行政院國家科學委員會補助專題研究計畫成果報告

※※※※※※※※※※※※※※※※※※※※※※※※※
※　　　　　　　　　　　　　　　　　　　　　　　※
※　　　　　　質環上斜導算之原始等式　　　　　　※
※　　　　　　　　　　　　　　　　　　　　　　　※
※※※※※※※※※※※※※※※※※※※※※※※※※

計畫類別：☑個別型計畫　　☐整合型計畫
計畫編號：NSC 90-2115-M-002-012
執行期間：90 年 8 月 1 日至 91 年 7 月 31 日

計畫主持人：莊正良
共同主持人：

本成果報告包括以下應繳交之附件：
☐赴國外出差或研習心得報告一份
☐赴大陸地區出差或研習心得報告一份
☐出席國際學術會議心得報告及發表之論文各一份
☐國際合作研究計畫國外研究報告書一份

執行單位：台大數學系

中　華　民　國九十一年十一月二十二日

ALGEBRAIC q–SKEW DERIVATIONS

Chen–Lian Chuang and Tsiu–Kwen Lee

Department of Mathematics
National Taiwan University
Taipei 106, TAIWAN
E–mail: chuang@math.ntu.edu.tw
E–mail: tklee@math.ntu.edu.tw

Abstract. Let R be a prime ring with extended centroid C. If a non–nilpotent q–skew σ–derivation is C–algebraic, we prove that its automorphism σ must also be C–algebraic.

1

† 2000 *Mathematics Subject Classification.* 16W20, 16W25, 16W55.

‡ *Key words and phrases.* Automorphism, prime ring, Martindale quotient ring, K–polynomial, skew derivation.

§0. Introduction

Throughout the paper, R is always a prime ring in the sense that, for any $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$. We let Q stand for the symmetric Martindale quotient ring of R (see [1] for its definition). The center C of Q is called the extended centroid of R. By a σ–derivation of R, where σ is an automorphism of R, we mean a map $\delta : R \to R$ such that $(x+y)^\delta = x^\delta + y^\delta$ and $(xy)^\delta = x^\delta y + x^\sigma y^\delta$ for $x, y \in R$. Generally, we call σ–derivations *skew* derivations. For $b \in R$, the map $\mathrm{ad}_\sigma(b) : x \mapsto x^\sigma b - bx$ defines a σ–derivation of R. Following [10], we must work in a more general context as follows: We topologize Q by endowing $x \in Q$ with the neighborhood system consisting of x plus a nonzero two–sided ideal of R. An automorphism σ of Q is called *bi–continuous* if both σ and σ^{-1} are continuous maps from Q into Q. Let $\mathcal{A}(R)$ be the set of bi–continuous automorphisms of Q. Let $\sigma \in \mathcal{A}(R)$. By a *continuous* σ–derivation of R we mean a σ–derivation δ of Q such that the map $\delta : Q \to Q$ is continuous. Following Kharchenko and Popov [**10**], we define, for $\sigma \in \mathcal{A}(R)$,

$$\mathcal{L}_\sigma(R) \stackrel{\text{def.}}{=} \text{the set of continuous } \sigma\text{–derivations of } R.$$

A continuous σ–derivation of R is called X–inner if it is of the form $\mathrm{ad}_\sigma(b)$ for some

---
1

$b \in Q$. Otherwise, it is said to be X–outer. All skew derivations considered here are continuous. We will study the following properties:

Definition: We let $R_{\mathcal{F}}$ denote the left Martindale quotient ring of R. A continuous σ–derivation δ of R is said to be *left $R_{\mathcal{F}}$–algebraic*, if, for all $x \in R$,

$$b_0 x^{\delta^n} + b_1 x^{\delta^{n-1}} + b_2 x^{\delta^{n-2}} + \cdots + b_{n-1} x^{\delta} = 0,$$

where $b_0 \neq 0, b_1, \cdots, b_n \in R_{\mathcal{F}}$. We say that δ is C–algebraic if all $b_i \in C$. The left $R_{\mathcal{F}}$–algebraicity and C–algebraicity for a bi-continuous automorphism σ of R are defined analogously.

But here we restrict our interest to the following interesting special class:

Definition: A continuous σ–derivation δ of R is called *q–skew*, where $q \in C$, if $q^{\sigma} = q$, $q^{\delta} = 0$ and $\sigma\delta\sigma^{-1} = q\delta$.

Our main aim is to prove that the automorphism σ of a C–algebraic q–skew σ–derivation must also be C–algebraic, unless the skew derivation is nilpotent. In some recent results on C–algebraic q–skew derivations, our result is useful in removing the assumption on the C–algebraicity of the associated automorphism. This work will be taken up somewhere else.

Our work here is strongly motivated by Leroy's work [11], where the following is shown:

Under the assumption that σ is left $R_{\mathcal{F}}$–algebraic, a σ–derivation δ satisfying $\sigma\delta = \delta\sigma$ is left $R_{\mathcal{F}}$–algebraic if and only if it is C–algebraic (Theorem 6 [11]).

On the other hand, Leroy and Matczuk constructed a q–skew derivation which is left $R_{\mathcal{F}}$–algebraic but which is not C–algebraic (Example 3.3 [12]). In this example, the associated automorphism of the q–skew derivation is not C–algebraic. It is thus natural to ask whether the C–algebraicity of a q–skew σ–derivation implies the C–algebraicity of σ. It is easy to construct nilpotent q–skew σ–derivations with σ not C–algebraic. It turns out that nilpotent q–skew derivations are the only exception. We actually prove the following generalization of Leroy's result:

For a non-nilpotent continuous q–skew σ–derivation δ of R, the C–algebraicity of δ is equivalent to the left $R_{\mathcal{F}}$–algebraicity of δ plus the C–algebraicity of σ (Theorem 4 below).

We remark that our result is false if the σ–derivation fails to be q–skew. This paper is organized as follows: In §1, we examine K–polynomials for q–skew derivations. In §2, we characterize left $R_{\mathcal{F}}$–algebraicity of q–skew derivations in terms of K–polynomials. In §3, we prove our main result.

§1. Minimal K–POLYNOMIALS

Our basic tools here are K–polynomials introduced in [3]. Let us recall some notions there. Let δ be a continuous σ–derivation of R. A direct expansion shows that

$$(xy)^{\delta^n} = \sum_{i=0}^{n} x^{D_{i,n-i}} y^{\delta^{n-i}},$$

where $D_{i,n-i}$ denote the sum of all distinct products with i δ's and $n-i$ σ's. Suppose that δ is q–skew. We have the identity [6, p.11]:

$$D_{i,n-i} = \binom{n}{i}_q \delta^i \sigma^{n-i},$$

where we define

$$\binom{n}{i}_q \overset{\text{def.}}{=} \frac{(n\,!)_q}{(i\,!)_q((n-i)\,!)_q}$$

with $(0\,!)_q \overset{\text{def.}}{=} 1$ and, for $j > 0$,

$$(j\,!)_q \overset{\text{def.}}{=} (1)(1+q)\cdots(1+q+\cdots+q^{j-3}+q^{j-2})(1+q+\cdots+q^{j-2}+q^{j-1}).$$

By Lemma 6.1 [5], if we regard q as an indeterminate, then $\binom{n}{i}_q$ is a polynomial in q and hence is always well–defined for any q. But we only need the following: Let n be the least integer such that $q^n = 1$. If $n > 1$, then $1 + q + \cdots + q^{n-1} = (q^n - 1)/(q-1) = 0$ and $1 + q + \cdots + q^{i-1} = (q^i - 1)/(q-1) \neq 0$ for $i < n$. So $(n\,!)_q = 0$ and, $(i\,!)_q \neq 0$ for $0 < i < n$. This implies, for $0 < i < n$, $\binom{n}{i}_q = 0$ and hence $D_{i,n-i} = 0$. So $(xy)^{\delta^n} = x^{\delta^n} y + x^{\sigma^n} y^{\delta^n}$ for $x, y \in Q$. Thus δ^n defines a σ^n–derivation. Also, $\sigma\delta^n\sigma^{-1} = (\sigma\delta\sigma^{-1})^n = (q\delta)^n = q^n\delta^n = \delta^n$. So $\sigma^n\delta^n = \delta^n\sigma^n$. We formulate this well–known fact in the following:

Lemma 1. *Let δ be a continuous q–skew σ–derivation of R. If ν is the least positive integer such that $q^\nu = 1$, then δ^ν defines a σ^ν–derivation satisfying $\sigma^\nu\delta^\nu = \delta^\nu\sigma^\nu$.*

We need a simple number theoretic result in the proof of Theorem 1:

Lemma 2. *Let p be a prime ≥ 2. Then $\binom{p^n k}{p^n} \equiv_p k$ for positive integer k.*

Proof. To see this, we work in $\mathbf{Z}_p[t]$, where $\mathbf{Z}_p$ denotes the ring of integers modulo p. Note that $(1+t)^{p^n} = 1 + t^{p^n}$. Using this, we have $(1 + t^{p^n})^k = (1+t)^{p^n k}$. The assertion follows by comparing the coefficients of t^{p^n} in their expansions.

Given $t \in Q$, we let $r(t)$, $\ell(t)$ denote respectively the right and the left multiplication by t:

$$r(t) : x \mapsto xt \qquad \text{and} \qquad \ell(t) : x \mapsto tx \qquad \text{for } x \in Q.$$

The following notion given in [3] provides the most important tool of this paper.

