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# GENERALIZED SKEW DERIVATIONS CHARACTERIZED BY ACTING ON ZERO PRODUCTS

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**Abstract.** Let $A$ be a prime ring such that its symmetric Martindale quotient ring contains a nontrivial idempotent. Then generalized skew derivations of $A$ are characterized by acting on zero products. Precisely, if $g, \delta\colon A \to A$ are additive maps such that $\sigma(x)g(y) + \delta(x)y = 0$ for all $x, y \in A$ with $xy = 0$ where $\sigma$ is an automorphism of $A$, then both $g$ and $\delta$ are characterized as specific generalized $\sigma$-derivations on a nonzero ideal of $A$.

## 1. Results

Let $B$ be a ring with a subring $A$. An additive map $\delta\colon A \to B$ is called a derivation if $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$. In a recent paper Jing, Lu and Li proved the following result [6, Theorem 6]:

*Let $\mathcal{B}$ be a standard operator algebra in a Banach space $X$ containing the identity operator $I$, and $\delta\colon \mathcal{B} \to \mathcal{B}$ be a linear map such that $\delta(AB) = \delta(A)B + A\delta(B)$ for any pair $A, B \in \mathcal{B}$ with $AB = 0$. Then $\delta(AB) = \delta(A)B + A\delta(B) - A\delta(I)B$ for all $A, B \in \mathcal{B}$. Moreover, if in addition $\delta(I) = 0$, then $\delta$ is a derivation.*

The result says that an additive map on a standard operator algebra is almost a derivation if it satisfies the expansion formula of derivations on pair elements with zero product. Since standard operator algebras involve many idempotents, from this point of view Chebotar, Ke and P.-H. Lee studied maps acting on zero products in the context of prime rings [2]. To give its precise statement we first fix some notation.

Throughout, unless specially stated, $A$ always denotes a prime ring with center $Z$, extended centroid $C$ and symmetric Martindale quotient ring $Q$. Moreover, let

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$Q_{mr}$ and $Q_{m\ell}$ be the maximal right and left quotient rings of $A$, respectively. See [1] for details. Chebotar, Ke and P.-H. Lee then proved the following([2, Theorem 2]):

*Let $\delta\colon A \to A$ be an additive map such that $\delta(x)y + x\delta(y) = 0$ for $x, y \in A$ with $xy = 0$. Suppose that $Q$ contains a nontrivial idempotent $e$ such that $eA \cup Ae \subseteq A$.*

(I) *If $1 \in A$, then $\delta(xy) = \delta(x)y + x\delta(y) - \lambda xy$ for all $x, y \in A$, where $\lambda = \delta(1) \in Z$. In particular, if $\delta(1) = 0$, then $\delta$ is a derivation of $A$.*

(II) *If $\deg(A) \geq 3$, then there exists $\lambda \in C$ such that $\delta(xy) = \delta(x)y + x\delta(y) - \lambda xy$ for all $x, y \in A$.*

Two natural generalizations of derivations are generalized derivations and $\sigma$-derivations (or skew derivations). Let $\sigma$ be an automorphism of $A$. An additive map $\delta\colon A \to Q_{m\ell}$ is called a *$\sigma$-derivation* if $\delta(xy) = \sigma(x)\delta(y) + \delta(x)y$ for all $x, y \in A$. Basic examples are derivations and $\sigma - 1$. Given $b \in A$, the map $\delta\colon x \in A \mapsto \sigma(x)b - bx$ obviously defines a $\sigma$-derivation, called the inner $\sigma$-derivation defined by $b$.

An additive mapping $g\colon A \to Q_{m\ell}$ is called a *generalized derivation* if there exists a derivation $d\colon A \to Q_{m\ell}$ such that $g(xy) = xg(y) + d(x)y$ for all $x, y \in A$. Basic examples are derivations, generalized inner derivations (i.e., maps of type $x \mapsto ax + xb$ for some $a, b \in A$) and left $A$-module mappings from $A$ into itself. From this version, the $\delta$ in the theorem above is indeed a generalized derivation. In this paper we will generalize the theorem above from a different point of view. Let us give a definition of generalized skew derivations, which gives a common generalization of both skew derivations and generalized derivations.

