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在含界面之異向彈性體內之集中力或差排的暫態分析（2/2）
Transient analysis of a line force or dislocation in an anisotropic elastic solid with boundary (2/2)

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一、中文摘要

本計劃分析在均勻半空間或雙材料無限域內之集中力或差排的暫態反應。以 Wu (2000)所提求解異向動彈力的方法為基礎來推廣處理該問題。此法之優點在於不需利用積分轉換即可求得顯式解。除此之外，顯式解中各個反射波及透射波亦可一目了然。本計劃第二年度考慮雙材料的問題。

關鍵詞：彈性波傳，異向性材料，掩埋波源，雙材料。

Abstract

A transient analysis of a line force or dislocation in an anisotropic elastic homogeneous halfspace or a bimaterial infinite space is made in this project. A new method proposed by Wu (2000) for anisotropic elastodynamics is extended to treat the problem. A major advantage of the method is that explicit solution can be derived without the use of integral transforms. Furthermore, individual reflected and refracted waves can be clearly identified in the solution. In the second year a bimaterial problem is considered.

Keywords: elastic wave propagation,
anisotropic material, buried source,
bimaterial.

2、INTRODUCTION

Wave propagation in stratified elastic media has been a subject of interest in the fields of acoustics, seismology and among others. Due to the existence of interfaces, the reflected and transmitted waves occur and interfacial waves may also arise. These physical phenomena, the mechanical properties of individual layers and so forth make analysis of the problems more complicated.

In two-dimensional anisotropic elastostatics, the Stroh’s formalism is a useful and powerful method to solve various problems. A distinctive feature of the Stroh’s formalism is that the general solution is expressed in terms of the eigenvalues and eigenvectors of a six-dimensional related eigenvalue problem. The readers are referred to [1] for more detailed discussions. An operative and useful method of elastodynamics is the integral transform technique. But, sometimes, the difficulty of inverting integral transform renders this method not practically easy. Cagniard-de Hoop technique [2] is one of workable methods for inversion of the Laplace transform. Especially for isotropic media, the Smirnov-Soloblev method [3] is an alternative method to solve self-similar elastodynamic problems without the need of integral transforms. The Smirnov-Soloblev method has been extended to general anisotropic elastic materials by Wu [4]. The formulation by Wu is similar to Stroh's formalism and maintains many of its advantages.

Self-similar problems were defined as problems to which the displacements are homogeneous functions of time t and position $\mathbf{x}$ in the study [4]. In other words, the physical systems thus involve neither a characteristic length nor a characteristic time. A characteristic length appears in the present problem of a dynamic buried source in a biomaterial. That problem is therefore non-self-similar and the formulation developed by Wu [3] cannot be applied. In this paper we modify Wu's formulation to treat the problem of interest. The present formulation also casts the two-dimensional anisotropic elastodynamic problem into a six-dimensional eigenvalue problem and the general solution is directly expressed in terms of the eigenvalues and eigenvectors in the time domain.

3、FORMULATION

The equations of motion expressed in terms of displacements are

(1) $\mathbf{Q}\mathbf{u}_{,11}+(\mathbf{S}+\mathbf{S}^T)\mathbf{u}_{,12}+\mathbf{W}\mathbf{u}_{,22}=\rho\ddot{\mathbf{u}}.$

where the matrices $\mathbf{Q}$, $\mathbf{S}$, and $\mathbf{W}$ are related to elastic constants C_{ijks} by $Q_{ik}=C_{i1k1}$, $S_{ik}=C_{i1k2}, W_{ik}=C_{i2k2}$. Let the displacement be assumed as $\mathbf{u}(x_1,x_2,t)=\mathbf{u}(w)$ with the variable $w(x_1,x_2,t)$ implicitly defined by

(2) $\phi(w,x_1,x_2,t)=wt-x_1-p(w)x_2-q(w)=0$

where $p(w)$ is the function of w stipulated by equation (1) and $q(w)$ is an arbitrary given function of w. The special case $q(w)=0$ has been discussed by Wu [4]. Let $\mathbf{u}'(w)$ be expressed as $\mathbf{u}'(w)=f(w)\mathbf{a}(w)$ where $f(w)$ is an arbitrary scalar function of w. It follows that $\mathbf{u}(w)$ is a solution of equation (1) if