Definition [3]: Given a continuous σ–derivation δ of R, let $D_{s,t}$, where $s, t \geq 0$, denote the sum of all distinct products with s δ's and t σ's. By a K–polynomial of δ with δ–order m, we mean an expression of the form:

$$\psi(x) = x^{\delta^m} + \sum_{j=1}^{m-1} t_j x^{\delta^{m-j}},$$

where $t_1, \ldots, t_{m-1} \in Q$, $m \geq 1$, satisfy

$$D_{k,m-k} + \sum_{j=1}^{k-1} D_{k-j,m-k}\ell(t_j) = \sigma^m r(t_k) - \sigma^{m-k}\ell(t_k) \qquad \text{for } k = 1, \ldots, m-1.$$

By the *minimal* K–identity of δ, we mean an identity of the form

$$\psi(x) = x^{\sigma^m} t_m - t_m x,$$

where $t_m \in Q$ and $\psi(x)$ is a K–polynomial with the minimal possible δ–order $m \geq 1$. We also call the corresponding $\psi(x)$ the *minimal* K–polynomial of δ.

Note that x^δ is the K–polynomial of δ with δ–order 1. By Lemma 1 [3], any K–polynomial $\psi(x)$ of δ–order m must define a σ^m–derivation of Q. If μ happens to be X–inner and if the δ–order m of $\psi(x)$ happens to be the minimum among such K–polynomials, then by the definition above, $\psi(x)$ is called the *minimal* K–polynomial of δ. An X–inner σ–derivation surely has the minimal K–polynomial, namely x^δ. But an X–outer σ–derivation δ may or may *not* have the minimal K–polynomial. Theorem 1 of [3] asserts the following: Let $\varphi(z_{ijk})$ be a GP in distinct indeterminates z_{ijk}, where $0 \leq j < m$ in case that δ has the minimal K–polynomial of δ–order m. If $\varphi(x_i^{\delta^j g_k})$, where g_k are mutually outer and x_i are distinct indeterminates, is a GPI of R, then so is $\varphi(z_{ijk})$. By this result, a left $R_{\mathcal{F}}$–algebraic skew derivation must have the minimal K–polynomial. Also, the minimal K–polynomial, if it exists, must be unique. In order to apply this here, we investigate minimal K–identities for q–skew derivations, which assume a particularly simple form:

Theorem 1. *Let δ be an X-outer continuous q-skew σ-derivation of R. Assume that δ has the minimal K-identity*

$$x^{\delta^m} + t_1 x^{\delta^{m-1}} + \cdots + t_{m-1} x^\delta = x^{\sigma^m} t_m - t_m x,$$

where $t_1, \ldots, t_m \in Q$. Then the following hold:

(1) There exists the least integer $\nu \geq 1$ such that $q^\nu = 1$ and δ^ν is a σ^ν–derivation such that $\delta^\nu\sigma^\nu = \sigma^\nu\delta^\nu$.

(2) In the case of $\operatorname{char} R = 0$, we have $m = \nu$ and $t_1 = \cdots = t_{m-1} = 0$.

(3) In the case of $\operatorname{char} R = p > 0$, m and $m - j$ for $1 \leq j \leq m-1$ with $t_j \neq 0$ are the form νp^n for some integer $n \geq 0$, and each nonzero t_j is a unit defining the X–inner automorphism σ^j. Also, $t_j^\sigma = t_j$ for $1 \leq j \leq m-1$ and, moreover, $t_j^\delta = 0$ for $1 \leq j \leq m-1$, unless $q = 1$ and σ is X–inner.

This theorem is best understood in terms of $\partial \overset{\text{def.}}{=} \delta^\nu$, where ν is the least integer ≥ 1 such that $q^\nu = 1$. The minimal K-polynomial of δ comes from the minimal K-polynomial of ∂ by substituting δ^ν for ∂. In the case of $\operatorname{char} R = 0$, the minimal K-polynomial of ∂ is just x^∂, that is, ∂ is X–inner. In the case of $\operatorname{char} R = p > 0$, the minimal K-polynomial of ∂ assumes the form

$$x^{\partial^{p^s}} + \sum_{i=1}^{s} t_i' x^{\partial^{p^{s-i}}},$$

where each nonzero t_j' is a unit defining the X–inner automorphism $\sigma^{\nu(p^s - p^{s-i})}$. Also all t_i' are constants of σ and, unless $q = 1$ and σ is X–inner, all t_i' are also constants of δ.

An interesting consequence is the following: In case of $\operatorname{char} R = 0$, if δ is a left $R_{\mathcal{F}}$–algebraic 1–skew σ–derivation, that is, if $\delta\sigma = \sigma\delta$, then $\delta = \delta^\nu$ must be X–inner. This is Proposition 1 of Leroy [11].

Proof of Theorem 1: We will often use the basic fact: If $aQ = Qa$ for some $0 \neq a \in Q$, then a is a unit. The proof is completed by a series of Claims. By the definition of K–polynomials, for each $1 \leq i \leq m-1$,

$$x^{D_{i,m-i}} + t_1 x^{D_{i-1,m-i}} + \cdots + t_{i-1} x^{D_{1,m-i}} = x^{\sigma^m} t_i - t_i x^{\sigma^{m-i}} \tag{2.1}$$

for $x \in Q$. This is actually an identity only in δ in view of the equality $\sigma\delta = q\delta\sigma$. Theorem 1 [3] asserts that any identities of δ–order $< m$ must be trivial. So (2.1) is trivial in the sense that, for each $j = 0, \ldots, i-1$, either $t_j = 0$ or $D_{i-j,m-i} = 0$, where $t_0 \overset{\text{def.}}{=} 1$.

Fix $t_j \neq 0$, where $0 \leq j \leq m-2$. We first examine (2.1) for $i = m-1$: The triviality of (2.1) says

$$D_{m-1-j,1} = (1 + q + \cdots + q^{m-1-j})\delta^{m-1-j}\sigma = 0.$$

If $q \neq 1$, then $0 = 1 + q + \cdots + q^{m-1-j} = (q^{m-j} - 1)/(q-1)$ implies $q^{m-j} = 1$. If $q = 1$, then $q^{m-j} = 1$ surely holds. So $q^{m-j} = 1$ always. There thus exists $\nu \geq 1$

such that $q^\nu = 1$. Let ν be the least such integer. Then ν divides $m-j$. Write $m-j = n\nu$. By Lemma 1, δ^ν is a 1–skew σ^ν–derivation. So we have

$$(xy)^{\delta^{m-j}} = (xy)^{\delta^{n\nu}} = \sum_{k=0}^{n} \binom{n}{k} x^{\delta^{\nu(n-k)}\sigma^{k\nu}} y^{\delta^{k\nu}}.$$

By Theorem 1 [3], comparing this with the expansion

$$(xy)^{\delta^{m-j}} = \sum_{s=0}^{m-j} x^{D_{m-j-s,s}} y^{\delta^s},$$

we have

$$D_{m-j-k\nu,k\nu} = D_{(n-k)\nu,k\nu} = \binom{n}{k} \delta^{\nu(n-k)}\sigma^{k\nu}.$$

For $0 < k < n$, $D_{m-j-k\nu,k\nu} = D_{(n-k)\nu,k\nu}$ occurs in (2.1) for $i = m - k\nu$ with the coefficient $t_j \neq 0$. Thus $D_{(n-k)\nu,k\nu} = 0$ and so $\binom{n}{k} = 0$ for all $0 < k < n$. In the case of $\operatorname{char} R = 0$, this implies $n = 1$. Therefore $m - j = \nu$ and this is the only nonzero t_j. This nonzero t_j must be $t_0 = 1$. So $m = \nu$ and all $t_j = 0$ for $j = 1, \ldots, m-2$. Also, m must be > 1, for otherwise, δ would be X–inner. In the case of $\operatorname{char} R = p > 0$, n must be a power of p by Lemma 2. Let us summarize what we have shown:

Claim 1. Let ν be the least integer ≥ 1 such that $q^\nu = 1$. (This ν must exist.) In the case of $\operatorname{char} R = 0$, $m = \nu$ and all $t_1 = \cdots = t_{m-2} = 0$. In the case of $\operatorname{char} R = p > 0$, if $t_j \neq 0$, where $0 \leq j \leq m-2$, then $m - j$ assumes the form $m - j = \nu p^n$ for some integer $n \geq 0$. In particular, $q^m = 1$.

Yet, nothing has been said about t_{m-1}, which occurs in the left side of (2.1) only for $i = m$. We remedy this in Claims 4 and 5 below. Again, let us explore the triviality of (2.1) for $i = 1, \ldots, m-1$.

Claim 2. If $t_j \neq 0$, where $0 \leq j \leq m-1$, then $x^{\sigma^j} = t_j x t_j^{-1}$ for $x \in Q$: The triviality of (2.1) for $i = j$ gives $x^{\sigma^m} t_j = t_j x^{\sigma^{m-j}}$ for $x \in Q$. In particular, $t_j Q = Q t_j$. So t_j is a unit and $x^{\sigma^j} = t_j x t_j^{-1}$ for $x \in Q$.

Claim 3. If a unit $u \in Q$ is such that $x^{\sigma^i} = uxu^{-1}$ for $x \in Q$, then $u^\sigma = q^i u$ and ν divides i^2. Furthermore, if σ is X–outer, $u^\delta = 0$: Observe that $(u^{-1})^\delta = -(u^\sigma)^{-1} u^\delta u^{-1}$. We compute

$$\begin{aligned} q^i u x^\delta u^{-1} &= q^i x^{\delta\sigma^i} = x^{\sigma^i \delta} = (uxu^{-1})^\delta \\ &= u^\delta x u^{-1} + u^\sigma x^\delta u^{-1} - u^\sigma x^\sigma (u^\sigma)^{-1} u^\delta u^{-1}. \end{aligned}$$

Since δ is X–outer, we have $u^\sigma = q^i u$. Therefore, $u^{\sigma^i} = q^{i^2} u$. But $u^{\sigma^i} = uuu^{-1} = u$. So $q^{i^2} = 1$ and hence ν divides i^2. Also, $u^\delta x u^{-1} - u^\sigma x^\sigma (u^\sigma)^{-1} u^\delta u^{-1} = 0$ implies $u^\delta = 0$ if σ is X–outer.

