SPLIT-RADIX FAST HARTLEY TRANSFORM

Indexing terms: Signal processing, Fast Fourier transforms, Transforms

The split radix is used to develop a fast Hartley transform algorithm, it is performed 'in-place', and requires the lowest number of arithmetic operations compared with other related algorithms.

Introduction: Bracewell[1,2] recently proposed a fast Hartley transform (FHT). This transform is closely related to the fast Fourier transform (FFT). It has two advantages over the FFT, however: (i) the forward and the inverse transforms are the same; (ii) the Hartley transformed outputs are real-valued rather than complex data, as with the FFT; also the Fourier spectrum can be calculated via the Hartley transform.

The split radix has been proposed for FFT computations by Duhamel and Hollmann.[3] This algorithm has the advantage of being performed 'in-place' in an FFT-like structure, and requires the lowest number of multiplications and additions for length $N = 2^n$.

In this letter we use split-radix decomposition to develop a fast Hartley transform; it is shown to be faster (in terms of multiplication and addition counts) than other current algorithms.

Split-radix algorithm: The split-radix algorithm[3] is based on both even-term radix-2 decompositions and odd-term radix-4 decompositions simultaneously. If the discrete Hartley transform (DHT)

$$X_k = \sum_{n=0}^{N-1} x_n \left(\cos \frac{2\pi}{N} nk + \sin \frac{2\pi}{N} nk \right)$$

is to be computed, it is decomposed as

$$X_{2k} = \sum_{n=0}^{(N/2)-1} (x_n + x_{n+(N/2)}) \times \left(\cos \frac{2\pi}{N/2} nk + \sin \frac{2\pi}{N/2} nk \right)$$

$$X_{4k+1} = \sum_{n=0}^{(N/4)-1} \left[(A_n + A_{(N/4)-n}) \cos \frac{2\pi}{\pi} n - (B_n - B_{(N/4)-n}) \sin \frac{2\pi}{N} n \right] \times \left(\cos \frac{2\pi}{N/4} nk + \sin \frac{2\pi}{N/4} nk \right)$$

$$X_{4k+3} = \sum_{n=0}^{(N/4)-1} \left[(A_n - A_{(N/4)-n}) \cos \frac{2\pi}{N} 3n + (B_n + B_{(N/4)-n}) \sin \frac{2\pi}{N} 3n \right] \times \left(\cos \frac{2\pi}{N/4} nk + \sin \frac{2\pi}{N/4} nk \right)$$

where

$$A_n = x_n - x_{n+(N/2)} \qquad B_n = x_{n+(N/4)} - x_{n+(3/4)N}$$

and

$$A_{(N/4)} = B_0$$

The above decimation-in-frequency decomposition reduces an N-point DHT into one $(N/2)$-point DHT and two $(N/4)$-point DHTs at the cost of $(N - 4)$ real multiplications and $(2N - 4)$ real additions, i.e.

$$N\text{-point DHT} \longrightarrow \frac{N}{2}\text{-point DHT} + 2 \times \frac{N}{4}\text{-point DHTs}$$

cost: $(N - 4)$ real 'multiplys' and $(2N - 4)$ real 'adds'

Fig. 1 shows the flow graph of the decomposition of a 16-point DHT computation into one 8-point DHT and two 4-point DHTs. The N-point DHT is then obtained by successive use of such decompositions up to the last stage. Fig. 2 shows a complete 8-point DHT altorithm. All decimation-in-frequency FHT algorithms provide the output points in bit-reversal order.

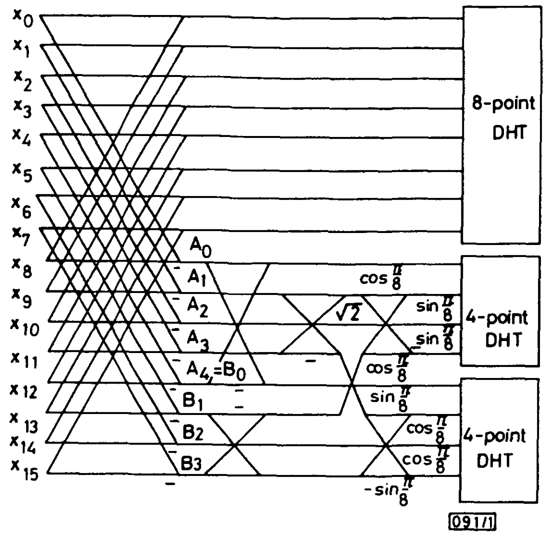

Fig. 1 *Flow graph of split-radix decomposition of a 16-point DHT into one 8-point DHT and two 4-point DHTs at a cost of 10 real multiplications and 26 real additions*

