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Fuzzy Inventory with Backorder

for

Fuzzy Order Quantity and Fuzzy Shortage Quantity

by

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Abstract:

In the problem of inventory with backorder, the total cost is $F(q,s)$ (in section 2, (1)), $0<q<s$, where q is the order quantity, s is the shortage quantity. For each fixed $q>0$, let $K_q(s)=F(q,s)$, $0<s<q$ and $G(q)=\min_{0<s<q} K_q(s)$. Also that the minimum of $F(q,s)$ with respect to q,s is the same as the minimum of $G(q)$ with respect to q and $s=\frac{a}{a+b}q$. Fuzzify both q and s, we will have the result of the fuzzy cost $G(\widetilde{q})$, $\widetilde{s}=\frac{a}{a+b}\widetilde{q}$, where $\widetilde{q}=(q_1,q_0,q_2)$ is a triangular fuzzy number. Also we will have the membership function $\mu_{\widetilde{G(q)}}(z)$ of the fuzzy cost $G(\widetilde{q})$ and its centroid.

Keywords: Inventory, fuzzy inventory, fuzzy cost.

1. Introduction

In the crisp inventory models, all the parameters in the total cost are known and have definite values without ambiguity, as well as the real variables of the total cost are positive. But in the reality, it is not so sure. Hence it is needed to consider the fuzzy inventory models. Due to the various fuzzy cases, one may consider different fuzzy inventory models as follows:

Petrovic and Sweeney (1994) [9], they fuzzified the demand, lead time and inventory level into triangular fuzzy numbers in an inventory control model, then decide the order quantity by the method of fuzzy proposition. Vojosevic, etc. (1996) [12], they fuzzified the ordering cost into trapezoidal fuzzy number in the total cost of an inventory without backorder model, obtained the fuzzy total cost. Then they did the defuzzification by using centroid and gained the total cost in the fuzzy sense. Chen and Wang (1996) [2] fuzzified the order cost, inventory cost, backorder cost into trapezoidal fuzzy numbers and used the functional principle to obtain the estimate of the total cost in the fuzzy sense. Gen, etc. (1997) [3], they considered the fuzzy input data expressed by fuzzy numbers. And the interval mean value concept is used there to help solving this problem. Roy and Maiti (1997) [10], they rewrote the problem of classic economic order quantity into a form of nonlinear programming problem, and introduced the fuzziness both in objective function and storage area. It was solved by fuzzify both nonlinear and geometric programming techniques for linear membership functions. Ishii and Konno (1998) [4], fuzzified the shortage cost L fuzzy number in a classical newsboy problem aimed to find an optimal ordering quantity in the sense of fuzzy ordering. A series of works about inventory problem in the fuzzy sense done by Yao's [1, 5-7, 13-16] can be divided into three classes: inventory without backorder, inventory with backorder and production inventory.

In [6, 14], they talked about the inventory without backorder in the fuzzy sense, in which they fuzzified the order quantity q into triangular fuzzy number, trapezoidal fuzzy number and got the optimal solution in the fuzzy sense. In [16], they considered the inventory without backorder for the fuzzy order quantity and the fuzzy total demand quantity. In [5, 7], they talked about the production inventory problem in the fuzzy sense, and got the optimal solution in the fuzzy sense. In [1, 13, 14, 15], they talked about the inventory with backorder in the fuzzy sense. In [13, 14, 15], they discussed about the total cost $F^*(q,s) = \frac{aTs^2}{2q} + \frac{bT(q-s)^2}{2q} + \frac{cR}{q}$, where a, b, c, R, T, q are illustrated in the section 2 of this article, and s is the maximal stock quantity, $q - s$ is the shortage quantity. In [13, 14] they fuzzified the order quantity q into triangular fuzzy number, trapzoidal fuzzy number, and s was a real variable. Then deduced the membership function of the fuzzy total cost, as well as its centroid. In [15], they fuzzified the total demand R into the interval-valued fuzzy set and q, s were real variables. The total cost in section 2, $F(q,s) = \frac{aT(q-s)^2}{2q} + \frac{bTs^2}{2q} + \frac{cR}{q}$, $0 < s < q$; a, b, c, T, R, q, s are illustrated in this article, and $q - s$ is the maximal stock quantity. In [1], they fuzzified the shortage quantity s into triangular fuzzy number $\widetilde{s}$, and the order quantity q was a positive real variable. and then deduced the membership function of the fuzzy total cost and its centroid. Through all [1, 5-7, 13-16], we found out that the smaller the fuzzification, the smaller the difference between the fuzzy case and the crisp case.

Note that in [1, 13, 14], they only fuzzified the q, s one at the time, and the other one left to be a positive real variable. In [15], they fuzzified the total demand R, and q, s are positive real variables. In this paper, the formula of the total cost function $F(q,s) = \frac{aT(q-s)^2}{2q} + \frac{bTs^2}{2q} + \frac{cR}{q}$ used here are the same as in [1]. But we fuzzify both q and s into triangular fuzzy number $\widetilde{q}$, $\widetilde{s}$.

In section 2, for each $q > 0$, let $K_q(s) = F(q,s), 0 < s < q$, and $G(q) = \min_{0<s<q} K_q(s)$. Then the minimum of $F(q,s)$ with respect to q, s is the same as the minimum of $G(q)$ with respect to q with $s = \frac{a}{a+b}q$. Therefore we shall fuzzify q into triangular fuzzy number $\widetilde{q} = (q_1, q_0, q_2)$, $0 < q_1 < q_0 < q_2$ and $\widetilde{s} = \frac{a}{a+b}\widetilde{q}$. Then we will get the fuzzy cost $G(\widetilde{q})$.

In order to do so, we shall divide the condition $0 < q_1 < q_0 < q_2$ into four cases;

(1): $q_* < q_1 < q_0 < q_2$; (2): $0 < q_1 < q_0 < q_2 < q_*$

(3): $0 < q_1 < q_* < q_0 < q_2$;   (4): $0 < q_1 < q_0 < q_* < q_2$

then find for each case the membership function $\mu_{G(\tilde{q})}(z)$ of $G(\tilde{q})$ and its centroid. Next, we shall find out the estimated value of the total cost in the fuzzy sense with (q_1, q_0, q_2) known, and also the estimated value of shortage quantity in the fuzzy sense.

In section 3, we shall give some examples.

2. The membership function and the centroid of the fuzzy total cost

Insert Fig. 1 here

We shall use the following notations in the discussion of inventory with backorder model.