An additive map $g\colon A \to Q_{m\ell}$ is called a *generalized $\sigma$-derivation*, where $\sigma$ be an automorphism of $A$, if there exists an additive map $\delta\colon A \to Q_{m\ell}$ such that $g(xy) = \sigma(x)g(y) + \delta(x)y$ for all $x, y \in A$. It is clear that $\delta$ is uniquely determined by $g$, which is called the associated additive map of $g$. It is easy to check that $\delta$ is always a $\sigma$-derivation (see [10]). We are now in a position to state our main result:

**Theorem 1.1.** *Let $A$ be a prime ring with symmetric Martindale quotient ring $Q$. Suppose that $Q$ contains a nontrivial idempotent and that $g$, $\delta\colon A \to A$ are additive maps. If $\sigma(x)g(y) + \delta(x)y = 0$ for all $x, y \in A$ with $xy = 0$ where $\sigma$ is an automorphism of $A$, then there exist a nonzero ideal $N$ of $A$ and a $\sigma$-derivation $d\colon A \to Q$ such that*

$$g(x) = d(x) + \sigma(x)b \quad \text{and} \quad \delta(x) = d(x) + ax$$

*for all $x \in N$, where $b \in Q_{m\ell}$ and $a \in Q_{mr}$. In addition, we can take $N = A$ if $eA \cup Ae \subseteq A$ for some nontrivial idempotent $e \in Q$.*

The proof of the theorem depends on both the Lie structure of rings and the theory of functional identities and will be given in the next section. As an immediate consequence of Theorem 1.1 we have the following generalization of Chebotar, Ke and P.-H. Lee's theorem [2, Theorem 2]:

**Corollary 1.2.** *Let $A$ be a prime ring with extended centroid $C$ and symmetric Martindale quotient ring $Q$. Suppose that $Q$ contains a nontrivial idempotent. If $\delta\colon A \to A$ is an additive map such that $x\delta(y) + \delta(x)y = 0$ for $x, y \in A$ with $xy = 0$, then there exists a nonzero ideal $N$ of $A$ such that $\delta(x) = d(x) + \lambda x$ for all $x \in N$, where $\lambda \in C$ and $d\colon A \to Q$ is a derivation. In addition, we can take $N = A$ if $eA \cup Ae \subseteq A$ for some nontrivial idempotent $e \in Q$.*

The following corollary gives Jing, Lu and Li's theorem [6, Theorem 6] in the context of prime rings:

**Corollary 1.3.** *Let $A$ be a prime ring with extended centroid $C$ and $c \in A$. Suppose that $A$ possesses a nontrivial idempotent. If $\delta\colon A \to A$ is an additive map such that $x\delta(y) + \delta(x)y + xcy = 0$ for $x, y \in A$ with $xy = 0$, then there exist a derivation $d\colon A \to AC$ and $\mu \in C$ such that $\delta(x) = d(x) + (\mu - c)x$ for all $x \in A$.*

*Proof.* By assumption, we have $x(\delta(y) + cy) + \delta(x)y = 0$ for all $x, y \in A$ with $xy = 0$. In view of Theorem 1.1, there exist a derivation $d\colon A \to Q$, $a \in Q_{mr}$ and $\mu \in Q_{m\ell}$ such that $\delta(x) = d(x) + ax$ and $\delta(x) + cx = d(x) + x\mu$ for all $x \in A$. Choose a dense right ideal $\rho$ and a dense left ideal $\lambda$ of $A$ such that $a\rho \subseteq A$ and $\lambda\mu \subseteq A$. Let $x \in \rho$, $z \in A$ and $y \in \lambda$. Then $xzy \in \rho A\lambda \subseteq \rho \cap \lambda$. Thus $(a + c)x, y\mu \in A$ and $((a + c)x)zy = xz(y\mu)$. Applying Martindale's Lemma [11], $y$ and $y\mu$ are $C$-dependent for $y \in \lambda$. It is now easy to prove that $\mu \in C$. Thus $a = \mu - c$ follows and so $\delta(x) = d(x) + (\mu - c)x$ for all $x \in A$. In particular, we have $d(A) \subseteq AC$, proving the corollary.