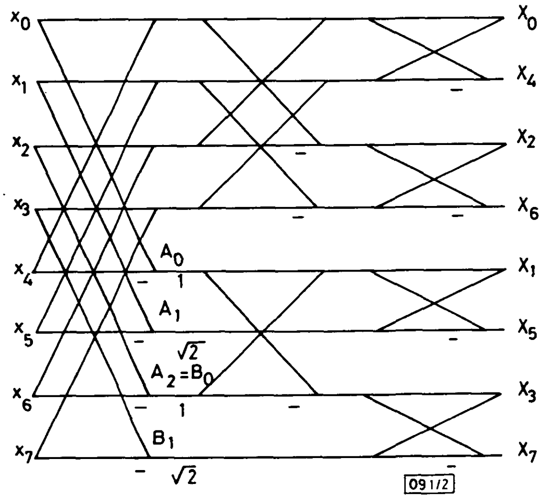

Fig. 2 *Flow graph of a complete 8-point split-radix FHT algorithm*

Computational complexity: The total numbers of operations of the split-radix FHT are

$$A = (2N - 4) \cdot \lceil \log_4 N \rceil \qquad M = (N - 4) \cdot \lceil \log_4 N \rceil \qquad \text{(split-radix FHT)}$$

where $\lceil u \rceil$ denotes the smallest integer greater than or equal to u. Compare this with other current algorithms:[2,4]

$$A = 2N \cdot (\log_2 N - 1) + 2 \qquad M = N \cdot (\log_2 N - 2) + 2 \qquad \text{(Bracewell}^2 \text{ decimation-in-time FHT)}$$

and

$$A = \frac{3}{2} N \cdot (\log_2 N - 1) + 2 \qquad M = N \cdot (\log_2 N - 3) + 4 \qquad \text{(Meckelburg and Lipka}^4 \text{ decimation-in-frequency FHT)}$$

In Tables 1 and 2 we list the number of multiplications and additions required to compute an N-point DHT. The split-radix algorithm has the lowest number of multiplications and additions compared with other related algorithms.

The split-radix algorithm needs less stages $\lceil \log_4 N \rceil$ than $\log_2 N$ in a conventional radix-2 DHT. This results in a much lower required number of arithmetic operations. Since the DHT is a symmetrical transform, additional savings can be gained (see Fig. 3):

(*a*) When

$$\theta = \frac{2\pi nk}{N} = \frac{1}{4}\pi \text{ or } \frac{5}{4}\pi \qquad ((2N-4)/N^2 \text{ to occur})$$

$$(\cos\theta + \sin\theta) = \sqrt{2} \text{ or } -\sqrt{2}$$

(save one multiplication)

(*b*) When

$$\theta = \frac{2\pi nk}{N} = \frac{3}{4}\pi \text{ or } \frac{7}{4}\pi \qquad ((2N-4)/N^2 \text{ to occur})$$

$$(\cos\theta + \sin\theta) = 0$$

(save 2 multiplications and 1 addition)

The practical problem is how to detect these additional savings without introducing too much complexity. This requires further study and research.