(3) $\mathbf{D}(p,w)\mathbf{a}(w)=\mathbf{0}$

where $\mathbf{D}(p,w)$ is given by $\mathbf{D}(p,w)=\mathbf{Q}+p(\mathbf{S}+\mathbf{S}^T)+p^2\mathbf{W}-\rho w^2\mathbf{I}.$ The general solution of equation (1) may be represented as

(4) $\mathbf{u}_{,1}=2\,\mathrm{Re}\{\sum_k f_k(w_k)(\partial w_k/\partial x_1)\mathbf{a}_k(w_k)\}$

(5) $\mathbf{u}_{,2}=2\,\mathrm{Re}\{\sum_k f_k(w_k)(\partial w_k/\partial x_2)\mathbf{a}_k(w_k)\}$

(6) $\dot{\mathbf{u}}=-2\,\mathrm{Re}\{\sum_k w_k f_k(w_k)(\partial w_k/\partial x_1)\mathbf{a}_k(w_k)\}$

where f_k is an arbitrary function of w, $k=1,2,3$ or $4,5,6$. The choice of the range of k depends on whether up-going rays or down-going rays are considered. The general solutions of the stress vectors $\mathbf{t}_1$ and $\mathbf{t}_2$ can be expressed as

(7) $$\begin{aligned}\mathbf{t}_1=2\,\mathrm{Re}\{\sum_k f_k[\rho w_k^2(\partial w_k/\partial x_1)\mathbf{a}_k(w_k)\\ -p_k(\partial w_k/\partial x_1)\mathbf{b}_k(w_k)]\}\end{aligned}$$

(8) $\mathbf{t}_2=2\,\mathrm{Re}\{\sum_k f_k(\partial w_k/\partial x_1)\mathbf{b}_k(w_k)\}$

where

(9) $\mathbf{b}_k(w)=(\mathbf{S}^T+p_k(w)\mathbf{W})\mathbf{a}_k(w).$

Equation (9) can be cast into the following six-dimensional eigenvalue problem

(10) $\mathbf{N}(w)\boldsymbol{\xi}(w)=p(w)\boldsymbol{\xi}(w)$

where $\mathbf{N}(w)=\begin{pmatrix}\mathbf{N}_1 & \mathbf{N}_2\\ \mathbf{N}_3(w) & \mathbf{N}_1^T\end{pmatrix}$, $\boldsymbol{\xi}(w)=\begin{pmatrix}\mathbf{a}(w)\\ \mathbf{b}(w)\end{pmatrix}$, $\mathbf{N}_1=-\mathbf{W}^{-1}\mathbf{S}^T$, $\mathbf{N}_2=\mathbf{W}^{-1}$, $\mathbf{N}_3(w)=\mathbf{S}\mathbf{W}^{-1}\mathbf{S}^T-\mathbf{Q}+\rho w^2\mathbf{I}$. Equation (10) is in the same form as that in Stroh's formalism for steady state motion [5]. The p and $\boldsymbol{\xi}$ are the eigenvalue and right eigenvector, respectively, of $\mathbf{N}$

4. BURIED DYNAMIC SOURCES IN A BIMATERIAL

Consider a bimaterial consisting of two dissimilar elastic half-spaces perfectly bonded together. Let the half-space $x_2 \geq 0$ be occupied by material 1 and the half-space $x_2 \leq 0$ be occupied by material 2. The bimaterial is initially stress-free and is subjected to a line force $H(t)\mathbf{F}$ and a dislocation of Burgers vector $H(t)\boldsymbol{\beta}$ at $x_1=0$ and $x_2=h>0$, $H(t)$ being a Heaviside step function. Figure 1 summarizes the configuration of this problem. The continuity conditions at the interface $x_2=0$ are given by

