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非線性系統之最佳運動規劃
Optimal Motion Planning Control for Nonlinear Systems

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中文摘要

傳統上，對最佳運動規劃的問題是用最佳控制理論，轉換成一個不同端點的邊界值問題，但是這種方法需要大量的數值計算。本計劃提出一種系統化的方法來解決最佳運動規劃的問題。第一步先解線性系統的規劃問題，第二步解非線性系統的規劃問題，第三步將價值函數視為一個額外的系統變數，來解非線性系統的最佳規劃問題。此種方法可避免大量的數值運算。

關鍵字: 運動規劃、最佳控制、非線性系統、微擾法。

Abstract

Conventionally, the optimal motion planning problem is solved by transformation into a two-point boundary value problem (TPBVP). The resultant TPBVP has twice the system dimension ($2n$), and requires complicated numerical scheme for its solution. In this project, we propose a different but systematic approach to the optimal motion planning problem. First, we solve the simple motion planning problem by perturbation refinement for a nonlinear system with dimension n. Second, we transform the *optimal* motion planning problem into a simple motion planning problem for an extended system with dimension $n+1$. A numerical procedure for the proposed optimal motion planning is developed accordingly, and verified through a simulation example.

Keywords: motion planning, optimal motion planning, optimal control, nonlinear system, perturbation.

1. Introduction

Motion planning problems attract a lot of research in recent years, especially for the class of systems without drift [1]. For a detailed review on the motion planning for driftless systems, the readers are refered to [2,3]. However, in application, the solutions to simple motion planning problem may not be practical since they often result in poor performance such as large excursion in state trajectory or extra demand on the control energy. Therefore, later research is directed to solving the *optimal* motion planning problem, in which a performance index is to be minimized in addition to finding a trajectory connecting two spefified end points. In theory, the optimal motion planning problem is equivalent to the open-loop optimal control problems with fixed end points [4]. Hence, a majority of research on the optimal motion planning rely on the calculus of variation as the mathematical tool for searching the optimal solution. This approach transforms the optimal motion planning problem into a two-point boundary value problem, which requires complicated numerical algorithms [5,6] for obtaining a solution. Another approach to optimal motion planning is to work in a finite dimensional space instead of the functional space by parameterizing the control input. For example, in [7,8], the control input is parameterized into a finite summation of some basis functions. The coefficients of the basis functions are calculated using different algorithms such as the Ritz Approximation Theory [7] or the Quasi-Newton method [8], so that a near-optimal solution is obtained. In order to obtain a solution that is close to the optimal solution, the number of basis functions, N, should be large, and this increases the burden on numerical calculation.

In this project, we develop a systematic approach to optimal motion planning, which is different from the two approaches mentioned above. We solve in the first stage the linear system motion planning problem, in the second stage the nonlinear system motion planning problem, and finally in the third stage the optimal motion planning problem. The key idea in our approach to optimal motion planning problem is to treat the cost function as an extended system state.

2. Motion Planning for Linear System

Consider the motion planning for a linear dynamic system :

$$\begin{aligned} \dot{z} &= A(t)z + B(t)v, \\ z(t_0) &= z_0 \end{aligned} \qquad (1)$$

find $v \triangleq v(t), t \in [t_0, t_f]$ such that $z(t_f) = z_f$, where $z \in R^n$ is the system state, $v \in R^p$ is the control input , $A(t)$ is a $n \times n$ matrix , and $B(t)$ a $n \times p$ matrix.

From linear system theory [9], the desired state transferring control is given by

$$v(t) = B^T(t)\Phi^T(t_f,t)W^{-1}(t_0,t_f)[z_f - \Phi(t_f,t_0)z_0], \quad (2)$$

where $W(t_0, t_f)$ is the controllability grammian [9] of the system (1),

$$W(t_0,t_f) = \int_{t_0}^{t_f} \Phi(t_f,t)B(t)B^T(t)\Phi^T(t_f,t)dt, \quad (3)$$

and Φ is the state transition matrix of the linear system.

Theorem 1 [9]: If the system (1) is controllable in the sense that $W(t_0, t_f)$ in (2) is positive definite, the control (3) transfers the state from $z(t_0) = z_0$ to $z(t_f) = z_f$.