The next application is applied to generalized polynomial identities (GPIs). An additive map $f\colon A \to A$ is called an *elementary operator* if there exist finitely many $a_i, b_i \in AC$ such that $f(x) = \sum_i a_i x b_i$ for all $x \in A$.

**Corollary 1.4.** *Let $A$ be a prime ring with extended centroid $C$. Suppose that its symmetric Martindale quotient ring $Q$ contains a nontrivial idempotent. If $f$ and $g$ are two elementary operators of $A$ satisfying $xf(y) + g(x)y = 0$ for $x, y \in A$ with $xy = 0$, then there exist $a, b, q \in AC$ and such that $f(y) = [q, y] + yb$ and $g(x) = [q, x] + ax$ for all $x, y \in A$.*

*Proof.* Since $f, g$ are elementary operators, there exist finitely many $a_i, b_i, c_i, d_i \in AC$

such that $f(x) = \sum_i a_i x b_i$ and $g(x) = \sum_j c_j x d_j$ for all $x \in A$. In view of Theorem 1.1, there exist a nonzero ideal $N$ of $A$, a derivation $d\colon A \to Q$, $a \in Q_{mr}$ and $b \in Q_{m\ell}$ such that

$$\sum_i a_i y b_i = d(y) + yb \quad \text{and} \quad \sum_j c_j x d_j = d(x) + ax$$

for all $x, y \in N$. Since $d$ can be uniquely extended to a derivation of $Q_{m\ell}$ into $Q_{m\ell}$ and $\sum_i a_i y b_i = d(y) + yb$ for all $y \in Q_{m\ell}$ (see, for instance, [9, Theorem 2]). In particular, we set $y = 1$, implying that $b = \sum_i a_i b_i \in AC$. An analogous argument proves $a \in AC$. Thus $d(AC) \subseteq AC$. Applying Kharchenko's Theorem ([7, Lemma 1] or [8, Theorem 2]), $d$ must be X-inner. That is, there exists $q \in Q$ such that $d(y) = [q, y]$ for all $y \in A$. Thus $qy - yq = \sum_i a_i y b_i - yb$ for all $y \in A$. Applying Martindale's Lemma [11], $q$ lies in the $C$-linear span of the elements $b_i$'s, $b$ and 1. Thus $q \in AC + C$. Since, for $\beta \in C$, $[q + \beta, y] = [q, y]$ for all $y \in A$, we may take $q \in AC$, proving the corollary.

The following example shows that the existence of nontrivial idempotents in $Q$ is essential to Theorem 1.1.

**Example 1.5.** Let $A$ be a prime ring, not a domain, with center $Z$ and let

$$\mathcal{M}_A = \{a \in A \mid xay = 0 \text{ whenever } x, y \in A \text{ with } xy = 0\}.$$

Suppose that $\mathcal{M}_A$ is noncentral. Choose an element $c \in \mathcal{M}_A \setminus Z$. Let $f, g\colon A \to A$ be additive maps defined by $f(x) = cxc$ and $g(x) = x$ for all $x \in A$. Then $xf(y) + g(x)y = 0$ for $x, y \in A$ with $xy = 0$. We claim that $f, g$ cannot assume the forms given in Theorem 1.1. Indeed, if there exist a derivation $d\colon A \to Q$ and a nonzero ideal $N$ of $A$ such that

$$f(x) = d(x) + xb \quad \text{and} \quad g(x) = d(x) + ax$$

for all $x \in N$, where $a \in Q_{mr}$ and $b \in Q_{m\ell}$. Then $d(x) = (1 - a)x$ for all $x \in A$, implying $d = 0$. Thus $xb = cxc$ for all $x \in A$. Applying Martindale's Lemma [11], we see that $c \in C$, the extended centroid of $A$, and so $c \in Z$, a contradiction.