One might speculate on the divisibility of i above by ν. But this is not always the case.

Claim 4. If $t_{m-1} \neq 0$, then $q = 1$ and $\operatorname{char} R = p > 0$: Assume $t_{m-1} \neq 0$. By Claim 2, $x^{\sigma^{m-1}} = t_{m-1} x t_{m-1}^{-1}$ for $x \in Q$. By Claim 3, ν divides $(m-1)^2 = m^2 - 2m + 1$. But ν also divides m by Claim 1. So $\nu = 1$ and hence $q = 1$. If $\operatorname{char} R = 0$, then, by Claim 1, $m = \nu = 1$ and δ is inner, absurd. So $\operatorname{char} R = p > 0$.

With this, the case of $\operatorname{char} R = 0$ is proved. We assume $\operatorname{char} R = p > 0$ from now on.

Claim 5. For $1 \leq j \leq m-1$, we have $t_j^\sigma = t_j$ and if σ is not X–inner, then $t_j^\delta = 0$: The assertions hold trivially if $t_j = 0$. So suppose $t_j \neq 0$. By Claim 2, t_j defines the inner automorphism σ^j. By Claim 3, $t_j^\sigma = q^j t_j$. Since ν divides j for $j \neq m-1$ by Claim 1 and $q = 1$ for $j = m-1$ by Claim 4, we have $q^j = 1$ always and hence $t_j^\sigma = t_j$. If σ is not X–inner, then $t_j^\delta = 0$ by Claim 3 again.

Our proof is completed by the final observation:

Claim 6. If σ is X–inner, then $q = 1$: Set $i = 1$ in Claim 3. Then $u^\sigma = qu$. But $u^\sigma = uuu^{-1} = u$. So $q = 1$ follows.

The following provides a useful reduction for the X–inner case.

Lemma 3. *Every q–skew X–inner σ–derivation δ has the form $\operatorname{ad}_\sigma(b)$ for some $b \in Q$ satisfying either $b^\sigma = b/q$, or $b^\sigma = b + u$, where, in the latter case, $q = 1$ and $u \in Q$ is a unit such that $x^\sigma = uxu^{-1}$ for $x \in Q$.*

Proof. Say $x^\delta = x^\sigma b - bx$, where $b \in Q$. Since δ is q–skew,

$$x^\sigma b^{\sigma^{-1}} - b^{\sigma^{-1}} x = x^{\sigma\delta\sigma^{-1}} = qx^\delta = q(x^\sigma b - bx).$$

So $(b^{\sigma^{-1}} - qb)x = x^\sigma (b^{\sigma^{-1}} - qb)$ for $x \in Q$. If $b^{\sigma^{-1}} - qb = 0$, then $b^\sigma = b/q$. We are done in this case. Suppose that $u \overset{\text{def.}}{=} b^{\sigma^{-1}} - qb \neq 0$. Then σ is the inner automorphism defined by u and $b^{\sigma^{-1}} = qb + u$. If $q = 1$, then $b^\sigma = b + u$. Suppose that $q \neq 1$. We see that $b' \overset{\text{def.}}{=} b + \frac{u}{q-1}$ also defines the same σ–derivation δ and satisfies

$$(b')^{\sigma^{-1}} = b^{\sigma^{-1}} + \frac{u}{q-1} = qb + u + \frac{u}{q-1} = qb + \frac{qu}{q-1} = q(b + \frac{u}{q-1}) = qb'.$$

Replacing b by b', we have $b^\sigma = b/q$, as asserted.

We observe an interesting

Corollary. *Let $\delta \in \mathcal{L}_\sigma(R)$ be q–skew and left $R_{\mathcal{F}}$–algebraic of degree n. If $q^\nu \neq 1$ for all integers $\nu \geq 1$, then δ is X–inner and $\delta^{4n-1} = 0$. Moreover, $\delta = \operatorname{ad}_\sigma(b)$ for some $b \in Q$ satisfying $b^{2n} = 0$ and $b^\sigma = b/q$.*

Proof. Let δ is a q–skew σ–derivation left $R_{\mathcal{F}}$–algebraic of degree n. Say,

$$a_0 x^{\delta^n} + a_1 x^{\delta^{n-1}} + \cdots + a_{n-1} x^{\delta} = 0 \tag{2.2}$$

holds for all $x \in R$, where $a_i \in R_{\mathcal{F}}$ and $a_0 \neq 0$. Suppose towards a contradiction that δ is X–outer. If δ has no the minimal K–identity, then applying Theorem 1 [3] to (2.2) we have $a_0 x_1 + a_1 x_2 + \cdots + a_{n-1} x_n = 0$ for all $x_i \in R$. In particular, $a_0 R = 0$ and so $a_0 = 0$, absurd. Thus δ has the minimal K–identity. Since $q^\nu \neq 1$ for all $\nu \geq 1$, δ must be X–inner by Theorem 1. This contradiction proves δ to be X–inner.

In view of Lemma 3, we may write $\delta = \mathrm{ad}_\sigma(b)$ for some $b \in Q$ with $b^\sigma = b/q$. So $(b^s)^\sigma = b^s/q^s$ for any $s \geq 0$. By Proposition 1 [11], we may choose $a_0 = 1$. A simple induction on $k \geq 1$ shows that

$$(b^s)^{\delta^k} = (\frac{1}{q^s} - 1)(\frac{1}{q^{s+1}} - 1)\cdots(\frac{1}{q^{s+k-1}} - 1)b^{s+k} = \gamma_{s,k} b^{s+k},$$

where $\gamma_{s,k} \overset{\text{def.}}{=} (\frac{1}{q^s} - 1)(\frac{1}{q^{s+1}} - 1)\cdots(\frac{1}{q^{s+k-1}} - 1)$. Since $q^\nu \neq 1$ for $\nu \geq 1$, $\gamma_{s,k} \neq 0$ for all $s, k \geq 1$. By Theorem 2 [3], (2.2) holds on $R_{\mathcal{F}}$. In particular, setting $x = b^s$ in (2.2) we obtain

$$\gamma_{s,n} b^{s+n} + \gamma_{s,n-1} a_1 b^{s+n-1} + \cdots + a_{n-1}\gamma_{s,1} b^{s+1} = 0 \tag{2.3}$$

But the n by n matrix $(\gamma_{s,t})$ with the (s,t)–entry $\gamma_{s,t}$ is nonsingular. (See Appendix for a proof.) It follows from (2.3) that $b^{2n} = 0$. Since $b^\sigma = \frac{b}{q}$, we have $\delta^{4n-1} = 0$ as asserted.

§2. LEFT $R_{\mathcal{F}}$–ALGEBRAICITY

As explained at the beginning of §1, a left $R_{\mathcal{F}}$–algebraic continuous skew derivation always has the minimal K–polynomial by Theorem 1 [3]. But the reverse implication is false in general, since, for instance, an X–inner derivation always has the minimal K–polynomial but is not necessarily $R_{\mathcal{F}}$–algebraic. Our aim here is to characterize left $R_{\mathcal{F}}$–algebraic q–skew derivations. The first theorem below gives a characterization of left $R_{\mathcal{F}}$–algebraic skew derivations in terms of their minimal K–polynomials. Note that this result holds not only for q–skew derivations but for any skew derivations:

Theorem 2. *A continuous skew derivation is left $R_{\mathcal{F}}$–algebraic if and only if its minimal K-polynomial defines a left $R_{\mathcal{F}}$-algebraic X-inner skew derivation.*

To prove Theorem 2, we need the following lemma, which summarizes a step in applying Theorem 1 and the results in [3]:

Lemma 4. *If $\psi = \delta^m + \sum_{i=1}^{m-1} \delta^{m-i}\ell(t_i)$ defines a K-polynomial*

$$x^{\psi} = x^{\delta^m} + t_1 x^{\delta^{m-1}} + \cdots + t_{m-1}x^{\delta},$$

then we have the following expression for x^{δ^n}:

$$x^{\delta^n} = \sum_{km+s\leq n;\, 0\leq s,k;\, s<m} u_{k,s}(x^{\delta^s})^{\psi^k},$$

where $u_{k,s} \in Q$ and $u_{k,s} = 1$ in case of $n = mk + s$.

Proof: We proceed by induction on n. Assume that x^{δ^n} has been so expressed as above. Replacing x by x^{δ}, we have

$$x^{\delta^{n+1}} = \sum_{km+s\leq n;\, 0\leq s,k;\, s<m} u_{k,s}(x^{\delta^{s+1}})^{\psi^k},$$

Since $km+s \leq n$, we have $km+(s+1) \leq n+1$. If $s+1 < m$, the term $u_{k,s}(x^{\delta^{s+1}})^{\psi^k}$ is still in the right form. Assume $s + 1 = m$. We have

$$x^{\delta^{s+1}} = x^{\delta^m} = x^{\psi} - t_1 x^{\delta^{m-1}} - \cdots - t_{m-1}x^{\delta}.$$

So

$$\begin{aligned}(x^{\delta^{s+1}})^{\psi^k} &= (x^{\delta^m})^{\psi^k} = (x^{\psi} - t_1 x^{\delta^{m-1}} - \cdots - t_{m-1}x^{\delta})^{\psi^k} \\ &= x^{\psi^{k+1}} - (t_1 x^{\delta^{m-1}})^{\psi^k} - \cdots - (t_{m-1}x^{\delta})^{\psi^k}.\end{aligned}$$

Since ψ defines a σ^m-derivation, we have the expansion formula $(xy)^{\psi} = x^{\psi}y + x^{\sigma^m}y^{\psi}$. Using this, the above expression of $(x^{\delta^{s+1}})^{\psi^k}$ can be expanded into the asserted form.