3. Motion Planning for Nonlinear System

After solving the linear motion planning problem, we now try to solve the nonlinear motion planning problem based on the linear results. The basic idea is as follows:

- Use perturbation to find the linearized system of the original nonlinear system along a trajectory $x^{(i)}$ generated by some $u^{(i)}$.
- Do motion planning for the linearized system so that the perturbed final state $\delta x^{(i)}(t_f)$ moves in the direction of the desired final target x_f.
- Iterate the previous two steps so that the final state $x(t_f)$ of the nonlinear system moves progressively toward the target x_f.

Consider a nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x)u \\ x(t_0) &= x_0 \end{aligned} \qquad (4)$$

where $x \in R^n$ is the system state, $u \in R^p$ is the control input, and $f(x)$ and $g_i(x)$ (the ith column of $g(x)$) for $i = 1, 2......, p$ are smooth vector fields. In this problem, we wish to find the control input u that can transfer the system state from the initial condition x_0 to a specified final position x_f at a given time t_f.

Before presenting the iteration procedure, we should define the relationship between each consecutive trajectories. Define the notations

$$\begin{aligned} x^{(i+1)} &= x^{(i)} + \delta x^{(i)} \\ u^{(i+1)} &= u^{(i)} + \delta u^{(i)}, \end{aligned}$$

where index i stands for the $i'th$ iteration. Since both $(x^{(i)},\ u^{(i)})$ and $(x^{(i+1)},\ u^{(i+1)})$ satisfy (1), we have, for $j = i,\ i+1$,

$$\begin{aligned} \dot{x}^{(j)} &= f(x^{(j)}) + g(x^{(j)})u^{(j)} \\ x^{(j)}(t_0) &= x_0 \end{aligned} \qquad (5)$$

Assume that $\delta x^{(i)}$ and $\delta u^{(i)}$ are small; we can linearize the nonlinear system (5) with $j = i+1$ along the trajectory $x^{(i)}$ and $u^{(i)}$ to give

$$\begin{aligned} \delta\dot{x}^{(i)} &= A^{(i)}(t)\delta x^{(i)} + B^{(i)}(t)\delta u^{(i)} \\ \delta x^{(i)}(t_0) &= 0 \end{aligned} \qquad (6)$$

where

$$\begin{aligned} A^{(i)}(t) &= [\frac{\partial f(x^{(i)})}{\partial x} + \frac{\partial g(x^{(i)})u^{(i)}}{\partial x}] && (7) \\ B^{(i)}(t) &= g(x^{(i)}) && (8) \end{aligned}$$

We can now summarize the iteration procedure for motion planning of a nonlinear system.

1. Guess an initial control $u^{(0)}(t), t \in [t_0, t_f]$, and find its corresponding trajectory $x^{(0)}(t)$ from (4). Set the iteration index i be zero.
2. Evaluate the matrices $A^{(i)}(t)$ and $B^{(i)}(t)$ in (7) and (8) on the trajectory $x^{(i)}$ and $u^{(i)}$.
3. Set $\delta x^{(i)}(t_f) = h^{(i)} \times [x_f - x^{(i)}(t_f)]$, with a sufficiently small $0 < h^{(i)} < 1$.
4. Design the transferring control $\delta u^{(i)}(t)$ (Assume that the $[A^{(i)}(t),\ B^{(i)}(t)]$ pair is controllable)

   $$\delta u^{(i)}(t) = B^{(i)T}(t)\Phi^{(i)T}(t_f,t)W^{(i)-1}(t_0,t_f) \cdot \delta x^{(i)}(t_f),$$

   where $\Phi^{(i)}$ is the state transition matrix of $A^{(i)}$, and

   $$W^{(i)}(t_0,t_f) = \int_{t_0}^{t_f} \Phi^{(i)}(t_f,\tau)B^{(i)}(\tau)B^{(i)T}(\tau)\Phi^{(i)T}(t_f,\tau).$$

5. Determine the next control input $u^{(i+1)} = u^{(i)} + \delta u^{(i)}$.

6. Substitute $u^{(i+1)}$ in (1) to determine the $(i+1)$st trajectory $x^{(i+1)}$.

7. Compare the final state $x^{(i+1)}(t_f)$ of the $(i+1)$st trajectory $x^{(i+1)}$ with the specified final state x_f.

$$If \qquad \left\| x^{(i+1)}(t_f) - x_f \right\| \le \varepsilon_x, \qquad (9)$$

where ε_x is a small positive constant, stop the iteration procedure. If the termination criterion (9) is not satisfied, return to Step 2, using the trajectory $x^{(i+1)}, u^{(i+1)}$ in place of $x^{(i)}, u^{(i)}$. The value of the termination constant ε_x depends on the desired accuracy of state positioning.