We remark that there exist indeed such prime rings. See, for instance, Dubrovin's example [3]. Another such example is $A = K\{x, y\}/(x^2)$ [5, pp. 105-108], where $K\{x, y\}$ is the free algebra over a field $K$ in two noncommuting indeterminates $x$ and $y$. In this example, $A$ is a prime ring and $Kx + xAx + (x^2)/(x^2)$ coincides with the set of all elements of $A$ with square zero. Let $\overline{x} = x + (x^2) \in A$. Then $a\overline{x}b = 0$ whenever $a, b \in A$ with $ab = 0$. Thus $\overline{x} \in \mathcal{M}_A$ and is noncentral.

## 2. Proof of Theorem 1.1

We begin with some preliminary lemmas.

**Lemma 2.1.** *Let $I$ be a nonzero ideal of $A$ and let $f: I \to Q_{m\ell}$ be a left $I$-module map. Then there exists $q \in Q_{m\ell}$ such that $f(x) = xq$ for all $x \in I$.*

*Proof.* Notice that $Q_{m\ell}I$ is a dense left ideal of $Q_{m\ell}$. We define the map $\tilde{f}: Q_{m\ell}I \to Q_{m\ell}$ by $\sum_i x_i a_i \mapsto \sum_i x_i f(a_i)$, where $x_i \in Q_{m\ell}, a_i \in I$. Then $\tilde{f}$ is well-defined. Indeed, let $\sum_i x_i a_i = 0$, where $x_i \in Q_{m\ell}, a_i \in I$. Choose a dense left ideal $J$ of $A$ such that $Jx_i \subseteq I$ for each $i$. Then, for $y \in J$, we have

$$0 = f(y \sum_i x_i a_i) = f(\sum_i (yx_i)a_i) = \sum_i yx_i f(a_i) = y\big(\sum_i x_i f(a_i)\big),$$

implying $\sum_i x_i f(a_i) = 0$. Thus $\tilde{f}$ is well-defined. It is clear that $\tilde{f}$ is a left $Q_{m\ell}$-module map extending $f$. We remark that the maximal left quotient ring of $Q_{m\ell}$ coincides with itself. Thus there exists a $q \in Q_{m\ell}$ such that $\tilde{f}(z) = zq$ for all $z \in Q_{m\ell}I$. In particular, $f(x) = xq$ for all $x \in I$. This proves the lemma.

**Lemma 2.2.** *Let $I$ be a nonzero ideal of $A$ and let $f, g: I \to Q_{m\ell}$ be two additive maps. Suppose that $f(x)y + \sigma(x)g(y) = 0$ for all $x, y \in I$, where $\sigma$ is an automorphism of $A$. Then there exists $a \in Q_{m\ell}$ such that $f(x) = \sigma(x)a$ and $g(y) = -ay$ for all $x, y \in I$.*

*Proof.* Let $x, y, z \in I$. By assumption, we have

$$f(zx)y + \sigma(zx)g(y) = 0 \quad \text{and} \quad \sigma(z)(f(x)y + \sigma(x)g(y)) = 0.$$

Thus $(f(zx) - \sigma(z)f(x))y = 0$ and so $f(zx) = \sigma(z)f(x)$ by the primeness of $A$. Note that $\sigma$ can be uniquely extended to an automorphism of $Q_{m\ell}$. Consider the map $\phi: I \to Q_{m\ell}$ defined by $\phi(x) = \sigma^{-1}(f(x))$ for $x \in I$. Then $\phi$ is a left $I$-module map. By Lemma 2.1, there exists $b \in Q_{m\ell}$ such that $\phi(x) = xb$ for all $x \in I$. That is, $f(x) = \sigma(x)a$ for all $x \in I$, where $a = \sigma(b) \in Q_{m\ell}$. It is clear that $g(y) = -ay$ for all $y \in I$. This proves the lemma.