T : the whole period (month, quarter, or year) of the plan

q : the order quantity per cycle

a : the holding cost per unit quantity per cycle

b : the shortage cost per unit quantity per cycle

c : the order cost per cycle

R : the total demand quantity of the whole plan period

s : the shortage quantity per cycle

t_1 : the duration within a cycle during those inventory is held

t_2 : the duration within a cycle during a shortage exists

t_q : the length of the inventory cycle, $t_q = t_1 + t_2$

From Fig. 1, we have $\frac{R}{T} = \frac{q}{t_q} = \frac{q-s}{t_1} = \frac{s}{t_2}$ = demand quantity per time.

By [8,11], the total cost $F(q, s)$ during the plan period T is

$$F(q, s) = [\frac{1}{2}at_1(q - s) + \frac{1}{2}bt_2s + c]\frac{R}{q}$$

$$= \frac{aT(q-s)^2}{2q} + \frac{bTs^2}{2q} + \frac{cR}{q}$$
$$= \frac{(a+b)Ts^2}{2q} - aTs + \frac{aT}{2}q + \frac{cR}{q}, \quad 0 < s < q \tag{1}$$

Therefore the optimal solution in crisp case is

$$q_* = [\frac{2(a+b)cR}{abT}]^{\frac{1}{2}}, \quad s_* = [\frac{2acR}{b(a+b)T}]^{\frac{1}{2}} \quad \text{and} \quad F(q_*, s_*) = (\frac{2abcRT}{a+b})^{\frac{1}{2}} \tag{2}$$

Observed that $\quad s_* = \frac{a}{a+b} q_* \quad (< q_*)$ (3)

Since to fuzzify both q and s in (1) into triangular fuzzy numbers $\widetilde{q}, \widetilde{s}$ at the same time and find out the fuzzy total cost $F(\widetilde{q}, \widetilde{s})$. And obtain the membership function by using Extension Principle. This process is very tedius and difficult. Instead, we shall use the following step then apply the Extension Principle to obtain the membership function.

For each fixed $q > 0$, from (1),

$$\text{let} \quad K_q(s) = F(q,s) = \frac{(a+b)T}{2q}s^2 - aTs + \frac{aT}{2}q + \frac{cR}{q}, \quad 0 < s < q \tag{4}$$

Hence, $\frac{\partial}{\partial s}K_q(s) = \frac{(a+b)T}{q}s - aT = 0$. Then we have $s = \frac{a}{a+b}q\,(< q)$

$$\min_{0<s<q} K_q(s) = K_q(\frac{aq}{a+b}) = \frac{abT}{2(a+b)}q + \frac{cR}{q} (= G(q), \text{ say }), \quad 0 < q \tag{5}$$

and $\frac{d}{dq}G(q) = \frac{abT}{2(a+b)} - \frac{cR}{q^2} = 0$, then we have $q = (\frac{2(a+b)cR}{abT})^{\frac{1}{2}} = q_*$

$$\min_{0<q} \min_{0<s<q} K_q(s) = \min_{0<q} G(q) = G(q_*) \tag{6}$$

From $s = \frac{a}{a+b}q$ and (4), (5), we get

$$F(q, \frac{a}{a+b}q) = K_q(\frac{a}{a+b}q) = G(q) \tag{7}$$

Thus from (1), (2), (3), (7), we have

$$\min_{0<s<q} F(q,s) = F(q_*, s_*) = F(q_*, \frac{a}{a+b}q_*) = G(q_*) = \min_{0<q} G(q)$$

Property 1. The minimum total cost $F(q, s)$ with respect to q, s is the same as the minimum total cost $G(q) = F(q, \frac{a}{a+b}q)$ with respect to q and $s = \frac{a}{a+b}q$, i.e. $G(q_*) = F(q_*, s_*) = [\frac{2abcRT}{a+b}]^{\frac{1}{2}}$.

As mentioned in Introduction, in [1], the authors fuzzified s of $F(q, s)$ in (1) into triangular fuzzy numbers, and left q to be a positive real numbers. In [13], the authors fuzzified q of the total cost $F^*(q, s)$ into triangular fuzzy numbers, and left s to be a positive real variable. Hence in [1, 13], they only fuzzified one of the q, s. Here we shall fuzzify both q, s at the same time for the following reason.

Under the assumption that all the lead time (i.e. the period from the ordering time to the arrival time) in each cycle are the same will lead to the result of $F(q, s)$ in formula (1), where q is the order quantity and s is the shortage quantity. In the reality, such as the traffic condition may vary as well as other situations may affect the lead time among each cycle. Hence in (1) we can not assume the lead time are all the same in each cycle. This will affect to the certainty of order quantity q and shortage quantity s too. Therefore we shall fuzzify both q and s at the same time to suit the real situation better than just fuzzify one of those.

In the crisp case, the ordering quantity q and the shortage quantity s are chosen in order to have $F(q, s)$ reached its minimum. In the fuzzy case, we fuzzify q, s by the following: Since in the crisp case, we employed two variables, one is order quantity q and the other is shortage quantity s. And we learned that there is a relation between these two, i.e. $0s = \frac{a}{a+b}q$ in Property 1. So here we only need to deal with one variable " q " problem. Replace q by q_0, and define the order quantity by using a fuzzy language " round q_0", i.e. using a triangular fuzzy number $\widetilde{q} = (q_1, q_0, q_2)$ and $\widetilde{s} = \frac{a}{a+b}\widetilde{q}$ through Property 1.

$$\mu_{\underset{q}{\sim}}(q) = \begin{cases} \frac{q-q_1}{q_0-q_1} & , \quad q_1 \le q \le q_0 \\ \frac{q_2-q}{q_2-q_0} & , \quad q_0 \le q \le q_2 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{8}$$

where $0 < q_1 < q_0 < q_2$; q_1, q_0, q_2 are unknown.

From $s = \frac{a}{a+b}q$ and the Extension Principle, we have

$$\mu_{\underset{s}{\sim}}(s) = \max_{s=\frac{a}{a+b}q} \mu_{\underset{q}{\sim}}(q) = \mu_{\underset{q}{\sim}}(\frac{(a+b)s}{a}) = \begin{cases} \frac{(a+b)s-aq_1}{a(q_0-q_1)} & , \quad \frac{a}{a+b}q_1 \leq s \leq \frac{a}{a+b}q_0 \\ \frac{aq_2-(a+b)s}{a(q_2-q_0)} & , \quad \frac{a}{a+b}q_0 \leq s \leq \frac{a}{a+b}q_2 \\ 0 & , \quad \text{otherwise} \end{cases} \tag{9}$$

The centroid of $\widetilde{q}$, by (8) is

$$C(\widetilde{q}) = \frac{1}{3}(q_1 + q_0 + q_2) \tag{10}$$

where $C(\widetilde{q})$ denotes the estimated value of the order quantity in the fuzzy sense. The centroid of $\widetilde{s}$, by (9) is

$$C(\widetilde{s}) = \frac{1}{3}(\frac{a}{a+b})(q_1 + q_0 + q_2) = (\frac{a}{a+b})C(\widetilde{q}) \quad (< C(\widetilde{q})) \tag{11}$$

where $C(\widetilde{s})$ denotes the estimated value of the shortage quantity in the fuzzy sense.