Theorem 2: If the pair $(A^{(i)}(t), B^{(i)}(t))$ in (6) is controllable for all $i \ge 0$, and $h^{(i)}$ is chosen such that

$$0 < (1 - h^{(i)}) < \xi < 1, \quad \forall\, i, \qquad (10)$$

for some ξ, it is guaranteed that the above iteration procedure will terminate at some finite i.

4. Optimal Motion Planning

In Section 3, we successfully find the control inputs, by iterations, to generate a trajectory connecting the specified initial and final states. In addition, we often wish that the system response has better performance such as minimum time to achieve the target, minimum length of path, or minimum energy used. Therefore, in this Section, we consider the **optimal** motion planning problem: given a nonlinear system (4), find the control input $u \triangleq u(t), t \in [t_0, t_f]$ to place the final state to a specified position $x(t_f) = x_f$, and at the same time to minimize a cost function

$$J = \int_{t_0}^{t_f} l(x,u)d\tau, \qquad (11)$$

where $l(x,u)$ is a positive scalar function of the state x and the control u. To solve this problem, we will adopt an approach that will transform an optimal motion planning problem in this Section into a simple motion planning problem treated in the previous Section.

First, based on the cost function (11), define a new cost function state

$$J(t) = \int_{t_0}^{t} l(x,u)d\tau$$

Taking its time derivative, one can obtain the governing equation of $J(t)$

$$\begin{aligned} \dot{J}(t) &= l(x,u) \\ J(t_0) &= 0 \end{aligned} \qquad (12)$$

Thus, combining the original system (4) with the additional state $J(t)$, we have an extended system

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{J} \end{bmatrix} &= \begin{bmatrix} f(x) + g(x)u \\ l(x,u) \end{bmatrix} \\ \begin{bmatrix} x(t_0) \\ J(t_0) \end{bmatrix} &= \begin{bmatrix} x_0 \\ 0 \end{bmatrix} \end{aligned} \qquad (13)$$

The algorithm for optimal motion planning should be divided into two stages:

Stage I. *motion planning for the original system (4)*

Firstly, we follow the procedure of motion planning in Section 3 for the original system (1) to find a trajectory and control (x_m, u_m) satisfying

$$\dot{x}_m = f(x_m) + g(x_m)u_m$$

and

$$x_m(t_0) = x_0, \quad x_m(t_f) = x_f.$$

With (x_m, u_m), we calculate the cost function state $J_m(t)$ from (12) corresponding to $(x_m(t), u_m(t))$. Notice that final value of the cost function state $J_m(t_f)$ will be larger than the *optimal* value of the cost function (11), namely J^*. Therefore, when applying perturbation to the trajectory x_m in Stage II, the perturbation should be in the direction that will *decrease* the value of $J_m(t_f)$. If, instead of x_m, we randomly pick an initial trajectory x_0 that does not connect the specified end points, its correspondent $J_0(t_f)$ may be larger or smaller than J^*. In this case, there is no method to know in which direction to perturb the initial trajectory x_0.