Although the next lemma has a more generalized version, for our purpose we only state the following special form.

**Lemma 2.3.** *Let $d: M \to Q$ be a $\sigma$-derivation, where $M$ is a nonzero ideal of $A$ and $\sigma$ is an automorphism of $A$. Assume that there exists a nonzero ideal $J$ of $A$ such that $d(m)J + Jd(m) \subseteq A$ for all $m \in M$. Then $d$ can be uniquely extended to a $\sigma$-derivation from $A$ into $Q$.*

*Proof.* Replacing $J$ by $M \cap J$, we may assume from the start that $J \subseteq M$. Let $a \in A$.

Define a map $\psi_a: MJ \to A$ by the rule

$$\psi_a(\sum_i m_i x_i) = \sum_i d(am_i)x_i - \sum_i \sigma(a)d(m_i)x_i \in A,$$

where $m_i \in M$ and $x_i \in J$. We claim that $\psi_a$ is well-defined. Indeed, if $\sum_i m_i x_i = 0$, then $\sum_i (am_i)x_i = 0$ and so

$$\begin{aligned}0 = \sum_i d((am_i)x_i) =& \sum_i (\sigma(am_i)d(x_i) + d(am_i)x_i)\\ =& \sum_i d(am_i)x_i - \sum_i \sigma(a)d(m_i)x_i,\end{aligned}$$

since

$$\begin{aligned}\sum_i \sigma(am_i)d(x_i) =& \sum_i \sigma(a)\sigma(m_i)d(x_i) = \sum_i \sigma(a)(d(m_i x_i) - d(m_i)x_i)\\ =& \sigma(a)d(\sum_i m_i x_i) - \sum_i \sigma(a)d(m_i)x_i = -\sum_i \sigma(a)d(m_i)x_i.\end{aligned}$$

The claim is proved. It is clear that $\psi_a$ is a right $A$-module map. Thus $\psi_a$ is defined by an element $\tilde{d}(a) \in Q_r$, the right Martindale quotient ring of $A$. That is, $\tilde{d}: A \to Q_r$ has the following property: $\tilde{d}(a)m = d(am) - \sigma(a)d(m)$ for $a \in A$ and $m \in M$. Let $a, b \in A$ and $m \in M$. Then $\tilde{d}(ab)m = d(abm) - \sigma(ab)d(m)$. On the other hand, by the fact that $bm \in M$ we have

$$\begin{aligned}(\sigma(a)\tilde{d}(b) + \tilde{d}(a)b)m =& \sigma(a)(d(bm) - \sigma(b)d(m)) + d(abm) - \sigma(a)d(bm)\\ =& d(abm) - \sigma(ab)d(m).\end{aligned}$$

Thus $\tilde{d}(ab) = \sigma(a)\tilde{d}(b) + \tilde{d}(a)b$, proving $\tilde{d}$ to be a $\sigma$-derivation. Clearly, $\tilde{d}(m) = d(m)$ for all $m \in M$ and so $\tilde{d}$ is an extension of $d$. Notice that $J\sigma(M)$ is a nonzero ideal of $A$. Let $a \in A$ and $m \in M$. Then $d(ma) = \sigma(m)\tilde{d}(a) + d(m)a$ and so

$$J\sigma(M)\tilde{d}(a) \subseteq Jd(M) + Jd(M)A \subseteq A,$$

implying that $\tilde{d}(a) \in Q$. Hence, $\tilde{d}: A \to Q$ and the lemma is thus proved.

We are now ready to prove the main theorem stated in §1.