From Property 1 and (7), by using the total cost $G(q)$ and $s = \frac{a}{a+b}q$, we are able to find the minimum of $F(q,s)$.

Note 1: From (7) and $\widetilde{s} = \frac{a}{a+b}\widetilde{q}$, then we have the following:

$$G(\widetilde{q}) = F(\widetilde{q}, \frac{a}{a+b}\widetilde{q}) = F(\widetilde{q}, \widetilde{s}) \tag{*}$$

Or, we may derive (*) by the following:

From (7), let $x = \frac{a}{a+b}y$ and $z = F(y,x) = G(y)$. By $\widetilde{s} = \frac{a}{a+b}\widetilde{q}$, we have

$$\begin{aligned} \mu_{F(\widetilde{q},\widetilde{s})}(z) &= \sup_{(y,x)\in F^{-1}(z)} \mu_{\widetilde{q}}(y) \wedge \mu_{\widetilde{s}}(x) \\ &= \sup_{y\in G^{-1}(z)} \mu_{\widetilde{q}}(y) \wedge \mu_{\frac{a}{a+b}\widetilde{q}}(\frac{a}{a+b}y) \\ &= \sup_{y\in G^{-1}(z)} \mu_{\widetilde{q}}(y) = \mu_{G(\widetilde{q})}(z) \forall z \end{aligned}$$

Thus $G(\widetilde{q}) = F(\widetilde{q}, \widetilde{s})$. Hence (*) holds. Therefore from Property 1 and (*), our problem can be solved by considering $G(q)$, $s = \frac{a}{a+b}q$ and fuzzifies q to $\widetilde{q}$ with membership function (8).

First we fuzzify $G(q)$ to $G(\widetilde{q})$ (the fuzzy total cost). Let $G(q) = z$, from (5), we have

$$abTq^2 - 2(a+b)zq + 2(a+b)cR = 0 \tag{12}$$

The determinant of (12) is

$$D(z) = (a+b)^2 z^2 - 2ab(a+b)cTR = (a+b)^2[z^2 - G(q_*)^2] \geq 0, \quad \text{if } z \geq [\frac{2abcTR}{a+b}]^{\frac{1}{2}} = G(q_*)$$

The roots of (12) are

$$d_1(z) = \frac{(a+b)z - \sqrt{D(z)}}{abT} = \frac{a+b}{abT}[z - \sqrt{z^2 - G(q_*)^2}] \quad \text{and}$$

$$d_2(z) = \frac{(a+b)z + \sqrt{D(z)}}{abT} = \frac{a+b}{abT}[z + \sqrt{z^2 - G(q_*)^2}]$$

From $G(q) = z$ and the Extension Principle, we have the membership function of the fuzzy total cost $G(\widetilde{q})$ is

$$\mu_{G(\widetilde{q})}(z) = \begin{cases} \sup\limits_{q \in G^{-1}(z)} \mu_{\widetilde{q}}(q) = \max[\mu_{\widetilde{q}}(d_1(z)), \mu_{\widetilde{q}}(d_2(z))] & , \quad \text{if } z \geq G(q_*) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{13}$$

The z we considered here has to satisfy the condition $G(q_*) \leq z$. In order to solve (13), we use the following Table 1.

Table 1. The position of $d_1(z), d_2(z)$, and $\mu_{\underset{\sim}{G(q)}}(z)$ for $z \geq G(q_*)$

case		q_1		q_0		q_2		$\mu_{\underset{\sim}{G(q)}}(z)$
1	$d_1(z), d_2(z)$							0
2	$d_1(z)$		$d_2(z)$					$\frac{d_2(z)-q_1}{q_0-q_1}$
3	$d_1(z)$				$d_2(z)$			$\frac{q_2-d_2(z)}{q_2-q_0}$
4	$d_1(z)$						$d_2(z)$	0
5			$d_1(z), d_2(z)$					$\frac{d_2(z)-q_1}{q_0-q_1}$
6			$d_1(z)$		$d_2(z)$			max $[\frac{d_1(z)-q_1}{q_0-q_1}, \frac{q_2-d_2(z)}{q_2-q_0}]$
7			$d_1(z)$				$d_2(z)$	$\frac{d_1(z)-q_1}{q_0-q_1}$
8					$d_1(z), d_2(z)$			$\frac{q_2-d_1(z)}{q_2-q_0}$
9					$d_1(z)$		$d_2(z)$	$\frac{q_2-d_1(z)}{q_2-q_0}$
10							$d_1(z), d_2(z)$	0

From (5) , $G(q_j) = \dfrac{abT}{2(a+b)}q_j + \dfrac{cR}{q_j}, \quad j = 0, 1, 2$ (14)

From (6) , $G(q_*) \leq G(q_j) \quad$ for $j = 0, 1, 2$ (15)

$$\begin{aligned} G(q_j) - G(q_k) &= \frac{q_j - q_k}{q_j q_k}\left(\frac{abT}{2(a+b)}\right)\left[q_j q_k - \frac{2(a+b)cR}{abT}\right] \\ &= \frac{q_j - q_k}{q_j q_k}\left(\frac{abT}{2(a+b)}\right)\left[q_j q_k - q_*^2\right] \end{aligned}$$

$$\text{If } q_j \geq q_k \quad \text{and} \quad q_j q_k \geq q_*^2, \quad \text{then} \quad G(q_j) \geq G(q_k) \tag{16}$$

$$\text{If } q_j \geq q_k \quad \text{and} \quad q_j q_k \leq q_*^2, \quad \text{then} \quad G(q_j) \leq G(q_k) \tag{17}$$

In the following discussion, $z \geq G(q_*)$, and $\frac{(abTq_j)}{(a+b)} \leq G(q_*) \Longleftrightarrow q_j \leq q_*$

(1). For every $j = 0, 1, 2$; if $q_j \leq d_1(z)$, then

$$\sqrt{D(z)} \leq (a+b)z - abTq_j \tag{$*$}$$

we have

(i) If $\frac{abTq_j}{a+b} \leq z$ then $G(q_*) \leq z \leq G(q_j)$. Since $G(q_*) \leq z$, so, if $\frac{abTq_j}{a+b} \leq G(q_*)$. i.e. $q_j \leq q_*$, then $G(q_*) \leq z \leq G(q_j)$. Thus we have the following (18).

