials $\mathfrak{u}_{gf}$ and $\mathfrak{l}_{gf}$ one can generalize the results on $gl(m+n)$ to the $n = \infty$ situation.

The $U_q(sl(\infty))$-module $\Lambda^\infty(V)$ [KMS] has a basis of the form

$$v_{m_1} \wedge v_{m_2} \wedge v_{m_3} \wedge \cdots,$$

$m_1 > m_2 > m_3 > \cdots$ and $m_i = 1-i$, for $i >> 0$. (Our $\Lambda^\infty(V)$ actually is the zero sector of KMS.) It has another basis parameterized by partitions $\lambda = (\lambda_1, \lambda_2, \cdots)$:

$$|\lambda >:= v_{\lambda_1} \wedge v_{\lambda_2 - 1} \wedge v_{\lambda_3 - 2} \wedge \cdots .$$

Let $\mathbb{Z}_+^{m+\infty}$ consist of $f : I(m|\infty) \to \mathbb{Z}$ with $f(-m) > \cdots > f(-1)$, $f(1) > f(2)) > \cdots$ and $f(i) = 1-i$, for $i >> 0$. A basis of $\Lambda^m(V) \otimes \Lambda^\infty(V)$ is given by standard monomial basis

$$\mathcal{K}_f = v_{f(-m)} \wedge \cdots v_{f(-1)} \otimes v_{f(1)} \wedge v_{f(2)} \wedge \cdots , \quad f \in \mathbb{Z}_+^{m+\infty}.$$

As in the finite n case we have canonical basis elements $\mathcal{U}_f$ and dual canonical basis element $\mathcal{L}_f$. These basis elements correspond in a similar fashion to generalized Verma modules, tilting modules and irreducible module in the category $O_+^{m+\infty}$, which consists of $gl(m+\infty)$-modules that are finitely generated, locally finite over $gl(m+\infty)_{\geq 0}$, semisimple over $gl(m+\infty)_0$ and whose composition factors are $\mathcal{L}(\lambda)$, with $\lambda \in X_{++}^{m+\infty}$. That is Theorem 4.2 holds for $n = \infty$ as well. We want to mention that the $n = \infty$ version of the theorem implies the finite n theorem.

On the super side the $U_q(sl(\infty))$-module $\Lambda^\infty(W)$ has a basis of the form

$$w_{m_1} \wedge w_{m_2} \wedge w_{m_3} \wedge \cdots ,$$

$m_1 < m_2 < m_3 < \cdots$ and $m_i = i$, for $i >> 0$. It has another basis parameterized by partitions $\lambda = (\lambda_1, \lambda_2, \cdots)$ as well:

$$|\lambda'_* >:= w_{1-\lambda'_1} \wedge w_{2-\lambda'_2} \wedge w_{3-\lambda'_3} \wedge \cdots .$$

Let $\mathbb{Z}_+^{m|\infty}$ consist of $f : I(m|\infty) \to \mathbb{Z}$ with $f(-m) > \cdots > f(-1)$, $f(1) < f(2)) < \cdots$ and $f(i) = i$, for $i >> 0$. A basis of $\Lambda^m(V) \otimes \Lambda^\infty(W)$ is given by standard monomial basis

$$K_f = v_{f(-m)} \wedge \cdots v_{f(-1)} \otimes w_{f(1)} \wedge w_{f(2)} \wedge \cdots , \quad f \in \mathbb{Z}_+^{m|\infty}.$$

We have canonical basis elements U_f and dual canonical basis element L_f. These basis elements correspond to Kac modules, tilting modules and irreducible module in the category $O_+^{m|\infty}$, which consists of $gl(m|\infty)$-modules that are finitely generated, locally finite over $gl(m|\infty)_{\geq 0}$, semisimple over $gl(m|\infty)_0$ and whose composition factors are $L(\lambda)$, with $\lambda \in X_{++}^{m|\infty}$. That is, Brundan's Theorem 3.2 holds for $n = \infty$ as well. Again the $n = \infty$ version of the theorem implies the finite n theorem. The following theorem is crucial.

Theorem 5.1. [CWZ] *The map sending* $|\lambda >$ *to* $|\lambda'_* >$ *induces an isomorphism of* $U_q(sl(\infty))$*-modules* $\Lambda^m(V) \otimes \Lambda^\infty(V)$ *and* $\Lambda^m(V) \otimes \Lambda^\infty(W)$*. Furthermore it compatible with the respective bar involutions and partial orderings.*

An immediate consequence of the theorem is that the canonical bases and the dual canonical bases correspond and hence the Kazhdan-Lusztig polynomials correspond. That is, if we denote the map by $\natural$, then we have

Corollary 5.2. *In the* $n = \infty$ *case we have*

$$u_{\mu\lambda} = \mathfrak{u}_{\mu^\natural \lambda^\natural}, \quad \ell_{\mu\lambda} = \mathfrak{l}_{\mu^\natural \lambda^\natural}.$$

Corollary 5.3. *There exists an isomorphism between the Grothendieck groups of $O_+^{m|\infty}$ and $O_+^{m+\infty}$, which sends $K(\lambda)$, $U(\lambda)$ and $L(\lambda)$ to $\mathcal{K}(\lambda^\natural)$, $\mathcal{U}(\lambda^\natural)$ and $\mathcal{L}(\lambda^\natural)$, respectively.*

The next corollary is an application of Corollary 5.2 together with the symmetric and skew-symmetric Howe dualities for the dual pair (GL, GL).

Corollary 5.4. [CWZ] $\natural$ *is an isomorphism of Grothendieck rings. In particular* $\natural$ *preserves the composition factors of the tensor products in $O_+^{m+\infty}$ and $O_+^{m|\infty}$.*

We have mentioned earlier that the $n = \infty$ cases determine the finite n cases. This follows essentially from the stability properties of the Kazhdan-Lusztig polynomials on the combinatorial sides, while on the representation theory sides this follows from stability properties of certain truncation functors which we will not discuss here. From this and Corollary 5.2 we now derive Theorem 2.1: We have

$$\mathrm{ch}_{\mathcal{L}(\lambda)} = \sum_\mu \mathfrak{l}_{\mu\lambda}(1)\mathrm{ch}_{\mathcal{K}(\mu)}.$$

Applying the involution ω_+ we have

$$\omega_+(\mathrm{ch}_{\mathcal{L}(\lambda)}) = \sum_\mu \mathfrak{l}_{\mu\lambda}(1)\omega_+(\mathrm{ch}_{\mathcal{K}(\mu)}) = \sum_\mu \mathfrak{l}_{\mu\lambda}(1)\mathrm{ch}_{K(\mu^\natural)} = \sum_\mu \ell_{\mu^\natural\lambda^\natural}(1)\mathrm{ch}_{K(\mu^\natural)} = \mathrm{ch}_{L(\lambda^\natural)}.$$

Now setting $x_{n+1} = x_{n+2} = \cdots = 0$ and using the stability of $\ell_{\mu^\natural\lambda^\natural}$ we get

$$\sum_\mu \ell_{\mu^\natural\lambda^\natural}(1)\mathrm{ch}_{K_n(\mu)} = \mathrm{ch}_{L_n(\lambda^\natural)}.$$

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