Table 1 NUMBER OF REAL MULTIPLICATIONS TO COMPUTE AN N-POINT DHT

N	Bracewell	Meckelburg–Lipka	Split-radix
16	34	20	24
32	98	68	84
64	258	196	180
128	642	516	496
256	1538	1284	1008
512	3586	3076	2540
1024	8194	7172	5100

Table 2 NUMBER OF REAL ADDITIONS TO COMPUTE AN N-POINT DHT

N	Bracewell	Meckelburg–Lipka	Split-radix
16	98	74	56
32	258	194	180
64	642	482	372
128	1538	1154	1008
256	3586	2690	2032
512	8194	6146	5100
1024	18434	13826	10220

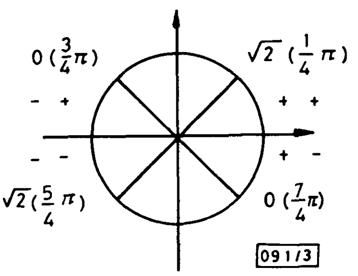

Fig. 3 *Additional DHT computational savings*

Conclusion: The split-radix algorithm is presented for the fast computation of the discrete Hartley transform. This decomposition combines radix-2 flexibility and radix-4 regularity in this algorithm, it is performed 'in-place' and requires the lowest number of arithmetic operations compared with other related algorithms.

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PROPERTIES OF MODES ON PERTURBED FIBRES

Indexing terms: Optical fibres, Optical waveguides

The terms in the Fourier series for the radius of a perturbed step-index optical fibre dictate which particular modes become truly LP and determine their optical axes; these axes need not coincide for different modes. Nonlinear optics experiments may be used to measure the perturbations.

The design and use of optical fibres requires an understanding of the properties of the modal fields. Snyder[1] showed that when core and cladding refractive indices, n_{co} and n_{cl}, differ only slightly, a simple, accurate analysis follows in the $\Delta \equiv (n_{co} - n_{cl})/n_{co} \sim 0$ limit. Gloge[2] exploited that analysis and introduced the linearly polarised LP modes, which are a consequence of a degeneracy when $\Delta \to 0$. For finite Δ, the broken degeneracy precludes the use of LP modes for accurate calculations if polarisation effect are important or unless fibre lengths are short, for example, in photoreceptor optics.[3] An LP form for the transverse field $\boldsymbol{E}$ of a mode with propagation constant β follows from the vector equation

$$\{\nabla^2 + k^2n^2 - \beta^2\}\boldsymbol{E} = -\nabla(\boldsymbol{E}\cdot\nabla \ln n^2) \tag{1}$$

if the right-hand side is negligible.[4,5] k is the wavenumber and ∇ is the transverse gradient operator. Then $\boldsymbol{E} = \psi(x, y)\hat{e}$, where $\hat{e}$ is a unit polarisation vector and ψ obeys the scalar wave equation.

Two authors[6,7] incorporated the effects of the neglected term on the right-hand side of eqn. 1 to correct the LP-plus-scalar-field scheme and to show when it is appropriate. We apply the theory to a step-index fibre with an arbitrary perturbed core/cladding boundary and indicate how the perturbations might be measured in the few-mode case.

Problem: A step fibre with radius ρ_0 and waveguide parameter $V = \rho_0 kn_{co}\sqrt{(2\Delta)}$ is perturbed to have radius ρ:

$$\rho = \rho_0\left\{1 + \sum_{j=1}^{\infty} a_j \cos j(\theta - \theta_j)\right\} \tag{2}$$

Polar angles θ_j and amplitudes a_j are arbitrary, but a_j is assumed small, a_j^2 is negligible and perturbation methods[6,7] apply. Axes x_j', y_j' (Fig. 1) have unit vectors

$$\hat{x}_j' = \hat{y}\sin\theta_j + \hat{x}\cos\theta_j \tag{3}$$
$$\hat{y}_j' = \hat{y}\cos\theta_j - \hat{x}\sin\theta_j$$

Fundamental, l = 0 modes: To first order, β is unchanged for the scalar field, bur corrections to the field ψ, for example by the Green function method,[8] involve terms propertional to $a_j\cos(j\theta)$. The optical axes are found by using ψ in the standard procedure (Appendix A of Reference 6 or Section 32-5 of Reference 8) for determining β corrections due to the right-hand side of eqn. 1. By symmetry, only the term in ψ proportional to a_2 has any effect; the fundamental modes have the form $\psi(x, y)\{\hat{x}_2' \text{ or } \hat{y}_2'\}$. The birefringence depends only on the a_2 or 'elliptical' term in the perturbation, and is therefore already known; see, for example, Section 18-10 of Reference 8.