Proof of Theorem 2. Let $\delta \in \mathcal{L}_{\sigma}(R)$. We may assume that δ is X-outer, for otherwise there is nothing to prove. If the X-inner skew derivation defined by the minimal K-polynomial of δ is left $R_{\mathcal{F}}$-algebraic, then obviously so is δ. Conversely, assume that δ is left $R_{\mathcal{F}}$-algebraic of degree n, say. By Theorem 1 [3], there exists the K-polynomial x^{ψ} of δ with minimal order $m > 1$ which defines an X-inner σ^m-derivation $\mathrm{ad}_{\sigma^m}(b)$. By Lemma 4, we may rewrite the left $R_{\mathcal{F}}$-algebraic dependence of δ in the form

$$\sum_{km+s\leq n;\, 0\leq k,s;\, s<m} u_{k,s}(x^{\delta^s})^{\psi^k} = 0 \qquad \text{for } x \in R.$$

Suppose that $n = \bar{k}m + \bar{s}$, where $\bar{k} \geq 0$ and $0 \leq \bar{s} < m$. Substitute $\mathrm{ad}_{\sigma^m}(b)$ for ψ and apply Theorem 1 [3] to the resulted expression by replacing $x^{\delta^{\bar{s}}}$ by a new indeterminate z and all other x^{δ^s} by 0. Since $u_{\overline{k},\overline{s}} = 1$, we thus obtain a left $R_{\mathcal{F}}$–integral dependence for $\mathrm{ad}_{\sigma^m}(b)$, as asserted.

We still have to characterize $R_{\mathcal{F}}$–algebraicity of X–inner q–skew σ–derivations. We need the following reformulation of the well–known Martindale's lemma [13]:

Lemma 6. *Let $a_i, b_i \in R_{\mathcal{F}}$, $1 \leq i \leq n$. Then $\sum_{i=1}^{n} a_i x b_i = 0$ for all $x \in R$ if and only if $\sum_{i=1}^{n} a_i \otimes b_i = 0$ in $R_{\mathcal{F}} \otimes_C R_{\mathcal{F}}$.*

Theorem 3. *Let $\mathrm{ad}_\sigma(b)$, where $\sigma \in \mathcal{A}(R)$ and $b \in Q$, be q-skew. Then $\mathrm{ad}_\sigma(b)$ is left $R_{\mathcal{F}}$-algebraic if and only if either (i) $b^\sigma = b/q$ and b is nilpotent or (ii) there exist $l \geq 1$ and a unit $u \in Q$ such that $x^{\sigma^l} = uxu^{-1}$ for $x \in R$ and such that $u^{-1}b^l$ is C-algebraic.*

Proof. Since $\mathrm{ad}_\sigma(b)$ is q–skew, we have

$$x^\sigma(b - qb^\sigma) = (b - qb^\sigma)x \qquad \text{for all } x \in Q. \tag{2.4}$$

If $b^\sigma = b/q$ and b is nilpotent, then $\mathrm{ad}_\sigma(b)$ is nilpotent and hence left $R_{\mathcal{F}}$–algebraic. So we assume that $b^\sigma \neq b/q$ or b is not nilpotent. Our argument is divided into two cases according as $b - qb^\sigma = 0$ or $b - qb^\sigma \neq 0$.

Case 1: $b - qb^\sigma = 0$: That is, $b^\sigma = b/q$. We may write

$$x^{\mathrm{ad}_\sigma(b)^k} = \gamma_k x^{\sigma^k} b^k + \gamma_{k-1} b x^{\sigma^{k-1}} b^{k-1} + \cdots + \gamma_1 b^{k-1} x^\sigma b + \gamma_0 b^k x$$

for some $\gamma_i \in C$. The term $\gamma_0 b^k x$ comes from $(-b)^k x$ and so $\gamma_0 = (-1)^k$. The term $\gamma_k x^{\sigma^k} b^k$ comes from $x^{\sigma^k} b^{\sigma^{k-1}} \cdots b^\sigma b$ and so $\gamma_k = \frac{1}{qq^2 \cdots q^{k-1}} = \frac{1}{q^{\frac{k(k-1)}{2}}} \neq 0$. Suppose that $\mathrm{ad}_\sigma(b)$ is left $R_{\mathcal{F}}$–algebraic. By Proposition 1 [11], $\mathrm{ad}_\sigma(b)$ is $R_{\mathcal{F}}$-integral. Say,

$$x^{\mathrm{ad}_\sigma(b)^n} + a_1 x^{\mathrm{ad}_\sigma(b)^{n-1}} + \cdots + a_{n-1} x^{\mathrm{ad}_\sigma(b)} = 0,$$

where $a_i \in R_{\mathcal{F}}$. Using the above expansion of $x^{\mathrm{ad}_\sigma(b)^k}$, we can write this in the form

$$\sum_{k=1}^{n} \sum_{i=0}^{k} a_{k,i} b^{k-i} x^{\sigma^i} b^i = 0, \tag{2.5}$$

where $a_{k,i} \in R_{\mathcal{F}}$. Observe that $a_{n,0} = (-1)^n$ and $a_{n,n} = q^{\frac{-n(n-1)}{2}} \neq 0$.

If $1, \sigma, \sigma^2, \ldots, \sigma^n$ are mutually outer, then by Proposition 1 [7], $a_{n,n} x^{\sigma^n} b^n = 0$ and hence $b^n = 0$, absurd. So there exists the least integer $l \geq 1$ such that σ^l is

X–inner. Say, $x^{\sigma^l} = uxu^{-1}$, where u is a unit in Q. Replace x^{σ^i} by $u^s x^{\sigma^r} u^{-s}$, where $i = ls + r$ and $0 \le r < l$. We thus transform (2.5) into a linear GPI with $1, \sigma, \dots, \sigma^{l-1}$. Suppose that $n = l\bar{s} + \bar{r}$, where $0 \le \bar{r} < l$. There is only one term involving σ^n, namely, $a_{n,n} x^{\sigma^n} b^n$. This term gives rise to the term

$$a_{n,n} u^{\bar{s}} x^{\sigma^{\bar{r}}} u^{-\bar{s}} b^n = a_{n,n} u^{\bar{s}} x^{\sigma^{\bar{r}}} u^{-\bar{s}} b^{\bar{s}l+\bar{r}}.$$

All other terms with $x^{\sigma^{\bar{r}}}$ come from x^{σ^i} with $i < n$ and hence assume the form

$$B_s u^s x^{\sigma^{\bar{r}}} u^{-s} b^{sl+\bar{r}},$$

where $B_s \in R_{\mathcal{F}}$ and where $i = sl + \bar{r} < n = \bar{s}l + \bar{r}$, that is, $s < \bar{s}$. Therefore, the sum of terms with $x^{\sigma^{\bar{r}}}$ assumes the following form

$$a_{n,n} u^{\bar{s}} x^{\sigma^{\bar{r}}} u^{-\bar{s}} b^{\bar{s}l+\bar{r}} + \sum_{0 \le s < \bar{s}} B_s u^s x^{\sigma^{\bar{r}}} u^{-s} b^{sl+\bar{r}} = 0.$$

This gives a trivial linear GPI by [7]. If $u^{-\bar{s}} b^{\bar{s}l+\bar{r}}$ did not fall in the C-linear span of $u^{-s} b^{sl+\bar{r}}$, $0 \le s < \bar{s}$, then $a_{n,n} u^{\bar{s}} = 0$. This is absurd, since $0 \ne a_{n,n} \in C$ and u is a unit. So $u^{-\bar{s}} b^{\bar{s}l+\bar{r}}$ is a C-linear combination of $u^{-s} b^{sl+\bar{r}}$. Say,

$$u^{-\bar{s}} b^{\bar{s}l+\bar{r}} + \sum_{0 \le s < \bar{s}} \alpha_s u^{-s} b^{sl+\bar{r}} = 0,$$

where $\alpha_i \in C$. Multiply this equality from the left hand side by u^{-1} and from the right hand side by $b^{l-\bar{r}}$. We have

$$u^{-\bar{s}-1} b^{(\bar{s}+1)l} + \sum_{0 \le s < \bar{s}} \alpha_s u^{-s-1} b^{(s+1)l} = 0.$$

Using $ubu^{-1} = b^{\sigma^l} = b/q^l$, we have $u^{-s} b^{sl} = \gamma (u^{-1} b^l)^s$ for some $0 \ne \gamma \in C$. So the above equality gives the asserted C–algebraicity of $u^{-1} b^l$.

Conversely, suppose that $u^{-1} b^l$ is C–algebraic. Let us write

$$x^{\mathrm{ad}_\sigma(b)^k} = \gamma_{k,k} x^{\sigma^k} b^k + \gamma_{k,k-1} b x^{\sigma^{k-1}} b^{k-1} + \gamma_{k,k-2} b^2 x^{\sigma^{k-2}} b^{k-2} + \cdots + \gamma_{k,0} b^k x,$$

where $\gamma_{k,i} \in C$. Note that $\gamma_{k,0} = (-1)^k$ and $\gamma_{k,k} = q^{-k(k-1)/2} \ne 0$.