**Theorem 1.1.** *Let $A$ be a prime ring with symmetric Martindale quotient ring $Q$. Suppose that $Q$ contains a nontrivial idempotent and that $g$, $\delta: A \to A$ are additive maps. If $\sigma(x)g(y) + \delta(x)y = 0$ for all $x, y \in A$ with $xy = 0$ where $\sigma$ is an automorphism of $A$, then there exist a nonzero ideal $N$ of $A$ and a $\sigma$-derivation $d: A \to Q$ such that*

$$g(x) = d(x) + \sigma(x)b \quad \text{and} \quad \delta(x) = d(x) + ax$$

*for all $x \in N$, where $b \in Q_{m\ell}$ and $a \in Q_{mr}$. In addition, we can take $N = A$ if $eA \cup Ae \subseteq A$ for some nontrivial idempotent $e \in Q$.*

*Proof.* Let $e$ be a nontrivial idempotent of $Q$. Choose a nonzero ideal $I$ of $A$ such that $Ie + eI \subseteq A$. We consider the additive subgroup $E$ of $Q$ generated by the set $\{f \in Q \mid If + fI \subseteq A \text{ and } f^2 = f\}$. Then $e \in E$. We claim that $0 \neq I^2[E,E]I^2 \subseteq E + E^2$.

We follow Herstein's argument [4, Proof of Lemma 1.3]. Let $f \in E$ and $x \in I$. Then $f + fx(1-f), f + (1-f)xf \in E$ and so

$$[f,x] = (f + fx(1-f)) - (f + (1-f)xf) \in E.$$

Thus $[E,I] \subseteq E$. Indeed, $[E,E]I^2 \subseteq [E,EI^2] + E[E,I^2] \subseteq E + E^2$, and so

$$I^2[E,E]I^2 \subseteq [I^2,[E,E]I^2] + [E,E]I^4 \subseteq [I^2, E + E^2] + E + E^2.$$

Since $[I^2, E + E^2] \subseteq E + E^2$, we see that $I^2[E,E]I^2 \subseteq E + E^2$. Finally, $0 \neq [e, e + eI^2(1-e)] \subseteq [E,E]$, proving our claim.

Let $x, y \in I$ and $f = f^2 \in E$. Then $xf, (1-f)y, x(1-f), fy \in A$. By assumption,

$$\begin{aligned} &\delta(xf)(1-f)y + \sigma(xf)g((1-f)y) = 0 \quad \text{and} \\ &\delta(x(1-f))fy + \sigma(x(1-f))g(fy) = 0. \end{aligned} \tag{1}$$

Solve the two equations by Lemma 2.2: there exist two unique elements $u, v \in Q_{m\ell}$, depending on $f$, such that

$$\begin{aligned} &\delta(xf)(1-f) = \sigma(x)u, \quad \sigma(f)g((1-f)y) = -uy \quad \text{and} \\ &\delta(x(1-f))f = \sigma(x)v, \quad \sigma(1-f)g(fy) = -vy. \end{aligned} \tag{2}$$

Thus, by (2), we see that

$$\delta(x)f - \delta(xf) = \sigma(x)(v-u) \quad \text{and} \quad \sigma(f)g(y) - g(fy) = (v-u)y. \tag{3}$$

By (3), it follows from the definition of $E$ that there exists an additive map $d\colon E \to Q_{m\ell}$ such that

$$\delta(x)m - \delta(xm) = -\sigma(x)d(m) \quad \text{and} \quad \sigma(m)g(y) - g(my) = -d(m)y \tag{4}$$

for all $x, y \in I$ and all $m \in E$. Let $m_1, m_2 \in E$; then

$$\delta(x)m_2 - \delta(xm_2) = -\sigma(x)d(m_2) \text{ and } \delta(x)m_1 - \delta(xm_1) = -\sigma(x)d(m_1) \tag{5}$$

for all $x \in I$. Let $x \in I^2$; then $xm_1 \in I$. By (5) we have

$$\delta(xm_1)m_2 - \delta(xm_1m_2) = -\sigma(xm_1)d(m_2) \text{ and}$$
$$-\delta(xm_1)m_2 + \delta(x)m_1m_2 = -\sigma(x)d(m_1)m_2.$$