$$\mathbf{u}_{,1}(x_1,0^+,t)=\mathbf{u}^*_{,1}(x_1,0^-,t) \tag{11}$$

$$\mathbf{t}_2(x_1,0^+,t)=\mathbf{t}^*_2(x_1,0^-,t) \tag{12}$$

where the superscript "*" denotes quantities referred to material 2. Let $\mathbf{u}_{,1}$ ant $\mathbf{t}_2$ in the upper half-space be expressed as

$$\mathbf{u}_{,1}=\mathbf{u}^{(0)}_{,1}+\mathbf{u}^{(1)}_{,1} \tag{13}$$

$$\mathbf{t}_2=\mathbf{t}^{(0)}_2+\mathbf{t}^{(1)}_2 \tag{14}$$

where $\mathbf{u}^{(0)}_{,1}$ and $\mathbf{t}^{(0)}_2$ are, respectively, the displacement gradient and the stress vector due to the sources in an infinite medium, $\mathbf{u}^{(1)}_{,1}$ and $\mathbf{t}^{(1)}_2$ are those due to the reflected waves from the interface. The solution for the line force in an infinite medium has been obtained by Wu (2000) and that for the line dislocation may be derived similarly. The result is

$$\mathbf{u}^{(0)}_{,1}=(-1/\pi)\,\mathrm{Im}\{\sum_{k=4}^{6}(c_k/w_k)(\partial w_k/\partial x_1)\mathbf{a}_k\}, \tag{15}$$

$$\mathbf{t}^{(0)}_2=(-1/\pi)\,\mathrm{Im}\{\sum_{k=4}^{6}(c_k/w_k)(\partial w_k/\partial x_1)\mathbf{b}_k\}, \tag{16}$$

where $c_k=\mathbf{a}_k^T\mathbf{F}+\mathbf{b}_k^T\boldsymbol{\beta}$ and w_k in equation (2) is determined by taking $q(w)=-p(w)h$ so that $w_k=y_1+p_k(w_k)y_2$ with $y_1=x_1/t$, $y_2=(x_2-h)/t$. Since up-going waves are generated in material 1 and down-going waves are in material 2, the expressions for $\mathbf{u}^{(1)}_{,1}$ and $\mathbf{t}^{(1)}_2$ in material 1 are given by

$$\begin{aligned}\mathbf{u}^{(1)}_{,1}&=(-1/\pi)\sum_{k=4}^{6}\sum_{j=1}^{3}\mathrm{Im}\{R_{kj}(c_k/w_{kj})\\ &(\partial w_{kj}/\partial x_1)\mathbf{a}_j\}\end{aligned} \tag{17}$$

$$\begin{aligned}\mathbf{t}^{(1)}_2&=(-1/\pi)\sum_{k=4}^{6}\sum_{j=1}^{3}\mathrm{Im}\{R_{kj}(c_k/w_{kj})\\ &(\partial w_{kj}/\partial x_1)\mathbf{b}_j\}\end{aligned} \tag{18}$$

where $w_{kj}t=x_1+p_j(w_{kj})x_2-p_k(w_{kj})h$ and R_{kj} may be regarded as the reflection coefficients. Those for material 2 are

$$\begin{aligned}\mathbf{u}^*_{,1}&=(-1/\pi)\sum_{k=4}^{6}\sum_{j=4}^{6}\mathrm{Im}\{T_{kj}(c_k/w^*_{kj})\\ &(\partial w^*_{kj}/\partial x_1)\mathbf{a}^*_j\}\end{aligned} \tag{19}$$

$$\begin{aligned}\mathbf{t}^*_2&=(-1/\pi)\sum_{k=4}^{6}\sum_{j=4}^{6}\mathrm{Im}\{T_{kj}(c_k/w^*_{kj})\\ &(\partial w^*_{kj}/\partial x_1)\mathbf{b}^*_j\}\end{aligned} \tag{20}$$

where $w^*_{kj}t=x_1+p^*_j(w^*_{kj})x_2-p_k(w^*_{kj})h$ and T_{kj} may be regarded as the transmission coefficients. Note that at the interface $x_2=0$, $w_{kj}(x_1,t)=w^*_{kj}(x_1,t)=w_k(x_1,t)$. The continuity conditions of equations (11) and (12) then lead to

$$\mathbf{A}(w_k)\mathbf{R}_k(w_k)-\hat{\mathbf{A}}^*(w_k)\hat{\mathbf{T}}_k(w_k)=-\mathbf{a}_k(w_k) \tag{21}$$