Let j be the largest integer, if there exists any, such that $1 < j < k$ and such that $\gamma_{k,j} \ne 0$. Observe that

$$x^{\mathrm{ad}_\sigma(b)^k} - \frac{\gamma_{k,j}}{\gamma_{j,j}} b^{k-j} x^{\mathrm{ad}_\sigma(b)^j} = \gamma_{k,k} x^{\sigma^k} b^k + \gamma'_{k,j-1} b^{k-j+1} x^{\sigma^{j-1}} b^{j-1} + \cdots + \gamma'_{k,0} b^k x,$$

where $\gamma'_i \in C$. Continue in this manner. We see that there exists uniquely $\beta_i \in C$ such that

$$x^{\mathrm{ad}_\sigma(b)^k} - \sum_{j=1}^{k-1} \beta_j b^{k-j} x^{\mathrm{ad}_\sigma(b)^j} = \gamma_{k,k} x^{\sigma^k} b^k + \gamma'_{k,k} b^k x$$

for some $\gamma'_{k,k} \in C$. For brevity, we set $\tilde{\gamma}_k \overset{\text{def.}}{=} \gamma_{k,k}$, $\tilde{\gamma}'_k \overset{\text{def.}}{=} \gamma'_{k,k}$ and

$$\psi_k(x) \overset{\text{def.}}{=} x^{\mathrm{ad}_\sigma(b)^k} - \sum_{j=1}^{k-1} \beta_j b^{k-j} x^{\mathrm{ad}_\sigma(b)^j}. \tag{2.6}$$

We then have, for $s \geq 1$,

$$\begin{aligned}\psi_{sl}(x) &= \tilde{\gamma}_{sl} x^{\sigma^{sl}} b^{sl} + \tilde{\gamma}'_{sl} b^{sl} x \\ &= \tilde{\gamma}_{sl} u^s x u^{-s} b^{sl} + \tilde{\gamma}'_{sl} b^{sl} x \\ &= u^s (\tilde{\gamma}_{sl} x u^{-s} b^{sl} + \tilde{\gamma}'_{sl} u^{-s} b^{sl} x).\end{aligned}$$

Thus we have

$$u^{-s} \psi_{sl}(x) = \tilde{\gamma}_{sl} x u^{-s} b^{sl} + \tilde{\gamma}'_{sl} u^{-s} b^{sl} x.$$

Since $ubu^{-1} = b^{\sigma^l} = b/q^l$, we see that $u^{-s} b^{sl} = \beta_s (u^{-1} b^l)^s$ for some $0 \neq \beta_s \in C$. Since $u^{-1} b^l$ is C–algebraic, $u^{-s} b^{sl}$, $s = 1, 2, \ldots$, are C-dependent. Thus the subset

$$\{1 \otimes \tilde{\gamma}_{sl} u^{-s} b^{sl} \mid s = 1, 2, \ldots\} \cup \{\tilde{\gamma}'_{sl} u^{-s} b^{sl} \otimes 1 \mid s = 1, 2, \ldots\}$$

of $R_{\mathcal{F}} \otimes_C R_{\mathcal{F}}$ generates a finite-dimensional C-subspace. In particular, the subset $\{1 \otimes \tilde{\gamma}_{sl} u^{-s} b^{sl} + \tilde{\gamma}'_{sl} u^{-s} b^{sl} \otimes 1 \mid s = 1, 2, \ldots\}$ generates a finite-dimensional C-subspace. Say, for some $\alpha_s \in C$ and $m \geq 1$,

$$(1 \otimes \tilde{\gamma}_{ml} u^{-m} b^{ml} + \tilde{\gamma}'_{ml} u^{-m} b^{ml} \otimes 1) + \sum_{s=1}^{m-1} \alpha_s \big(1 \otimes \tilde{\gamma}_{sl} u^{-s} b^{sl} + \tilde{\gamma}'_{sl} u^{-s} b^{sl} \otimes 1\big) = 0.$$

Then, by Lemma 6,

$$u^{-m} \psi_{sm}(x) + \sum_{s=1}^{m-1} \alpha_s u^{-s} \psi_{sl}(x) = 0.$$

By (2.6), this gives the asserted left $R_{\mathcal{F}}$–algebraicity of $\mathrm{ad}_\sigma(b)$.

Case 2: $b - qb^\sigma \neq 0$: Set $u \overset{\text{def.}}{=} b - qb^\sigma$. By (2.4), u is a unit in Q such that $x^\sigma = uxu^{-1}$ for all $x \in Q$. We thus have $l = 1$ and must show the equivalence of the left $R_{\mathcal{F}}$–algebraicity of $\mathrm{ad}_\sigma(b)$ and the C–algebraicity of $a \overset{\text{def.}}{=} u^{-1} b$ for non-nilpotent *b*. If $q \neq 1$, then $b' \stackrel{\text{def.}}{=} b + \frac{u}{q-1}$ satisfies $b' - q(b')^\sigma = 0$. If b' is nilpotent, say, $(b')^n = 0$, then we compute

$$\begin{aligned} 0 = (b')^n = (b + \frac{u}{q-1})^n &= \left(u(a + \frac{1}{q-1})\right)^n \\ &= u^n(a^{\sigma^{-(n-1)}} + \frac{1}{q-1})(a^{\sigma^{-(n-2)}} + \frac{1}{q-1}) \cdots (a^{\sigma^{-1}} + \frac{1}{q-1})(a + \frac{1}{q-1}). \end{aligned}$$

But $a^{\sigma^{-1}} = u^{-1}b^{\sigma^{-1}} = u^{-1}(u+qb) = 1 + qu^{-1}b = qa + 1$ and hence

$$a^{\sigma^{-k}} = q^k a + q^{k-1} + q^{k-2} + \cdots + 1 = q^k a + \frac{q^k - 1}{q-1}.$$

Substitute these expressions of $a^{\sigma^{-k}}$ in the above and note that u is a unit. We obtain

$$(q^{n-1}a + \frac{q^{n-1}}{q-1})(q^{n-2}a + \frac{q^{n-2}}{q-1}) \cdots (a + \frac{1}{q-1}) = 0$$

or equivalently $(a + \frac{1}{q-1})^n = 0$. This shows the C–algebraicity of a, as asserted. So we assume that b' is not nilpotent. Applying the previous case, we have the equivalence of the left $R_{\mathcal{F}}$–algebraicity of $\mathrm{ad}_\sigma(b')$ and the C–algebraicity of $u^{-1}b'$. But $\mathrm{ad}_\sigma(b') = \mathrm{ad}_\sigma(b)$. Also, since $u^{-1}b' = u^{-1}b + \frac{1}{q-1} = a + \frac{1}{q-1}$, the C–algebraicities of $u^{-1}b'$ and of $u^{-1}b$ are the same. It follows from the equivalence of the left $R_{\mathcal{F}}$–algebraicity of $\mathrm{ad}_\sigma(b)$ and the C–algebraicity of $u^{-1}b$.

We hence assume $q = 1$. So $b^\sigma = b - u$. We compute

$$x^{\mathrm{ad}_\sigma(b)} = x^\sigma b - bx = uxu^{-1}b - bx = u(xu^{-1}b - u^{-1}bx) = u(xa - ax).$$

Set $d \stackrel{\text{def.}}{=} \mathrm{ad}(a)$. Note that $u^d = ua - au = u(u^{-1}b) - (u^{-1}b)u = b - u^{-1}bu = b - b^{\sigma^{-1}} = -u$. With this and by a simple induction, we see that $\mathrm{ad}_\sigma(b)^k = \ell(u^k)d(d-1)\cdots(d-k+1)$. Let us write

$$x^{\mathrm{ad}_\sigma(b)^k} = u^k(x^{d^k} + \sum_{j=1}^{k-1} \alpha_{jk} x^{d^j}) \tag{2.7}$$

for some $\alpha_{jk} \in C$. Note that $x^{d^j} = \sum_{i=0}^{j} \binom{j}{i}(-a)^{j-i}xa^i$. Assume that $\mathrm{ad}_\sigma(b)$ is left $R_{\mathcal{F}}$–algebraic. Say, for all $x \in R$,

$$x^{\mathrm{ad}_\sigma(b)^n} + a_1 x^{\mathrm{ad}_\sigma(b)^{n-1}} + \cdots + a_{n-1}x^{\mathrm{ad}_\sigma(b)} = 0,$$

where $a_i \in R_{\mathcal{F}}$. With the above expansion of $x^{\mathrm{ad}_\sigma(b)^k}$ in terms of u and d, this equality can be expressed in the form

$$u^n xa^n + (\cdot)xa^{n-1} + \cdots + (\cdot)xa + (\cdot)x = 0,$$

where $(\cdot)$ denote elements in $R_{\mathcal{F}}$. If $1, a, \ldots, a^{n-1}, a^n$ are C-independent, then particularly $u^n = 0$, absurd. So $1, a, \ldots, a^{n-1}, a^n$ are C-dependent. Their dependence relation gives the asserted C–algebraicity of a.

Conversely, suppose that a is C–algebraic. Then so is $d \overset{\text{def.}}{=} \operatorname{ad}(a)$. Say,

$$x^{d^n} + \gamma_1 x^{d^{n-1}} + \cdots + \gamma_{n-1} x^d = 0$$

for some $\gamma_i \in C$. With (2.7), a simple induction shows that

$$x^{d^k} = u^{-k} x^{\operatorname{ad}_\sigma(b)^k} + \sum_{j=1}^{k-1} \beta_{jk} u^{-j} x^{\operatorname{ad}_\sigma(b)^j}$$

for some $\beta_{jk} \in C$. In the C–algebraic dependence of d, we replace each d^k by its expression in terms of $\operatorname{ad}_\sigma(b)$ given above. The left $R_{\mathcal{F}}$–algebraicity of $\operatorname{ad}_\sigma(b)$ follows as asserted.