(ii) If $z \leq \frac{abTq_j}{a+b}$, then $(*)$ has no solution. Since $G(q_*) \leq z$, therefore $G(q_*) \leq \frac{abTq_j}{a+b}$. i.e. $q_j \geq q_*$. Thus we have the following (19).

$$\text{If } (q_j \leq d_1(z)) \wedge (q_j \leq q_*), \text{ then } G(q_*) \leq z \leq G(q_j) \tag{18}$$

$$\text{If } (q_j \leq d_1(z)) \wedge (q_j \geq q_*), \text{ then there is no solution} \tag{19}$$

(2). For every $j = 0, 1, 2$; if $d_1(z) \leq q_j$, then $(a+b)z - abTq_j \leq \sqrt{D(z)}$, and we have

$$\text{If } (d_1(z) \leq q_j) \wedge (q_j \leq q_*), \text{ then } G(q_j) \leq z \tag{20}$$

$$\text{If } (d_1(z) \leq q_j) \wedge (q_j \geq q_*), \text{ then } G(q_*) \leq z \tag{21}$$

(3). For every $j = 0, 1, 2$; If $q_j \leq d_2(z)$, then $abTq_j - (a+b)z \leq \sqrt{D(z)}$, and we have

$$\text{If } (q_j \leq d_2(z)) \wedge (q_j \leq q_*), \text{ then } G(q_*) \leq z \tag{22}$$

$$\text{If } (q_j \leq d_2(z)) \wedge (q_j \geq q_*), \text{ then } G(q_j) \leq z \tag{23}$$

(4). For every $j = 0, 1, 2$; if $d_2(z) \leq q_j$, then $\sqrt{D(z)} \leq abTq_j - (a+b)z$, and we have

$$\text{If } (d_2(z) \leq q_j) \wedge (q_j \leq q_*), \text{ then there is no solution} \tag{24}$$

$$\text{If } (d_2(z) \leq q_j) \wedge (q_j \geq q_*), \text{ then } G(q_*) \leq z \leq G(q_j) \tag{25}$$

Let

$$\begin{aligned}
D_1(p_1,p_2) =& \int_{p_1}^{p_2} d_1(z)\,dz \\
=& \frac{a+b}{2abT}(p_2^2 - p_1^2) - \frac{a+b}{2abT}[\,p_2\sqrt{p_2^2 - G(q_*)^2} - p_1\sqrt{p_1^2 - G(q_*)^2}\,] \\
&+ cR\ln\frac{p_2 + \sqrt{p_2^2 - G(q_*)^2}}{p_1 + \sqrt{p_1^2 - G(q_*)^2}} \\
D_2(p_1,p_2) =& \int_{p_1}^{p_2} d_2(z)\,dz \\
=& \frac{a+b}{2abT}(p_2^2 - p_1^2) + \frac{a+b}{2abT}[\,p_2\sqrt{p_2^2 - G(q_*)^2} - p_1\sqrt{p_1^2 - G(q_*)^2}\,] \\
&- cR\ln\frac{p_2 + \sqrt{p_2^2 - G(q_*)^2}}{p_1 + \sqrt{p_1^2 - G(q_*)^2}}
\end{aligned}$$

Let

$$V_1(p_1,p_2) = \frac{1}{q_0 - q_1}\int_{p_1}^{p_2}[d_1(z) - q_1]\,dz = \frac{1}{q_0 - q_1}[D_1(p_1,p_2) - q_1(p_2 - p_1)] \tag{26}$$

$$V_2(p_1,p_2) = \frac{1}{q_0 - q_1}\int_{p_1}^{p_2}[d_2(z) - q_1]\,dz = \frac{1}{q_0 - q_1}[D_2(p_1,p_2) - q_1(p_2 - p_1)] \tag{27}$$

$$V_3(p_1,p_2) = \frac{1}{q_2 - q_0}\int_{p_1}^{p_2}[q_2 - d_1(z)]\,dz = \frac{1}{q_2 - q_0}[q_2(p_2 - p_1) - D_1(p_1,p_2)] \tag{28}$$

$$V_4(p_1,p_2) = \frac{1}{q_2 - q_0}\int_{p_1}^{p_2}[q_2 - d_2(z)]\,dz = \frac{1}{q_2 - q_0}[q_2(p_2 - p_1) - D_2(p_1,p_2)] \tag{29}$$

Let

$$\begin{aligned}
D_{12}(p_1,p_2) =& \int_{p_1}^{p_2} z d_1(z)\,dz \\
=& \frac{a+b}{3abT}(p_2^3 - p_1^3) - \frac{a+b}{3abT}[\,(p_2^2 - G(q_*)^2)^{\frac{3}{2}} - (p_1^2 - G(q_*)^2)^{\frac{3}{2}}\,] \\
D_{22}(p_1,p_2) =& \int_{p_1}^{p_2} z d_2(z)\,dz \\
=& \frac{a+b}{3abT}(p_2^3 - p_1^3) + \frac{a+b}{3abT}[\,(p_2^2 - G(q_*)^2)^{\frac{3}{2}} - (p_1^2 - G(q_*)^2)^{\frac{3}{2}}\,]
\end{aligned}$$

$$V_{12}(p_1,p_2) = \frac{1}{q_0-q_1}\int_{p_1}^{p_2} z[d_1(z)-q_1]\,dz = \frac{1}{q_0-q_1}[D_{12}(p_1,p_2) - \frac{q_1}{2}(p_2^2-p_1^2)] \quad (30)$$

$$V_{22}(p_1,p_2) = \frac{1}{q_0-q_1}\int_{p_1}^{p_2} z[d_2(z)-q_1]\,dz = \frac{1}{q_0-q_1}[D_{22}(p_1,p_2) - \frac{q_1}{2}(p_2^2-p_1^2)] \quad (31)$$

$$V_{32}(p_1,p_2) = \frac{1}{q_2-q_0}\int_{p_1}^{p_2} z[q_2-d_1(z)]\,dz = \frac{1}{q_2-q_0}[\frac{q_2}{2}(p_2^2-p_1^2)] - D_{12}(p_1,p_2)] \quad (32)$$

$$V_{42}(p_1,p_2) = \frac{1}{q_2-q_0}\int_{p_1}^{p_2} z[q_2-d_2(z)]\,dz = \frac{1}{q_2-q_0}[\frac{q_2}{2}(p_2^2-p_1^2)] - D_{22}(p_1,p_2)] \quad (33)$$

Now, in order to find $\mu_{\widetilde{G(q)}}(z)$ easier, we shall divide the region $0 < q_1 < q_0 < q_2$ into the following four cases:

(1°) $q_* \leq q_1 < q_0 < q_2$;   (2°) $0 < q_1 < q_0 < q_2 \leq q_*$

(3°) $0 < q_1 \leq q_* < q_0 < q_2$;   (4°) $0 < q_1 < q_0 \leq q_* < q_2$

Then we proceed to find the membership function $\mu_{\widetilde{G(q)}}(z)$ and its centroid accordingly:

§2.1. Find $\mu_{\widetilde{G(q)}}(z)$ and its centroid under the condition $q_* \leq q_1 < q_0 < q_2$.

$$q_* \leq q_1 < q_0 < q_2 \quad (2.1.1)$$

From (15), (16), (2.1.1), we have

$$G(q_*) < G(q_1) < G(q_0) < G(q_2) \quad (2.1.2)$$

Then by Table 1, (18) - (25), (2.1.1) and (2.1.2), we have

(i) For case 1,4,10,

$$\mu_{\widetilde{G(q)}}(z) = 0 \quad (2.1.3)$$

(ii) For case 5 - 7, $q_1 < d_1(z)$ and case 8, 9, $q_0 \leq d_1(z)$.