(22) $\mathbf{B}(w_k)\mathbf{R}_k(w_k) - \hat{\mathbf{B}}^*(w_k)\hat{\mathbf{T}}_k(w_k) = -\mathbf{b}_k(w_k)$

where $\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3]$, $\mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3]$, $\mathbf{R}_k = [R_{k1} \quad R_{k2} \quad R_{k3}]^T$, $\hat{\mathbf{A}}^* = [\mathbf{a}_4^* \quad \mathbf{a}_5^* \quad \mathbf{a}_6^*]$, $\hat{\mathbf{B}}^* = [\mathbf{b}_4^* \quad \mathbf{b}_5^* \quad \mathbf{b}_6^*]$, $\hat{\mathbf{T}}_k = [T_{k4} \quad T_{k5} \quad T_{k6}]^T$.

With solving equations (21) and (22) yields

(23) $\mathbf{R}_k = \mathbf{A}^{-1}\mathbf{M}^{-1}[\mathbf{M}_2^*\mathbf{a}_k + i\mathbf{b}_k]$,

(24) $\hat{\mathbf{T}}_k = (\hat{\mathbf{A}}^*)^{-1}\mathbf{M}^{-1}[\mathbf{M}_1\mathbf{a}_k + i\mathbf{b}_k]$

where $\mathbf{M}_1 = -i\mathbf{B}\mathbf{A}^{-1}$, $\mathbf{M}_2^* = -i\hat{\mathbf{B}}^*(\hat{\mathbf{A}}^*)^{-1}$ and $\mathbf{M} = \mathbf{M}_1 - \mathbf{M}_2^*$. The functions $R_{kj}(w_{kj})$ and $T_{kj}(w_{kj}^*)$ can be obtained from $\mathbf{R}_k$ and $\hat{\mathbf{T}}_k$, respectively, as ($k = 4,5,6$)

$$R_{kj} = \mathbf{e}_j^T\mathbf{R}_k,\ j = 1,2,3;\ T_{kj} = \mathbf{e}_{j-3}^T\hat{\mathbf{T}}_k,\ j = 4,5,6.$$

where $\mathbf{e}_j$ is the unit vector in the x_j – direction.

5. NUMERICAL EXAMPLE

For fixed x_1, x_2 and t expand $\phi(w,t)$ in equation (2) about w_0 up to the second order terms by Taylor's series,

(25) $\phi(w) = \phi_0 + (\frac{\partial\phi}{\partial w})_0\Delta w + \frac{1}{2}(\frac{\partial^2\phi}{\partial w^2})_0(\Delta w)^2 = 0$

where $\Delta w = w - w_0$, and $(f)_0 = f(w_0)$. Equation (25) can be regarded as a quadratic equation of Δw:

(26) $a(\Delta w)^2 + 2b\Delta w + c = 0$

In equation (26), a, b, and c are given by

$a = -p_j''(w_0)x_2 + p_k''(w_0)h$,

$b = t - p_j'(w_0)x_2 + p_k'(w_0)h$, $c = 2\phi_0$.

Equation (26) yields $\Delta w = (-b + \sqrt{b^2 - ac})/a$ for a given w_0. Let $w_1 = w_0 + \Delta w$. If $|\phi(w_1)| < \varepsilon$, where ε is a preset error, then w_1 is accepted as the solution of w for given x_1, x_2 and t. Otherwise equation (26) is solved again with w_0 replaced by w_1. The process is repeated until the error criterion is met.