§3. C–ALGEBRAICITY

We first observe an easy characterization of C–algebraic automorphisms:

Lemma 5. *For $\sigma \in \mathcal{A}(R)$ the following statements are equivalent:*

(1) σ is C–algebraic.

(2) σ is $R_{\mathcal{F}}$–algebraic.

(3) there exist $n \geq 1$ and an invertible element $u \in Q$ such that $x^{\sigma^n} = uxu^{-1}$ for $x \in R$ and u is C–algebraic.

This follows immediately from the theory of identities with automorphisms [7] and Martindale's theorem for linear identities [13]. (See also Theorem 7.9.10 [1].) In view of this, the C–algebraicity of σ for a nonnilpotent C-algebraic q–skew σ derivation follows immediately from the following more general

Theorem 4. *Let δ be a continuous q-skew σ-derivation of a prime ring R. Assume that δ is not nilpotent. The following are equivalent:*

(1) δ is C–algebraic.

(2) δ and σ are both left $R_{\mathcal{F}}$–algebraic.

We need one more

Lemma 7. *Let $\operatorname{ad}_\sigma(b)$ be an inner q-skew σ-derivation, where $b \in Q$ satisfies $b^\sigma = \frac{1}{q}b$. If $\nu \geq 1$ is the least integer such that $q^\nu = 1$, then $\operatorname{ad}_\sigma(b)^\nu = (-1)^{\nu-1} \operatorname{ad}_{\sigma^\nu}(b^\nu)$.*

Proof. Write $\mathrm{ad}_\sigma(b) = \sigma r(b) - \ell(b)$, where $r(b)$ and $\ell(b)$ denote respectively the right and left multiplications by b. Set $X \stackrel{\text{def.}}{=} \sigma r(b)$ and $Y \stackrel{\text{def.}}{=} \ell(b)$. For any $z \in Q$, we have

$$z^{YX} = (bz)^\sigma b = b^\sigma z^\sigma b = \frac{1}{q} b z^\sigma b = \frac{1}{q} z^{\sigma r(b)\ell(b)} = \frac{1}{q} z^{XY}.$$

So $YX = \frac{1}{q}XY$ and we have

$$\mathrm{ad}_\sigma(b)^\nu = (\sigma r(b) - \ell(b))^\nu = (X-Y)^\nu = \sum_{i=0}^{\nu} \binom{\nu}{i}_{1/q} X^i(-Y)^{n-i} = X^\nu + (-Y)^\nu,$$

since $\binom{\nu}{i}_{1/q} = 0$ for all $i = 1, \dots, \nu - 1$. But

$$\begin{aligned} X^\nu =& (\sigma r(b))^\nu = \sigma^\nu r(b^{\sigma^{\nu-1}} b^{\sigma^{\nu-2}} \cdots b^\sigma b) \\ =& (1/q)^{(\nu-1)+(\nu-2)+\cdots+1} \sigma^\nu r(b^\nu) = (1/q)^{\frac{\nu(\nu-1)}{2}} \sigma^\nu r(b^\nu). \end{aligned}$$

If ν is odd, then $\left((1/q)^\nu\right)^{(\nu-1)/2} = 1$. If ν is even, say $\nu = 2k$, then $1 = q^\nu = q^{2k}$ implies $q^k = -1$. and hence $(1/q)^{\frac{\nu(\nu-1)}{2}} = (1/q)^{k(2k-1)} = (-1)^{2k-1} = -1$. It follows that

$$\mathrm{ad}_\sigma(b)^\nu = (-1)^{\nu-1}\sigma^\nu r(b^\nu) + (-1)^\nu \ell(b^\nu) = (-1)^{\nu-1}\,\mathrm{ad}_{\sigma^\nu}(b^\nu),$$

proving the lemma.

We are now ready to give the

Proof of Theorem 4. (2) $\Rightarrow$ (1): Let δ be a continuous q-skew σ-derivation which is left $R_{\mathcal{F}}$–algebraic of degree $n \geq 1$. If $q = 1$, then $\sigma\delta = \delta\sigma$ and the C–algebraicity of δ follows from Theorem 6 [11]. So we assume $q \neq 1$. By the Corollary to Theorem 1, $q^\nu = 1$ for some $\nu \geq 1$ since δ is not nilpotent. Let ν be the least such integer. By Lemma 1, δ^ν is a σ^ν-derivation and $\delta^\nu\sigma^\nu = \sigma^\nu\delta^\nu$.

We *claim* that δ^ν is also $R_{\mathcal{F}}$–algebraic: If δ is X–outer, then, by Theorem 1 [3], its minimal K-polynomial x^ψ exists and defines a left $R_{\mathcal{F}}$–algebraic X–inner σ^m-derivation by Theorem 2. By Theorem 1 and the remark following, the minimal K–polynomial x^ψ of δ comes from the minimal K–polynomial of $\partial \stackrel{\text{def.}}{=} \delta^\nu$ by substituting δ^ν for ∂. So the minimal K–polynomial of $\partial \stackrel{\text{def.}}{=} \delta^\nu$ also defines a left $R_{\mathcal{F}}$–algebraic X–inner skew derivation and hence is left $R_{\mathcal{F}}$–algebraic by Theorem 2 again. If δ is X–inner, by Lemma 3 we may write $\delta = \mathrm{ad}_\sigma(b)$, where $b \in Q$ satisfies $b^\sigma = b/q$. By Lemma 7, $\delta^\nu = (-1)^{\nu-1}\,\mathrm{ad}_{\sigma^\nu}(b^\nu)$. Since δ is not nilpotent, b is also not nilpotent. Since $\mathrm{ad}_\sigma(b)$ is $R_{\mathcal{F}}$–algebraic, by Theorem 3 there exists $l \geq 1$ and a unit $u \in Q$ such that $x^{\sigma^l} = uxu^{-1}$ for $x \in R$ and such that $u^{-1}b^l$ is C–algebraic. But $ubu^{-1} = b^{\sigma^l} = b/q^l$; this implies $ub^l = b^l u/q^{l^2}$. Thus $u^{-\nu}b^{\nu l}$ is C–algebraic and $x^{\sigma^{\nu l}} = u^\nu x u^{-\nu}$

for $x \in Q$. Theorem 3 implies that $\mathrm{ad}_{\sigma^\nu}(b^\nu)$ is left $R_{\mathcal{F}}$–algebraic and so is δ^ν, as claimed.

We have thus shown that δ^ν is left $R_{\mathcal{F}}$–algebraic. Since σ is left $R_{\mathcal{F}}$–algebraic, by Lemma 5 we see that σ and hence σ^ν also are C–algebraic. We now apply Theorem 6 [11] to δ^ν and obtain the C–algebraicity of δ^ν. The C–algebraicity of δ follows immediately from that of δ^ν.

(1) $\Rightarrow$ (2): Assume that δ is C–algebraic. Then δ is surely left $R_{\mathcal{F}}$–algebraic. So it suffices to show that σ is left $R_{\mathcal{F}}$–algebraic or equivalently C–algebraic by Lemma 5. By the Corollary to Theorem 1, $q^\nu = 1$ for some $\nu \geq 1$ since δ is not nilpotent. Let ν be the least such integer. By Lemma 3, δ^ν is a σ^ν-derivation satisfying $\delta^\nu\sigma^\nu = \sigma^\nu\delta^\nu$. We show the C-algebraicity of δ^ν: Let n be the minimal integer ≥ 1 such that

$$x^{\delta^n} + \beta_1 x^{\delta^{n-1}} + \cdots + \beta_{n-1} x^{\delta} = 0 \tag{2.8}$$

holds for all $x \in R$, where $\beta_i \in C$. So $\delta, \ldots, \delta^{n-1}$ span C-linearly δ^n. Replacing x by x^δ, we see that $\delta^2, \ldots, \delta^n$ span C-linearly δ^{n+1} and hence $\delta, \ldots, \delta^{n-1}$ also span C-linearly δ^{n+1}. Continuing in this manner, we see that $\delta, \ldots, \delta^{n-1}$ span C-linearly all powers of δ. This implies that the infinite set $\{\delta^\nu, \delta^{2\nu}, \ldots\}$ is C–linearly dependent. So some $\delta^{k\nu}$ is a C-linear combination of $\delta^\nu, \delta^{2\nu}, \ldots, \delta^{(k-1)\nu}$. This gives a C–algebraic dependence of δ^ν. It suffices to show the C–algebraicity of σ^ν, which obviously implies the C–algebraicity of σ. Replacing δ by δ^ν and σ by σ^ν, we may assume that $\delta\sigma = \sigma\delta$ from the start.

