行政院國家科學委員會專題研究計畫　成果報告

連續動力系統裡 Silnikov 式同宿軌道的建造

計畫類別：個別型計畫
計畫編號：NSC91-2115-M-002-020-
執行期間：91 年 12 月 01 日至 92 年 10 月 31 日
執行單位：國立臺灣大學數學系暨研究所

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報告類型：精簡報告
報告附件：出席國際會議研究心得報告及發表論文
處理方式：本計畫可公開查詢

中　華　民　國 93 年 5 月 14 日

The purpose of the project was to construct Silnikov type homoclinic orbits in continuous dynamical systems. I have succeeded in doing this in the context of singularly perturbed systems. This is joint work with Flaviano Battelli of the University of Ancona. It has been published as "Singular perturbations, transversality and Silnikov saddle-focus homoclinic orbits" in *J. Dynamics and Diff.Eqns.* **15**(2003), 357-425. I describe this work below.

We consider a singularly perturbed system

$$\begin{aligned} \dot{x} &= \varepsilon f(x, y, \varepsilon, \lambda) \\ \dot{y} &= g(x, y, \varepsilon, \lambda) \end{aligned} \tag{1}$$

where $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$ and $\varepsilon \in \mathbf{R}$ are small parameters. We assume that $y = 0$ is a normally hyperbolic centre manifold for system (1). Next we make the hypothesis that $(0, 0)$ is a hyperbolic equilibrium on the centre manifold which, when the parameters are zero, has an associated homoclinic orbit $(0, y_0(t))$, which is nondegenerate in the sense that, up to a scalar multiple, $\dot{y}_0(t)$ is the unique nontrivial bounded solution of the variational equation

$$\dot{y} = g_y(0, y_0(t), 0, 0)y.$$

Note that under this condition the adjoint equation

$$\dot{y} = -g_y^*(0, y_0(t), 0, 0)y$$

has, up to a scalar multiple, a unique nontrivial bounded solution $\psi(t)$.

In our main theorem we find a curve in the (λ, ε) parameter space along which the homoclinic orbit to the hyperbolic equilibrium is preserved under perturbation.

MAIN THEOREM. *Suppose that*

$$\int_{-\infty}^{\infty} \psi^*(t) g_\lambda(0, y_0(t), 0, 0) dt \neq 0.$$

Then there exists a function $\lambda(\varepsilon)$ with $\lambda(0) = 0$ such that for ε sufficiently small and nonnegative, system (1) with $\lambda = \lambda(\varepsilon)$ has a homoclinic solution $(x(t, \varepsilon), y(t, \varepsilon))$, that is,

$$(x(t, \varepsilon), y(t, \varepsilon)) \neq (0, 0) \quad \text{but} \quad (x(t, \varepsilon), y(t, \varepsilon) \to (0, 0) \quad \text{as} \quad t \to \pm\infty$$

such that

$$x(t,0) = 0, \quad y(t,0) = y_0(t).$$

Note that the existence part of this theorem is in Szmolyan [4] who gives a mostly geometric proof. Beyn and Stiefenhofer [1] gave an analytic proof of a similar theorem which enabled them to provide more information about the homoclinic solutions as functions of ε. However, they had to impose non-resonance conditions on the eigenvalues when $m > 1$. In this paper we give a different analytic proof but without the imposition of non-resonance conditions. Our analytic proof is closer to Szmolyan's proof in that we find the homoclinic solution as the intersection of certain leaves in the foliated centre stable and centre unstable manifolds but instead of using transversality in a geometric way we find our solutions using the implicit function theorem. Like Beyn and Stiefenhofer, our analytic approach allows us to find additional properties of the homoclinic solution as a function of ε. Moreover we give some additional properties and applications as described below.

Next we consider the transversality properties of the homoclinic orbits.

Theorem 2. *Let the conditions of the Main Theorem hold and let Q be the projection onto the stable subspace along the unstable subspace of the hyperbolic equilibrium on the centre manifold. Then if*

$$\int_{-\infty}^{\infty} \psi^*(t) g_x(0, y_0(t), 0, 0) dt \left[f_x(0,0,0,0)(\mathbf{I} - Q) \int_{-\infty}^{+\infty} f(0, y_0(t), 0, 0) dt \right] \neq 0,$$

the centre-stable manifold $\mathcal{W}^{cs}$ and the unstable manifold $\mathcal{W}^u$ intersect transversely at $(x(0,\varepsilon), y(0,\varepsilon))$; if

$$\int_{-\infty}^{\infty} \psi^*(t) g_x(0, y_0(t), 0, 0) dt \left[f_x(0,0,0,0) Q \int_{-\infty}^{+\infty} f(0, y_0(t), 0, 0) dt \right] \neq 0,$$

the centre-unstable manifold $\mathcal{W}^{cu}$ and the stable manifold $\mathcal{W}^s$ intersect transversely at $(x(0,\varepsilon), y(0,\varepsilon))$.