Stage II. *motion planning for the* **extended** *system (13)*

Secondly, linearize the *extended* system (13) along the trajectory (x_m, J_m, u_m) to obtain

$$\begin{bmatrix} \dot{\delta x} \\ \dot{\delta J} \end{bmatrix} = \begin{bmatrix} [\frac{\partial f(x_m)}{\partial x} + \frac{\partial g(x_m)}{\partial x}] & 0 \\ \frac{\partial l(x_m,u_m)}{\partial x} & 0 \end{bmatrix} \begin{bmatrix} \delta x \\ \delta J \end{bmatrix} + \begin{bmatrix} g(x_m) \\ \frac{\partial l(x_m,u_m)}{\partial u} \end{bmatrix} \delta u, \quad \begin{bmatrix} \delta x(t_0) \\ \delta J(t_0) \end{bmatrix} = 0 \qquad (14)$$

where $(\delta x, \delta J, \delta u)$ is defined by $\delta x = x - x_m$, $\delta J = J - J_m$, and $\delta u = u - u_m$. Based on the linearized extended system model (14), we can use the transferring control δu to place the final state $\delta x(t_f) = 0$ and $\delta J(t_f) < 0$. In other words, while maintaining the fixed two end-points x_0 and x_f, we can find a new control $u = u_m + \delta u$ to bring down the value of the cost function. By repeating this procedure, we can eventually reach the minimum value of $J(t_f)$.

Before presenting the iteration procedure, define the notions

$$\begin{aligned} x^{(j+1)} &= x^{(j)} + \delta x^{(j)} \\ J^{(j+1)} &= J^{(j)} + \delta J^{(j)} \\ u^{(j+1)} &= u^{(j)} + \delta u^{(j)} \end{aligned}$$

Since $x^{(j)}(t)$, $J^{(j)}(t)$ and $u^{(j)}(t)$ satisfy (21), we have

$$\begin{aligned} \begin{bmatrix} \dot{x}^{(j)} \\ \dot{J}^{(j)} \end{bmatrix} &= \begin{bmatrix} f(x^{(j)}) + g(x^{(j)})u^{(j)} \\ l(x^{(j)}, u^{(j)}) \end{bmatrix} \quad (15) \\ \begin{bmatrix} x^{(j)}(t_0) \\ J^{(j)}(t_0) \end{bmatrix} &= \begin{bmatrix} x_0 \\ 0 \end{bmatrix}. \end{aligned}$$

Assuming that $\delta x^{(j)}$, $\delta J^{(j)}$ and $\delta u^{(j)}$ are small, we can linearize the extended system (23) along the trajectory $(x^{(j)}(t), J^{(j)}(t), u^{(j)}(t))$ to yield

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} \delta x^{(j)} \\ \delta J^{(j)} \end{bmatrix} &= A_e^{(j)}(t)\begin{bmatrix} \delta x^{(j)} \\ \delta J^{(j)} \end{bmatrix} + B_e^{(j)}(t)\delta u^{(j)} \quad (16) \\ \begin{bmatrix} \delta x^{(j)}(t_0) \\ \delta J^{(j)}(t_0) \end{bmatrix} &= 0 \end{aligned}$$

where

$$\begin{aligned} A_e^{(j)}(t) &= \begin{bmatrix} [\frac{\partial f(x^{(j)})}{\partial x} + \frac{\partial g(x^{(j)})}{\partial x}] & 0 \\ \frac{\partial l(x^{(j)}, u^{(j)})}{\partial x} & 0 \end{bmatrix}, \quad (17) \\ B_e^{(j)}(t) &= \begin{bmatrix} g(x^{(j)}) \\ \frac{\partial l(x^{(j)}, u^{(j)})}{\partial u} \end{bmatrix} \quad (18) \end{aligned}$$

We now summarize the procedure of optimal motion planning for a nonlinear system.

1. Find the trajectory $(x_m(t), J_m(t), u_m(t))$ introduced in Stage I.
2. Let the iteration index j be zero, and set $(x^{(0)}(t), u^{(0)}(t), J^{(0)}(t)) = (x_m(t), u_m(t), J_m(t))$.
3. Evaluate the matrices $A_e^{(j)}(t)$ and $B_e^{(j)}(t)$ in (17) and (18) on the trajectory $(x^{(j)}, J^{(j)}, u^{(j)})$.
4. Evaluate the controllability grammain $W_e^{(j)}(t_0, t_f)$

$$W_e^{(j)}(t_0, t_f) = \int_{t_0}^{t_f} \Phi_e^{(j)}(t_f, \tau) B_e^{(j)}(\tau) B_e^{(j)T}(\tau) \Phi_e^{(j)T}(t_f, \tau) d\tau,$$

where $\Phi_e^{(j)}$ is the state transition matrix of $A_e^{(j)}$. If

$$|detW_e^{(j)}(t_0, t_f)| < \epsilon_w, \quad (19)$$

where ϵ_w is a small positive number, stop the iteration procedure. Otherwise, continous to Step 5.