From (2.1.1) and (19), there is no solution (2.1.4)

(iii) For case 2, $d_1(z) \leq q_1 \leq d_2(z) \leq q_0$

From $d_1(z) \leq q_1, q_* \leq q_1$ and (21), we have $G(q_*) \leq z$ (∗1)

From $q_1 \le d_2(z), q_* \le q_1$ and (23), we have $G(q_1) \le z$ (*2)

From $d_2(z) \le q_0, q_* \le q_0$ and (25), we have $G(q_*) \le z \le G(q_0)$ (*3)

From (*1) - (*3) and (2.1.2), we have

$$\mu_{\tilde{G(q)}}(z) = \frac{d_2(z) - q_1}{q_0 - q_1}, \quad G(q_1) \le z \le G(q_0) \tag{2.1.5}$$

(iv) For case 3, $d_1(z) \le q_1$, $q_0 \le d_2(z) \le q_2$. From (2.1.1), $d_1(z) \le q_1$, and (21), $G(q_*) \le z$, $q_0 \le d(z)$ and (23), $G(q_0) \le z, d_2(z) \le q_2$; also from (25), $G(q_*) \le z \le G(q_2)$. Finally by (2.1.2), we have

$$\mu_{\tilde{G(q)}}(z) = \frac{q_2 - d_2(z)}{q_2 - q_0}, \; G(q_0) \le z \le G(q_2) \tag{2.1.6}$$

Let $T_1 = \{(q_1, q_0, q_2) \,|\, q_* \le q_1 < q_0 < q_2\}$ By (2.1.3) - (2.1.6), we have the following Property.

Property 2. Let $(q_1, q_0, q_2) \in T_1$. Then the membership function is given by

$$\mu_{\tilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q_1) \le z \le G(q_0) \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad G(q_0) \le z \le G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.1.7}$$

By (27), (29), (31), (33), we have

Property 3. Let $(q_1, q_0, q_2) \in T_1$, then the centroid of $\mu_{\tilde{G(q)}}(z)$ in (2.1.7) of Property 1 is given by

$$E_1(q_1, q_0, q_2) = \frac{R_1}{P_1} \tag{2.1.8}$$

where

$$\begin{aligned} P_1 &= \int_{-\infty}^{\infty} \mu_{\tilde{G(q)}}(z)\,dz \\ &= \frac{1}{q_0 - q_1} \int_{G(q_1)}^{G(q_0)} [d_2(z) - q_1]\,dz + \frac{1}{q_2 - q_0} \int_{G(q_0)}^{G(q_2)} [q_2 - d_2(z)]\,dz \\ &= V_2(G(q_1), G(q_0)) + V_4(G(q_0), G(q_2)) \\ R_1 &= \int_{-\infty}^{\infty} z\mu_{\tilde{G(q)}}(z)\,dz \\ &= \frac{1}{q_0 - q_1} \int_{G(q_1)}^{G(q_0)} z[d_2(z) - q_1]\,dz + \frac{1}{q_2 - q_0} \int_{G(q_0)}^{G(q_2)} z[q_2 - d_2(z)]\,dz \\ &= V_{22}(G(q_1), G(q_0)) + V_{42}(G(q_0), G(q_2)) \end{aligned}$$

§2.2. Find $\mu_{\underset{\sim}{G(q)}}(z)$ and its centroid under the condition $0 < q_1 < q_0 < q_2 \le q_*$

$$0 < q_1 < q_0 < q_2 \le q_* \tag{2.2.1}$$

From ((15), (17) and (2.2.1), we have

$$G(q_*) < G(q_2) < G(q_0) < G(q_1) \tag{2.2.2}$$

Then by Table 1. (18) - (25), (2.2.1) and (2.2.2) and similar to section 2.1, we have the following Property.

Let $T_2 = \{(q_1, q_0, q_2) \,|\, 0 < q_1 < q_0 < q_2 \le q_*\}$

Property 4. Let $(q_1, q_0, q_2) \in T_2$. Then the membership function $\mu_{\underset{\sim}{G(q)}}(z)$ is given by

$$\mu_{\underset{\sim}{G(q)}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_2) \le z \le G(q_0) \\ \frac{d_1(z) - q_1}{q_0 - q_1} & , \quad G(q_0) \le z \le G(q_1) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.2.3}$$

By (26), (28), (30), (32), we have

Property 5. Let $(q_1, q_0, q_2) \in T_2$. Then the centroid of $\mu_{\underset{\sim}{G(q)}}(z)$ in (2.2.3) of Property 2 is given by

$$E_2(q_1, q_0, q_2) = \frac{R_2}{P_2}$$

where

$$\begin{aligned} P_2 &= \frac{1}{q_2 - q_0} \int_{G(q_2)}^{G(q_0)} [q_2 - d_1(z)]\, dz + \frac{1}{q_0 - q_1} \int_{G(q_0)}^{G(q_1)} [d_1(z) - q_1]\, dz \\ &= V_3(G(q_2), G(q_0)) + V_1(G(q_0), G(q_1)) \end{aligned}$$

$$\begin{aligned} R_2 &= \frac{1}{q_2 - q_0} \int_{G(q_2)}^{G(q_0)} z[q_2 - d_1(z)]\, dz + \frac{1}{q_2 - q_0} \int_{G(q_0)}^{G(q_1)} z[d_1(z) - q_1]\, dz \\ &= V_{32}(G(q_2), G(q_0)) + V_{12}(G(q_0), G(q_1)) \end{aligned}$$

§2.3. Find $\mu_{\underset{\sim}{G(q)}}(z)$ and its centroid under the condition $0 < q_1 \le q_* < q_0 < q_2$.