Case 1: $\delta = \mathrm{ad}(b)\ell(u)$, where $u, b \in Q$ and u is a unit such that $ub = bu$: Substitute the expansion

$$x^{\mathrm{ad}(b)^k} = xb^k + \binom{k}{1}(-b)xb^{k-1} + \cdots + \binom{k}{k}(-b)^k x$$

into (2.8). Collect terms according to right coefficients $1, b, \ldots, b^n$ and pay special attention to terms with u^n, which is contributed only by $u^n x^{\mathrm{ad}(b)^n}$:

$$u^n x b^n + \big(-nu^n b + \cdots\big)xb^{n-1} + \cdots + \big(u^n(-b)^n + \cdots\big)x = 0, \tag{2.9}$$

where dots denote sum of terms with u^j, $j < n$. The set of left coefficients $\{1, b, \ldots, b^n\}$ is C-dependent, for otherwise the left coefficient u^n of xb^n would be 0 by [13]. Let $1 \leq s \leq n$ be the maximal integer such that $1, b, \ldots, b^{s-1}$ be C-independent. Then $b^s = \alpha_1 b^{s-1} + \cdots + \alpha_s$ for some $\alpha_1, \ldots, \alpha_s \in C$. The minimal polynomial of b over C is thus equal to $p(X) = X^s - \alpha_1 X^{s-1} - \cdots - \alpha_{s-1}X - \alpha_s$. Let $\overline{C}$ be the algebraic closure of C. The minimal polynomial of b over $\overline{C}$ remains to be $p(X)$. Work in $\overline{Q} \stackrel{\text{def.}}{=} Q \otimes_C \overline{C}$. By [4], the extended centroid of $\overline{Q}$ is exactly $\overline{C}$. Let d also denote the inner derivation of $\overline{Q}$ defined by b. The linear GPI (2.9) holds on Q by [2] and hence on $\overline{Q}$ by linearity. Let $\lambda \in \overline{C}$ be a root of $p(X)$. Obviously, $1, b-\lambda, \ldots, (b-\lambda)^{s-1}$ are $\overline{C}$-independent and $\overline{C}$-linearly span all powers of $b-\lambda$. Since $\mathrm{ad}(b) = \mathrm{ad}(b-\lambda)$ on $\overline{Q}$, we also have

$$u^n x(b-\lambda)^n + \big(nu^n(\lambda - b) + \cdots\big)x(b-\lambda)^{n-1} + \cdots + \big(u^n(\lambda-b)^n + \cdots\big)x = 0.$$

Express all $(b-\lambda)^k$ in the right coefficients as C-linear combinations of $1, b-\lambda, \ldots, (b-\lambda)^{s-1}$. In the resulted expression, the corresponding left coefficients of $x, xb-\lambda, \ldots, x(b-\lambda)^{s-1}$ must be all vanishing by [14]. Suppose that

$$(b-\lambda)^n = \mu_0 + \mu_1(b-\lambda) + \cdots + \mu_{s-1}(b-\lambda)^{s-1},$$

where $\mu_i \in \overline{C}$. Since d is not nilpotent, neither is $b-\lambda$. So $\mu_i \neq 0$ for some i. Let us fix such an i. For this particular i with $\mu_i \neq 0$, the left coefficient of $x(b-\lambda)^i$ assumes the form

$$u^n(\mu_i + \mu_i'(b-\lambda) + \cdots + \mu_i^{(n)}(b-\lambda)^n) + u^{n-1}(\cdot) + \cdots + u(\cdot),$$

where $\mu_i', \ldots, \mu_i^{(n)} \in \overline{C}$ and where $(\cdot)$ denote polynomials in $b-\lambda$. Let

$$f_\lambda(X) \stackrel{\text{def.}}{=} \mu_i + \mu_i'(X-\lambda) + \cdots + \mu_i^{(n)}(X-\lambda)^n \in \overline{C}[X].$$

We then have $f_\lambda(\lambda) = \mu_i \neq 0$ and $u^n f_\lambda(b) + u^{n-1}(\cdot) + \cdots + u(\cdot) = 0$. Let λ range over all distinct roots $\lambda_1, \lambda_2, \ldots \in \overline{C}$ of the minimum polynomial $p(X)$ of b. Write $f_j(X)$ for the corresponding $f_{\lambda_j}(X)$. For each j, we have an equality of the form

$$u^n f_j(b) + u^{n-1}(\cdot) + \cdots + u(\cdot) = 0, \tag{2.10}$$

where $(\cdot)$ denote polynomials in $b-\lambda_j$. Since $f_i(\lambda_i) \neq 0$, $p(X), f_1(X), f_2(X), \ldots$ have no nontrivial common factors in $\overline{C}[X]$. There exist $g_j(X) \in \overline{C}[X]$ such that

$$g_1(X)f_1(X) + g_2(X)f_2(X) + \cdots \equiv 1 \qquad \text{modulo } p(X)\overline{C}[X].$$

This implies $1 = g_1(b)f_1(b) + g_2(b)f_2(b) + \cdots$. Multiply (2.10) above by $g_j(b)$ from the right and add them up. It follows that

$$u^n + u^{n-1}(\cdot) + \cdots + u(\cdot) = 0,$$

where $(\cdot)$ are polynomials in b over $\overline{C}$. Since $[b, u] = 0$ and b is $\overline{C}$–algebraic, this implies that u is $\overline{C}$–algebraic and hence C–algebraic. Thus σ is thus C–algebraic by Lemma 5.

Case 2: $\delta = d\,\ell(u)$, where d is an X–outer derivation and where $u \in Q$ is a unit such that $u^d = 0$: In this case, the associated automorphism σ of δ is defined by $x^\sigma = uxu^{-1}$ for $x \in Q$. To prove the C–algebraicity of σ, it suffices to show the C–algebraicity of u in view of Lemma 5. By Kharchenko's theorem [8], $\operatorname{char} R = p > 0$ and there exists the least $m \geq 0$ such that $d, d^p, \dots, d^{p^{m-1}}$ are C-independent modulo X–inner derivations and such that

$$d^{p^m} + \alpha_1 d^{p^{m-1}} + \cdots + \alpha_m d = \operatorname{ad}(b)$$

for some $\alpha_i \in C$ and $b \in Q$. By the minimality of m, we verify easily that $\alpha_i^d = 0$ and $b^d \in C$. Denote $g(X) \overset{\text{def.}}{=} X^{p^m} + \alpha_1 X^{p^{m-1}} + \cdots + \alpha_m X$. Then $x^{g(d)} = [x, b] = x^{\operatorname{ad}(b)}$ for $x \in R$. We may write, for each $j \geq 0$,

$$X^j = \sum_{i=0}^{k_j} \gamma_{ji}(X) g(X)^i,$$

where the degree of each $\gamma_{ji}(X)$ is $< p^m$ and $\gamma_{jk_j} \neq 0$ for each j. Note that all coefficients of each $\gamma_{ji}(X)$ are constants of d. We thus have

$$x^{d^j} = \sum_{i=0}^{k_j} \left(x^{\gamma_{ji}(d)}\right)^{\operatorname{ad}(b)^i}.$$

Applying these to (2.8), we see that

$$u^n \sum_{i=0}^{k_n} (x^{\gamma_{ni}(d)})^{\operatorname{ad}(b)^i} + \sum_{j=1}^{n-1} \beta_{n-j} u^j \sum_{i=0}^{k_j} (x^{\gamma_{ji}(d)})^{\operatorname{ad}(b)^i} = 0. \tag{2.11}$$

Assume first that $g(X)$ is not a power of X. For $j < n$, $g(X)$ is also not a divisor of X^j. So $\gamma_{j0}(X) \neq 0$ for all j. Suppose that $\gamma_{n0}(X) = \mu_n X^e + \cdots$, where $0 \neq \mu_n \in C$ and where dots denote a sum of terms of degree $< e$. For $j < n$, we write

$$\gamma_{j0}(X) = \cdots + \mu_j X^e + \cdots$$

and, for $i > 0$,

$$\gamma_{ji}(X) = \cdots + \mu_{ji} X^e + \cdots,$$

where $\mu_j, \mu_{ji} \in C$. Since $e < p^m$, we may, in view of Theorem 2 [9], replace x^{d^e} by a new indeterminate y and x^{d^t} by 0 for all $e \neq t < p^m$ in (2.11). We conclude

$$u^n \Big(\mu_n y + \sum_{i \geq 1} \mu_{ni} y^{\operatorname{ad}(b)^i}\Big) + \sum_{j=1}^{n-1} \beta_{n-j} u^j \Big(\mu_j y + \sum_{i \geq 1} \mu_{ji} y^{\operatorname{ad}(b)^i}\Big) = 0$$

for all $y \in R$ and so for all $y \in Q$ [2]. Setting $y = 1$, we see that $\mu_n u^n + \sum_{j=1}^{n-1} \beta_{n-j}\mu_j u^j = 0$, as desired.

Suppose next that $g(X)$ divides X^n. Then $g(X) = X^{p^m}$ and so $x^{g(x)} = [x, b]$ for all $x \in R$. For $1 \le j \le n$, write $j = p^m s_j + t_j$, where $s_j \ge 0$ and $0 \le t_j < p^m$. Note that $s_n \ge 1$, since $n \ge p^m$ by the minimality of n. Then $X^j = (X^{p^m})^{s_j} X^{t_j}$ and so $x^{d^j} = (x^{d^{t_j}})^{\mathrm{ad}(b)^{s_j}}$ for all $x \in R$. We rewrite (2.8) as

$$u^n (x^{d^{t_n}})^{\mathrm{ad}(b)^{s_n}} + \sum_{j=1}^{n-1} \beta_{n-j} u^j (x^{d^{t_j}})^{\mathrm{ad}(b)^{s_j}} = 0$$

for all $x \in R$. Apply Kharchenko's theorem [9] by replacing $x^{d^{t_n}}$ with a new indeterminate y and $x^{d^{t_j}}$ with 0 for $t_j \ne t_n$ in the above. We see that

$$u^n y^{\mathrm{ad}(b)^{s_n}} + \sum_{j=1, t_j = t_n}^{n-1} \beta_{n-j} u^j y^{\mathrm{ad}(b)^{s_j}} = 0.$$

Write $u^j = (u^{p^m})^{s_j} u^{t_j}$ for each j:

$$(u^{p^m})^{s_n} u^{t_n} y^{\mathrm{ad}(b)^{s_n}} + \sum_{j=1, t_j = t_n}^{n-1} \beta_{n-j} (u^{p^m})^{s_j} u^{t_j} y^{\mathrm{ad}(b)^{s_j}} = 0.$$

Cancelling u^{t_n}, we have

$$(u^{p^m})^{s_n} y^{\mathrm{ad}(b)^{s_n}} + \sum_{j=1, t_j = t_n}^{n-1} \beta_{n-j} (u^{p^m})^{s_j} y^{\mathrm{ad}(b)^{s_j}} = 0,$$

for all $y \in R$. Since $[b, u] = 0$, this implies that the skew derivation $\mathrm{ad}(b)\ell(u^{p^m})$ is C–algebraic. Applying Case 1, we conclude that u^{p^m} is C–algebraic and so is u, unless $\mathrm{ad}(b)$ is nilpotent. But if $\mathrm{ad}(b)$, which is equal to d^{p^m}, is nilpotent, then so is d, a contradiction. This proves the case.