5. Determine the step size

$$\begin{bmatrix} \delta x^{(j)}(t_f) \\ \delta J^{(j)}(t_f) \end{bmatrix} = \begin{bmatrix} x_f - x^{(j)}(t_f) \\ -h^{(j)} \times J^{(j)}(t_f) \end{bmatrix}, \quad (20)$$

where $0 < h^{(j)} < 1$.

6. Design the state transferring control $\delta u^{(j)}(t)$

$$\delta u^{(j)}(t) = B_e^{(j)T}(t)\Phi_e^{(j)T}(t_f, t) W_e^{(j)-1}(t_0, t_f)\delta x^{(j)}(t_f)$$

7. Determine $u^{(j+1)} = u^{(j)} + \delta u^{(j)}$.
8. Use $u^{(j+1)}$ to determine the $(j+1)$st trajectory $(x^{(j+1)}, J^{(j+1)})$.
9. If

$$\left\| x^{(j+1)}(t_f) - x_f \right\| \leq \varepsilon_x, \text{ and } J^{(j+1)}(t_f) < J^{(j)}(t_f), \quad (21)$$

where ε_x is a small positive constant, return to Step 3, using the trajectory $x^{(j+1)}, u^{(j+1)}, h^{(j+1)}$ in place of $x^{(j)}, u^{(j)}, h^{(j)}$ and $h^{(j+1)} = h^{(j)}$. Otherwise, return to Step 5 using $h^{(j)} = 0.5 \times h^{(j)}$ in place of $h^{(j)}$.

Remark 1: Although in the above iteration we start with the trajectory $x^{(0)} = x_m$, which has two end points at x_0 and x_f, but it is difficult to keep the two end points fixed exactly at x_0 and x_f during the iteration because of linearization error and numerical error. Hence, in the perturbed control (20), we still need to make compensation $\delta x^{(j)}(t_f) = x_f - x^{(j)}(t_f)$ in each iteration to confine the end point close to the specified position x_f.

Remark 2: To judge if $J^{(j)}(t_f)$ has reached its minimum, one can check if the extended linearized system (16) is controllable, or equivalently if $detW_e^{(j)}(t_0, t_f) \neq 0$. Since controllability of the system (16) suggests that one can always find small enough perturbation in control $\delta u^{(j)}$ that moves the final state of the extended system (15) in any direction. In particular, one can always find $\delta u^{(j)}$ so that the cost function state at t_f moves in the decreasing direction; that is, $\delta J^{(i)}(t_f) < 0$ or $J^{(j+1)}(t_f) < J^{(j)}(t_f)$. Therefore, $detW_e^{(j)}(t_0, t_f) = 0$ is a necessary condition for $x^{(j)}(t)$ to be an optimal solution. We thus use condition (19) as the termination criterion for the iteration procedure.

Remark 3: When the step size $h^{(j)}$ in (20) is too large such that the perturbed control (21) is trying to decrease $J^{(j)}(t_f)$ to a value smaller than the optimal value J^*, which is impossble, two situations may take place: (1) the end point will leave the specified position x_f, or (2) the next iterated cost function value $J^{(j+1)}(t_f)$ will increase instead of decrease. Hence, if any of the situations above takes place, we have to reduce the step size by half until condition (22) is satisfied. This explains the mechanism of adjusting the step size $h^{(j)}$ in Step 9.

5. Conclusions

In this project, we develop a systematic approach to motion planning. We solve in the first stage the linear system motion planning problem, in the second stage the nonlinear system motion planning problem, and finally in the third stage the optimal motion planning problem. The approach is systematic because the solution at each stage depends on that of the previous solution. The major contribution is that the optimal motion planning problem with dimension n is transformed into a simple motion planning problem with dimentions $n+1$ by regarding the cost function as an extended system state. We thus successfully avoid the numerically complicated two-point boundary value problem, which is inevitable in other approachs to optimal motion planning problem.

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