$$0 < q_1 \le q_* < q_0 < q_2 \tag{2.3.1}$$

From (15), (16) and (2.3.1), we have

$$G(q_*) < G(q_0) < G(q_2) \tag{2.3.2}$$

Under the condition (2.3.2), all the permutations of $G(q_1), G(q_0), G(q_2)$ by (15) - (17) are

$$G(q_1) < G(q_0) < G(q_2), \text{ if } q_0 q_1 > q_*^2 \tag{2.3.3}$$

$$G(q_0) < G(q_1) < G(q_2), \text{ if } q_1 q_0 < q_*^2 \text{ and } q_2 q_1 > q_*^2 \tag{2.3.4}$$

$$G(q_0) < G(q_2) < G(q_1), \text{ if } q_1 q_2 < q_*^2 \tag{2.3.5}$$

Then by Table 1, (18) - (25), and (2.3.1) - (2.3.5), we have

(i) For case 1, 4, 10,

$$\mu_{\widetilde{G(q)}} = 0 \tag{2.3.6}$$

(ii) For case 2, $d_1(z) \le q_1 \le d_2(z) \le q_0$. From (20), (22), (25), each we have $G(q_1) \le z, G(q_*) \le z, G(q_*) \le z \le G(q_0)$. By (2.3.3), we have

$$\text{When } q_0 q_1 > q_*^2, \quad \mu_{\widetilde{G(q)}}(z) = \frac{d_2(z) - q_1}{q_0 - q_1}, \ G(q_1) \le z \le G(q_0) \tag{2.3.7}$$

(iii) For case 3, $d_1(z) \le q_1$, $q_0 \le d_2(z) \le q_2$. From (20), (23), (25), each we have $G(q_1) \le z, G(q_0) \le z, G(q_*) \le z \le G(q_2)$. By (2.3.3) and (2.3.4), we have

$$\text{When } q_0 q_1 > q_*^2, \quad \mu_{\widetilde{G(q)}}(z) = \frac{q_2 - d_2(z)}{q_2 - q_0}, \ G(q_0) \le z \le G(q_2) \tag{2.3.8}$$

$$\text{When } q_1 q_0 < q_*^2, \quad \text{and } q_2 q_1 > q_*^2,$$
$$\mu_{\widetilde{G(q)}}(z) = \frac{q_2 - d_2(z)}{q_2 - q_0}, \quad G(q_1) \le z \le G(q_2) \tag{2.3.9}$$

(iv) For case 5, $q_1 \le d_1(z), d_2(z) \le q_0$. From (18), (25), each we have $G(q_*) \le z \le G(q_1), G(q_*) \le z \le G(q_0)$. By (2.3.3) - (2.3.5), we have

$$\text{When } q_0 q_1 > q_*^2, \ \mu_{\widetilde{G(q)}}(z) = \frac{d_2(z) - q_1}{q_0 - q_1}, \ G(q_*) \le z \le G(q_1) \tag{2.3.10}$$

When $q_1 q_0 < q_*^2$, and $q_2 q_1 > q_*^2$

$$\mu_{\underset{G(q)}{\sim}}(z) = \frac{d_2(z) - q_1}{q_0 - q_1},\ G(q_*) \le z \le G(q_0) \tag{2.3.11}$$

$$\text{When}\ \ q_1 q_2 < q_*^2,\ \mu_{\underset{G(q)}{\sim}}(z) = \frac{d_2(z) - q_1}{q_0 - q_1},\ G(q_*) \le z \le G(q_0) \tag{2.3.12}$$

(v) For case 6, $q_1 \le d_1(z) \le q_0 \le d_2(z) \le q_2$. From (18), (21), (23) (25), each we have $G(q_*) \le z \le G(q_1)$, $G(q_*) \le z, G(q_0) \le z, G(q_*) \le z \le G(q_2)$. By (2.3.4) and (2.3.5), we have

When $q_1 q_0 < q_*^2$, and $q_2 q_1 > q_*^2$

$$\mu_{\underset{G(q)}{\sim}}(z) = \max[\frac{d_1(z) - q_1}{q_0 - q_1}, \frac{q_2 - d_2(z)}{q_2 - q_0}],\quad G(q_0) \le z \le G(q_1) \tag{2.3.13}$$

When $q_1 q_2 < q_*^2$,

$$\mu_{\underset{G(q)}{\sim}}(z) = \max[\frac{d_1(z) - q_1}{q_0 - q_1}, \frac{q_2 - d_2(z)}{q_2 - q_0}],\quad G(q_0) \le z \le G(q_2) \tag{2.3.14}$$

(vi) For case 7, $q_1 \le d_1(z) \le q_0, q_2 \le d_2(z)$. From (18), (21), (23), each we have $G(q_*) \le z \le G(q_1)$, $G'(q_*) \le z$, $G(q_2) \le z$. By (2.3.5), we have

$$\text{When}\ \ q_1 q_2 < q_*^2\ \mu_{\underset{G(q)}{\sim}}(z) = \frac{d_1(z) - q_1}{q_0 - q_1}, G(q_2) \le z \le G(q_1) \tag{2.3.15}$$

(vii) For case 8, 9, $q_0 \le d_1(z)$.

$$\text{Since}\quad q_0 \ge q_*,\ \text{by (19), there is no solution .} \tag{2.3.16}$$

Let $T_3 = \{(q_1, q_0, q_2) \,|\, 0 \le q_1 \le q_* < q_0 < q_2\}$. Then by (2.3.6) - (2.3.16), we have

Property 6. Let $(q_1, q_0, q_2) \in T_3$. Then the membership function of $\mu_{\underset{G(q)}{\sim}}(z)$ is given by

(1°) If $q_0 q_1 > q_*^2$, then

$$\mu_{\underset{G(q)}{\sim}}(z) = \begin{cases} \frac{d_2(z) - q_1}{q_0 - q_1} &, \quad G(q_*) \le z \le G(q_0) \quad [(2.3.10),\ (2.3.7)] \\ \frac{q_2 - d_2(z)}{q_2 - q_0} &, \quad G(q_0) \le z \le G(q_2) \quad [(2.3.8)] \\ 0 &, \quad \text{otherwise} \end{cases} \tag{2.3.17}$$

(2°) If $q_1 q_0 < q_*^2$, and $q_2 q_1 > q_*^2$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q_*) \le z \le G(q_0) \quad [(2.3.11)] \\ \max[\frac{d_1(z)-q_1}{q_0-q_1}, \frac{q_2-d_2(z)}{q_2-q_0}] & , \quad G(q_0) \le z \le G(q_1) \quad [(2.3.13)] \\ \frac{q_2-d_2(z)}{q_2-q_0}, & , \quad G(q_1) \le z \le G(q_2) \quad [(2.3.9)] \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.18}$$