Case 3. General: Consider the skew polynomial ring $S = Q[t; \sigma]$ with the multiplication rule: $tv = v^\sigma t$ for all $v \in Q$. We denote by $Q_s(S)$ the symmetric Martindale quotient ring of S and by F the extended centroid of S. Since $tS = St$, t is invertible in $Q_s(S)$. The inner automorphism $x \mapsto txt^{-1}$ of $Q_s(S)$ extends σ and will be also denoted by σ. We extend δ to S (and hence to $Q_s(S)$ also) by the rule: $(\sum_{i\ge 0} r_i t^i)^\delta = \sum_{i \ge 0} r_i^\delta t^i$ for $r_i \in S$. We verify easily that δ is a σ-derivation of S such that $t^\delta = 0$ and $\sigma\delta = \delta\sigma$. The minimality of n in (2.8) then implies that $\beta_i^\sigma = \beta_i$ for each i. By Theorem 2 [3], (2.8) holds for all $x \in Q$. We verify easily that (2.8) holds for all $x \in S$ and so for all $x \in Q_s(S)$. Since $\beta_i^\sigma = \beta_i$, we have

$\beta_i \in F$, the extended centroid of S by Theorem 3.3 [14]. Since δ is not nilpotent on R, neither is δ on S. In S, σ is the inner automorphism defined by t. By Case 2, σ is F–algebraic. That is, there exist $\mu_i \in F$ such that for $x \in Q$,

$$x^{\sigma^\ell} + \mu_1 x^{\sigma^{\ell-1}} + \cdots + \mu_{\ell-1} x^{\sigma} = 0. \tag{2.12}$$

Let $C_\sigma \overset{\text{def.}}{=} \{x \in C \mid x^\sigma = x\}$. We divide our argument into two subcases:

(I) All $\sigma^j, j > 0$, are outer in the left Martindale quotient ring of Q: By Proposition 2.3 and Theorem 3.3 [14], F is the quotient field of C_σ. Since $C_\sigma \subseteq C$, $F \subseteq C$. So $\mu_i \in F \subseteq C$ for each i and σ is C–algebraic, as asserted.

(II) Some σ^j, $j > 0$, is inner in the left Martindale quotient ring of Q: By the bi–continuity of σ, σ^j is inner in the left Martindale quotient ring of Q if and only if it is inner in Q. By Proposition 2.3, there exists the least integer $e \geq 1$ such that σ^e is an inner automorphism of Q defined by a σ-invariant unit of Q. Let $b \in Q$ is such a σ-invariant unit. Theorem 3.3 [14], F is the quotient field of $C_\sigma[bt^e]$. Multiplying (2.12) by a common multiple of denominators of $\mu_1, \ldots, \mu_{l-1}$, we have

$$\lambda_0 x^{\sigma^\ell} + \lambda_1 x^{\sigma^{\ell-1}} + \cdots + \lambda_{\ell-1} x^{\sigma} = 0,$$

where λ_i are polynomials in bt^e. Say, $\lambda_0 = c_0(bt^e)^j + \cdots$, where $0 \neq c_0 \in C_\sigma$. For each $i \geq 1$, we write $\lambda_i = \cdots + c_i(bt^e)^j + \cdots$, where $c_i \in C_\sigma$. Since the sum $Q + Qt + Qt^2 + \cdots$ is direct, so is the sum $Q + Qbt^e + Q(bt^e)^2 + \cdots$. We have, for $x \in Q$,

$$c_0 x^{\sigma^\ell} + c_1 x^{\sigma^{\ell-1}} + \cdots + c_{\ell-1} x^{\sigma} = 0.$$

This proves the C–algebraicity of σ. So (1) implies (2).

Appendix

Lemma. *Let C be a field and let $a_n = \beta^n - 1$ for $n \geq 1$, where $\beta \in C$. For $n, i \geq 1$, let $A_{n,i}$ denote the following $n \times n$ matrix:*

$$\begin{pmatrix} 1 & a_i & \cdots & a_i a_{i+1} \cdots a_{i+n-2} \\ 1 & a_{i+1} & \cdots & a_{i+1} a_{i+2} \cdots a_{i+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_{i+n-1} & \cdots & a_{i+n-1} a_{i+n} \cdots a_{i+2n-3} \end{pmatrix}.$$

Then $\det(A_{n,i}) = \beta^{i+(i+1)+\cdots+(i+n-2)} a_1 a_2 \cdots a_{n-1}, \det(A_{n-1,i+1})$. *In particular,*

$$\det \begin{pmatrix} a_1 & a_1 a_2 & \cdots & a_1 a_2 \cdots a_n \\ a_2 & a_2 a_3 & \cdots & a_2 a_3 \cdots a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_n a_{n+1} & \cdots & a_n a_{n+1} \cdots a_{2n-1} \end{pmatrix} = a_1^n a_2^{n-1} \cdots a_n \beta^{\frac{n(n^2-1)}{3}}.$$

Proof. We note first that if $t > s$, then $a_t - a_s = \beta^i a_{t-s}$. For $1 \le j \le n$, we denote by B_j the j-th row of the matrix $A_{n,i}$. By replacing the j-th row in $A_{n,i}$ for $2 \le j \le n$ with $B_j - B_{j-1}$, we see that

$$\begin{aligned}
&\det(A_{n,i}) \\
&= \det\begin{pmatrix} 1 & a_i & \cdots & a_i a_{i+1} \cdots a_{i+n-2} \\ 0 & \beta^i a_1 & \cdots & \beta^i a_{n-1} a_{i+1} \cdots a_{i+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \beta^{i+n-2} a_1 & \cdots & \beta^{i+n-2} a_{n-1} a_{i+n-1} \cdots a_{i+2n-2} \end{pmatrix} \\
&= \det\begin{pmatrix} \beta^i a_1 & \beta^i a_2 a_{i+1} & \cdots & \beta^i a_{n-1} a_{i+1} \cdots a_{i+n-1} \\ \beta^{i+1} a_1 & \beta^{i+1} a_2 a_{i+2} & \cdots & \beta^{i+1} a_{n-1} a_{i+2} \cdots a_{i+n} \\ \vdots & \vdots & \ddots & \\ \beta^{i+n-2} a_1 & \beta^{i+n-2} a_2 a_{i+n-1} & \cdots & \beta^{i+n-2} a_{n-1} a_{i+n-1} \cdots a_{i+2n-2} \end{pmatrix} \\
&= \beta^{i+(i+1)+\cdots+(i+n-2)} a_1 a_2 \cdots a_{n-1} \det(A_{n-1,i+1}),
\end{aligned}$$

as asserted. The latter case can be easily derived from this formula.

REFERENCES

1. K. I. Beidar, W. S. Martindale 3rd and A. V. Mikhalev, "Rings with Generalized Identities", Marcel Dekker, Inc., New York–Basel–Hong Kong, 1996.

2. C.-L. Chuang, GPI*s having coefficients in Utumi quotient rings*, Proc. Amer. Math. Soc. **103** (1988), 723–728.

3. C.-L. Chuang and T.-K. Lee, *Identities with a single skew derivations*, submitted.

4. T. S. Erickson, W. S. Martindale 3rd, and J. Osborn, *Prime nonassociative algebras*, Pacific J. Math. **60** (1975), 49–63.

5. K. R. Goodearl, *Prime ideals in skew polynomial rings and quantized Weyl algebras*, J. Algebra **150** (1992), 324–377.

6. K. R. Goodearl and E. R. Letzter, "Prime ideals in skew and q–skew polynomial rings", Memoirs Amer. Math. Soc. **109**, No. 521 (1997).

7. Kharchenko V. K. *Generalized Identities with Automorphisms*, Algebra i Logika **1975**, 14, 132–148.

8. V. K. Kharchenko, *Differential identities of prime rings*, Algebra i Logika **17** (1978), 220–238. (Engl. Transl., Algebra and Logic **17** (1978), 154–168.)

9. V. K. Kharchenko, *Differential identities of semiprime rings*, Algebra i Logika **18** (1979), 86–119 (Engl. Transl., Algebra and Logic **18** (1979), 58–80.)

10. V. K. Kharchenko and A. Z. Popov, *Skew derivations of prime rings*, Comm. Algebra **20**(11), (1992), 3321–3345.

11. A. Leroy, *S-derivations algebriques sur les anneaux premiers.* (French) [*Algebraic S-derivations over prime rings*] Ring theory (Antwerp, 1985), 114–120, Lecture Notes in Math., 1197, Springer, Berlin, 1986.

12. A. Leroy and J. Matczuk, *Quelques remarques a propos des S-derivations*, Comm. Algebra **13**(6) (1985), 1229–1244.

13. W. S. Martindale 3rd, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584.

14. J. Matczuk, *Extended centroids of skew polynomial rings*, Math. J. Okayama Univ. **30** (1988), 13–20.