Thus we have

$$\delta(x)m_1m_2 - \delta(x(m_1m_2)) = -\sigma(x)(\sigma(m_1)d(m_2) + d(m_1)m_2). \tag{6}$$

On the other hand, let $y \in I^2$; then $m_2y \in I$. By (4) we have

$$\sigma(m_1)g(m_2y) - g(m_1m_2y) = -d(m_1)m_2y \quad \text{and}$$
$$\sigma(m_1)\sigma(m_2)g(y) - \sigma(m_1)g(m_2y) = -\sigma(m_1)d(m_2)y,$$

implying that

$$\sigma(m_1m_2)g(y) - g((m_1m_2)y) = -(d(m_1)m_2 + \sigma(m_1)d(m_2))y \tag{7}$$

This means that $d$ can be extended to $E + E^2$ such that

$$\delta(x)m - \delta(xm) = -\sigma(x)d(m) \quad \text{and} \quad \sigma(m)g(y) - g(my) = -d(m)y$$

for all $x, y \in I^2$ and all $m \in E + E^2$. Moreover, $d(m_1m_2) = \sigma(m_1)d(m_2) + d(m_1)m_2$ for all $m_1, m_2 \in E$. Repeating the same argument above, we can extend $d$ to $E + E^2 + E^3 + E^4$ such that

$$\delta(x)m - \delta(xm) = -\sigma(x)d(m) \quad \text{and} \quad \sigma(m)g(y) - g(my) = -d(m)y \tag{8}$$

for all $x \in I^4$ and all $m \in E + E^2 + E^3 + E^4$. Moreover,

$$d(uv) = \sigma(u)d(v) + d(u)v \tag{9}$$

for all $u, v \in E + E^2$. Let $M = I^2[E, E]I^2 \neq 0$. Then $M$ is a nonzero ideal of $A$ contained in $E + E^2$.

Let $m \in M$. By (8) we see that $\sigma(I^4)d(m) \subseteq A$ and $d(m)I^4 \subseteq A$. Thus $d(m) \in Q$ follows. Thus $d\colon M \to Q$ is a $\sigma$-derivation satisfying

$$\delta(x)m - \delta(xm) = -\sigma(x)d(m) \quad \text{and} \quad \sigma(m)g(y) - g(my) = -d(m)y \tag{10}$$

for all $x, y \in I^4$ and all $m \in M$. Set $J = \sigma(I^4) \cap I^4$. Then $J$ is a nonzero ideal of $A$ and, moreover, $d(m)J + Jd(m) \subseteq A$ for all $m \in M$. In view of Lemma 2.3, $d$ can be uniquely extended to a $\sigma$-derivation from $A$ into $Q$. Thus we can rewrite (10) as

$$\delta(xm) - d(xm) = (\delta(x) - d(x))m \quad \text{and} \quad \sigma(m)(g(y) - d(y)) = g(my) - d(my) \tag{11}$$

for all $x, y \in I^4$ and all $m \in M$. Consider the map $\phi: I^4 \to Q$ defined by

$$x \in I^4 \mapsto \phi(x) = \delta(x) - d(x).$$

By (11), $\phi$ is a right $M$-module map. Thus it is a right $A$-module map by the primeness of $A$. So, by Lemma 2.1, there exists $a \in Q_{mr}$ such that $\phi(x) = ax$ and so $\delta(x) = d(x) + ax$ for all $x \in I^4$. By an analogous argument, there exists $b \in Q_{m\ell}$, such that $g(x) = d(x) + \sigma(x)b$ for all $x \in I^4$. We are now done by setting $N = I^4$.

Suppose, in addition, that $eA \cup Ae \subseteq A$ for some nontrivial idempotent $e \in Q$. Then $I = A$ in our construction. Moreover, we have $EA + AE \subseteq A$. Therefore, (11) remains true for all $x \in A$. So the map $\phi: I^4 \to Q$ can be replaced by $\phi: A \to Q$. Now our conclusion trivially holds. This proves the theorem.

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