(3°) If $q_1 q_2 < q_*^2$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q_*) \le z \le G(q_0) \quad [(2.3.12)] \\ \max[\frac{d_1(z)-q_1}{q_0-q_1}, \frac{q_2-d_2(z)}{q_2-q_0}] & , \quad G(q_0) \le z \le G(q_2) \quad [(2.3.14)] \\ \frac{d_1(z)-q_1}{q_0-q_1}, & , \quad G(q_2) \le z \le G(q_1) \quad [(2.3.15)] \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.19}$$

The value $\max[\frac{d_1(z)-q_1}{q_0-q_1}, \frac{q_2-d_2(z)}{q_2-q_0}] =$ MAX (say) in (2.3.18) and (2.3.19) can be altered by the following process:

Since the determinant of (12) is

$$D(z) = (a+b)^2 z^2 - 2ab(a+b)cTR = (a+b)^2(z^2 - G(q_*)^2),$$

therefore $d_1(z) = \frac{a+b}{abT}(z - \sqrt{z^2 - G(q_*)^2})$, $d_2(z) = \frac{a+b}{abT}(z + \sqrt{z^2 - G(q_*)^2})$, if $G(q_*) \le z$.

Let $\quad f(z) = \dfrac{d_1(z) - q_1}{q_0 - q_1} = \dfrac{(a+b)[z - \sqrt{z^2 - G(q_*)^2}] - abTq_1}{abT(q_0 - q_1)}, \quad G(q_*) \le z$

$$\text{then} \quad f(G(q_*)) = \frac{(a+b)G(q_*) - abTq_1}{abT(q_0 - q_1)} \ge 0 \quad (\text{ by } (2.3.1)) \tag{2.3.20}$$

$$f'(z) = \frac{a+b}{abT(q_0 - q_1)}[1 - \frac{z}{\sqrt{z^2 - G(q_*)^2}}] < 0 \quad \text{for } G(q_*) < z$$

$$f''(z) = \frac{a+b}{abT(q_0 - q_1)} \frac{G(q_*)^2}{[z^2 - G(q_*)^2]^{\frac{3}{2}}} > 0 \quad \text{for } G(q_*) < z$$

Hence $f(z)$is decreasing and concave upward when $z \ge G(q_*)$ (2.3.21)

$$\text{Let } g(z) = \frac{q_2 - d_2(z)}{q_2 - q_0} = \frac{abTq_2 - (a+b)[z + \sqrt{z^2 - G(q_*)^2}]}{abT(q_2 - q_0)}$$

$$g(G(q_*)) = \frac{abTq_2 - (a+b)G(q_*)}{abT(q_2 - q_0)} > 0 \quad (\text{ by } (2.3.1)) \tag{2.3.22}$$

$$g'(z) = -\frac{(a+b)}{abT(q_2 - q_0)}[1 + \frac{z}{\sqrt{z^2 - G(q_*)^2}}] < 0 \quad \text{for} \quad z > G(q_*)$$

$$g''(z) = \frac{(a+b)G(q_*)^2}{abT(q_2 - q_0)[z^2 - G(q_*)^2]^{\frac{3}{2}}} > 0 \quad \text{for} \quad z > G(q_*)$$

Hence $g(z)$ is decreasing and concave upward when $z \geq G(q_*)$ (2.3.23)

$$g(G(q_*)) - f(G(q_*)) = \frac{(q_2 - q_1)[abTq_0 - (a+b)G(q_*)]}{abT(q_0 - q_1)(q_2 - q_0)} > 0 \ (\text{ by } (2.3.1)\ ).$$

$$\text{So} \quad 0 < f(G(q_*)) < g(G(q_*)) \tag{2.3.24}$$

Now we shall solve $f(z) = g(z)$ and find its root.

From $\frac{d_1(z) - q_1}{q_0 - q_1} = \frac{q_2 - d_2(z)}{q_2 - q_0}$, we get

$$[(a+b)z - abTq_0](q_2 - q_1) = (a+b)(q_2 - 2q_0 + q_1)\sqrt{z^2 - G(q_*)^2} \tag{2.3.25}$$

Let $A = 4(a+b)^2(q_2 - q_0)(q_0 - q_1)$

$B = ab(a+b)Tq_0(q_2 - q_1)^2$

$C = a^2b^2q_0^2(q_2 - q_1)^2T^2 + 2ab(a+b)cTR(q_2 - 2q_0 + q_1)^2$

Then, after we sqare both sides of (2.3.25), we have

$$Az^2 - 2Bz + C = 0 \tag{2.3.26}$$

The determinant of (2.3.26) is $D^*(q_1, q_0, q_2) = B^2 - AC$. If $D^*(q_1, q_0, q_2) > 0$, then the two roots of (2.3.26) are

$$s_1 = \frac{B - \sqrt{D^*(q_1, q_0, q_2)}}{A}, \quad s_2 = \frac{B + \sqrt{D^*(q_1, q_0, q_2)}}{A}, \tag{2.3.27}$$

If $D^*(q_1, q_0, q_2) = 0$, then we have the double root

$$s_3 = \frac{B}{A} \tag{2.3.28}$$

[A]. In (2.3.18), $G(q_0) \le z \le G(q_1)$. Since $(\frac{a+b}{abT}G(q_*))^2 = \frac{2(a+b)cR}{abT}$ and from (2.3.1) $q_1 \le \frac{a+b}{abT}G(q_*)$, $G(q_1)^2 - G(q_*)^2 = (\frac{cR}{q_1} - \frac{abT}{2(a+b)}q_1)^2$, therefore $\sqrt{G(q_1)^2 - G(q_*)^2} = \frac{cR}{q_1} - \frac{abT}{2(a+b)}q_1$.

$$\begin{aligned} f(G(q_1)) &= \frac{1}{abT(q_0-q_1)}[(a+b)(G(q_1) - \sqrt{G(q_1)^2 - G(q_*)^2}) - abTq_1] \\ &= \frac{1}{abT(q_0-q_1)}(abTq_1 - abTq_1) = 0 \\ g(G(q_1)) &= \frac{1}{abT(q_2-q_0)}[abTq_2 - (a+b)(G(q_1) + \frac{cR}{q_1} - \frac{abT}{2(a+b)}q_1)] \\ &= \frac{1}{abT(q_2-q_0)}[abTq_2 - \frac{2(a+b)cR}{q_1}] \end{aligned}$$

Since $q_2q_1 > (\frac{a+b}{abT}G(q_*))^2 = \frac{2(a+b)cR}{abT}$ holds in (2°) therefore

$$g(G(q_1)) = \frac{1}{abT(q_2-q_0)}[abTq_2 - \frac{2(a+b)cR}{q_1}] > 0 = f(G(q_1)) \tag{2.3.29}$$