Next we state a theorem giving certain properties that the solutions we found in the Main Theorem possess.

Theorem 3. *Let the conditions of the Main Theorem be satisfied. Then the solution $(x,t,\varepsilon), y(t,\varepsilon)$ has the following properties:*

(i) *it is in general position, that is, the tangent spaces to the stable manifold and unstable manifold of the hyperbolic equilibrium* $(0,0)$ *of the system*

$$\begin{aligned}\dot{x} &= \varepsilon f(x,y,\varepsilon,\lambda(\varepsilon))\\ \dot{y} &= g(x,y,\varepsilon,\lambda(\varepsilon))\end{aligned}$$

at $(x(t,\varepsilon),y(t,\varepsilon))$ *intersect in the one-dimensional subspace spanned by* $(\dot{x}(t,\varepsilon),\dot{y}(t,\varepsilon))$*;*

(ii) *the solution* $(x(t,\varepsilon),y(t,\varepsilon))$ *is asymptotically tangent to the centre manifold* $y=0$ *as* $t\to-\infty$*, provided that*

$$(\mathbf{I}-Q)\int_{-\infty}^{\infty} f(0,y_0(t),0,0)dt \neq 0;$$

(iii) *the solution* $(x(t,\varepsilon),y(t,\varepsilon))$ *is asymptotically tangent to the centre manifold* $y=0$ *as* $t\to\infty$*, provided that*

$$Q\int_{-\infty}^{\infty} f(0,y_0(t),0,0)dt \neq 0;$$

(iv) *if we denote the unstable fibre over* $p(t,\varepsilon)=(x(t,\varepsilon),y(t,\varepsilon))$ *by* $\mathcal{W}^u_{p(t,\varepsilon)}$ *and the unstable fibre over* $(0,0)$ *by* $\mathcal{W}^u_{(0,0)}$ *then*

$$\lim_{t\to\infty} T_{p(t,\varepsilon)}\mathcal{W}^u_{p(t,\varepsilon)} = T_{(0,0)}\mathcal{W}^u_{(0,0)},$$

provided that $\mathcal{W}^u$ *intersects* $\mathcal{W}^{cs}$ *transversely along* $p(t,\varepsilon)$*;*

(v) *if we denote the stable fibre over* $p(t,\varepsilon)=(x(t,\varepsilon),y(t,\varepsilon))$ *by* $\mathcal{W}^s_{p(t,\varepsilon)}$ *and the stable fibre over* $(0,0)$ *by* $\mathcal{W}^s_{(0,0)}$ *then*

$$\lim_{t\to-\infty} T_{p(t,\varepsilon)}\mathcal{W}^s_{p(t,\varepsilon)} = T_{(0,0)}\mathcal{W}^s_{(0,0)},$$

provided that $\mathcal{W}^s$ *intersects* $\mathcal{W}^{cu}$ *transversely along* $p(t,\varepsilon)$*.*

It follows that if the first condition in Theorem 2 holds, then Theorem 3 (iv) is satisfied. It turns out that this last condition implies the "strong inclination" condition in Deng [2], thus allowing us to construct examples of systems in four dimensions which have Sil'nikov saddle-focus homoclinic orbits and hence are chaotic.

We now show how to construct these examples. Consider a system

$$\begin{aligned}\dot{x} &= \varepsilon[f(x)+y]\\ \dot{y} &= g(y)+\lambda y+h(x)y,\end{aligned}$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$ are in $\mathbf{R}^2$. The function f has the property that 0 is an unstable focus for

$$\dot{x} = f(x).$$

Then we take

$$h(x) = \begin{bmatrix} \sin x_2 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad g(y) = \begin{bmatrix} y_2 \\ g_0(y_1) \end{bmatrix},$$

where $g(0) = 0$ and $g_0'(0) > 0$ so that 0 is a saddle for

$$\dot{y} = g(y).$$

Also we assume the latter equation has a solution

$$y_0(t) = \begin{bmatrix} p(t) \\ \dot{p}(t) \end{bmatrix}$$

such that $(p(t), \dot{p}(t)) \to (0, 0)$ as $t \to \pm\infty$.

References

[1] Beyn, W.-J. and Stiefenhofer, M., A direct approach to homoclinic orbits in the fast dynamics of singularly perturbed systems, J. Dynamics Diff. Eqns. **99**, (1999), 671-709.

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[3] Sil'nikov, L. P., A contribution to the problem of the structure of an extended neighborhood of a rough equilibrium state of saddle-focus type, Math. USSR Sbornik **10**, (1970), 91-106.

[4] Szmolyan, P., Transversal heteroclinic and homoclinic orbits in singular perturbation problems, J. Diff. Equations **92**, (1991), 252-281.