The vertical displacement due to a unit vertical line impulse has been calculated for GaAs. The result is expressed in the following dimensionless form:

$$\bar{\mathbf{G}}_f^* = \pi\rho c_0 h\mathbf{G}_f^*(x_1, x_2, t),$$

where $\tau = tc_0/h$, $c_0 = \sqrt{C_{44}/\rho}$, ρ is density and

$$\mathbf{G}_f^*(x_1, x_2, t) = \frac{1}{\pi}\sum_{k=4}^{6}\sum_{j=4}^{6}\mathrm{Im}\{T_{kj}(w_{kj}^*)$$
$$\frac{\partial w_{kj}^*}{\partial x_1}\mathbf{a}_j^*(w_{kj}^*)\mathbf{a}_k^T(w_{kj}^*)\}.$$

The elastic constants of GaAs with respect to the symmetry axes in units of 100 GPa are C_{11} = 1.19, C_{12} = 0.538, and C_{44} = 0.595. For a start, the bimaterial is an infinite GaAs crystal. Then it is cut into two regions: $x_2 \geq 0$ and $x_2 \leq 0$. Let the region $x_2 \geq 0$ (material 1) be rotated by $10°$ and the region $x_2 \leq 0$ (material 2) by $-10°$ about the normal to the interface of the bimaterial. The result of $(\bar{\mathbf{G}}_f^*)_{22}$ as a function of τ for $x_1/h = 1.3$ and $x_2/h = -1.2$ is given in Figure 2. It is seen that all wave arrivals are accurately captured.

7. CONCLUSIONS

This study based on a previous investigation proposed by Wu (2000) develops a method to deal with the two-dimensional problem of a line force or dislocation in an anisotropic elastic medium. We can obtain analytic solutions for the problem of interest via the method without the need of performing integral transforms.

In this study the method is applied to treat the problem of a time-dependent buried source in an elastic bimaterial. Numerical results show complicated wave phenomena in the anisotropic elastic bimaterial.

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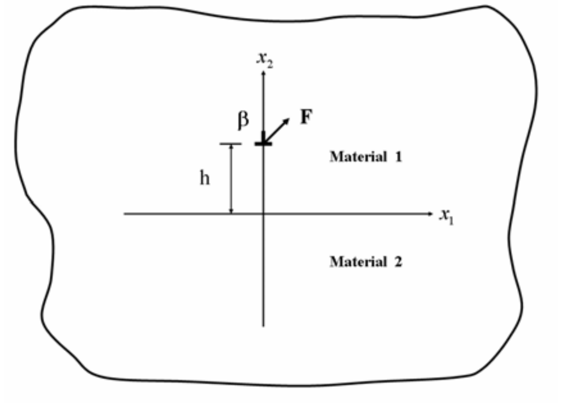

Figure 1. Configuration of the problem of interest.

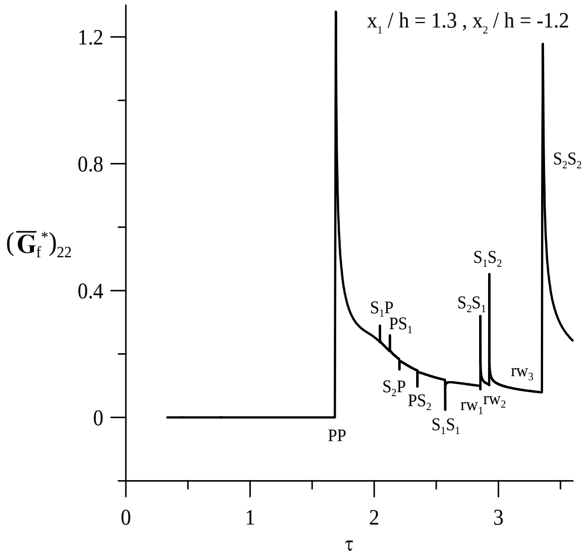

Figure 2. Dimensionless vertical displacement $(\overline{\mathbf{G}}_f^*)_{22}$ for GaAs at $x_1/h = 1.3$ and $x_2/h = -1.2$.