Thus, from (2.2.21), (2.3.23), (2.3.24), (2.3.27), (2.3.28) and (2.3.29) we have the following:

Insert Fig. 2 here

Insert Fig. 3 here

Insert Fig. 4 here

Hence now the maximum in (2.3.18) can be altered as:

(A1). If $D^*(q_1, q_0, q_2) \le 0$, from Fig. 2, 3, we have

$$\text{MAX } = \frac{q_2 - d_2(z)}{q_2 - q_0}, \quad G(q_0) \le z \le G(q_1) \tag{2.3.30}$$

(A2). If $D^*(q_1, q_0, q_2) > 0$, and satisfies each of the following condition, then from Fig. 4, we have

(A2.1) $s_1 < s_2 \le G(q_0) < G(q_1)$

$$\text{MAX } = \frac{q_2 - d_2(z)}{q_2 - q_0}, \quad G(q_0) \le z \le G(q_1) \tag{2.3.31}$$

(A2.2) $s_1 \le G(q_0) < s_2 \le G(q_1)$

$$\text{MAX } = \begin{cases} \frac{d_1(z)-q_1}{q_0-q_1}, & G(q_0) \le z \le s_2 \\ \frac{q_2-d_2(z)}{q_2-q_0}, & s_2 \le z \le G(q_1) \end{cases} \tag{2.3.32}$$

(A2.3) $\quad s_1 \le G(q_0) < G(q_1) \le s_2$

$$\text{MAX} = \frac{d_1(z) - q_1}{q_0 - q_1} \quad G(q_0) \le z \le G(q_1) \tag{2.3.33}$$

(A2.4) $\quad G(q_0) \le s_1 < s_2 \le G(q_1)$

$$\text{MAX} = \begin{cases} \frac{q_2 - d_2(z)}{q_2 - q_0}, & G(q_0) \le z \le s_1 \\ \frac{d_1(z) - q_1}{q_0 - q_1}, & s_1 \le z \le s_2 \\ \frac{q_2 - d_2(z)}{q_2 - q_0}, & s_2 \le z \le G(q_1) \end{cases} \tag{2.3.34}$$

(A2.5) $\quad G(q_0) \le s_1 < G(q_1) \le s_2$

$$\text{MAX} = \begin{cases} \frac{q_2 - d_2(z)}{q_2 - q_0}, & G(q_0) \le z \le s_1 \\ \frac{d_1(z) - q_1}{q_0 - q_1}, & s_1 \le z \le G(q_1) \end{cases} \tag{2.3.35}$$

(A2.6) $\quad G(q_0) < G(q_1) \le s_1 < s_2$

$$\text{MAX} = \frac{q_2 - d_2(z)}{q_2 - q_0}, \quad G(q_0) \le z \le G(q_1) \tag{2.3.36}$$

[B] In (2.3.19), $G(q_0) \le z \le G(q_2)$. From (2.3.1), $q_* \le q_2$ and $G(q_2)^2 - G(q_*)^2 = (\frac{abT}{2(a+b)} q_2 - \frac{cR}{q_2})^2$, therefore $\sqrt{G(q_2)^2 - G(q_*)^2} = \frac{abT}{2(a+b)} q_2 - \frac{cR}{q_2}$.

$$\begin{aligned} f(G(q_2)) &= \frac{1}{abT(q_0 - q_1)}[(a+b)(G(q_2) - \frac{abT}{2(a+b)} q_2 + \frac{cR}{q_2}) - abTq_1] \\ &= \frac{1}{abT(q_0 - q_1)}[\frac{2(a+b)cR}{q_2} - abTq_1] \end{aligned}$$

Since $q_1 q_2 < q_*^2 = \frac{2(a+b)cR}{abT}$ holds by the condition (3°), $f(G(q_2)) > 0$ and

$$\begin{aligned} g(G(q_2)) &= \frac{1}{abT(q_2 - q_0)}[abTq_2 - (a+b)(G(q_2) + \frac{abT}{a(a+b)} q_2 - \frac{cR}{q_2})] \\ &= \frac{1}{abT(q_2 - q_0)}[abTq_2 - abTq_2] = 0 \end{aligned}$$

$$\text{Therefore} \quad f(G(q_2)) > 0 = g(G(q_2)) \qquad (\text{ see Fig. 5 }) \tag{2.3.37}$$

Thus, from (2.3.21), (2.3.23), (2.3.24), (2.3.28) and (2.3.37), we have

Insert Fig.5 here

(B1). If $D^*(q_1, q_0, q_2) = 0$ and satisfies each of the following condition, then we have

(B1.1) $\quad s_3 \le G(q_0) < G(q_2)$

$$\text{MAX} = \frac{d_1(z) - q_1}{q_0 - q_1}, \quad G(q_0) \le z \le G(q_2) \tag{2.3.38}$$

(B1.2) $\quad G(q_0) < s_3 \leq G(q_2)$

$$\text{MAX} = \begin{cases} \frac{q_2 - d_2(z)}{q_2 - q_0}, & G(q_0) \leq z \leq s_3 \\ \frac{d_1(z) - q_1}{q_0 - q_1}, & s_3 \leq z \leq G(q_2) \end{cases} \tag{2.3.39}$$

(B1.3) $\quad G(q_0) < G(q_2) \leq s_3$

$$\text{MAX} = \frac{q_2 - d_2(z)}{q_2 - q_0}, \quad G(q_0) \leq z \leq G(q_2) \tag{2.3.40}$$

Let

$$\begin{aligned}
T_{31} &= \{(q_1, q_0, q_2) \,|\, q_0 q_1 > q_*^2\} \\
T_{32} &= \{(q_1, q_0, q_2) \,|\, q_1 q_0 < q_*^2 \text{ and } q_2 q_1 > q_*^2\} \\
T_{33} &= \{(q_1, q_0, q_2) \,|\, q_1 q_2 < q_*^2\} \\
T_{321} &= \{(q_1, q_0, q_2) \,|\, D^*(q_1, q_0, q_2) \leq 0\} \\
T_{322} &= \{(q_1, q_0, q_2) \,|\, D^*(q_1, q_0, q_2) > 0 \text{ and } s_1 < s_2 \leq G(q_0) < G(q_1)\} \\
T_{323} &= \{(q_1, q_0, q_2) \,|\, D^*(q_1, q_0, q_2) > 0 \text{ and } s_1 \leq G(q_0) < s_2 \leq G(q_1)\} \\
T_{324} &= \{(q_1, q_0, q_2) \,|\, D^*(q_1, q_0, q_2) > 0 \text{ and } s_1 \leq G(q_0) < G(q_1) \leq s_2\} \\
T_{325} &= \{(q_1, q_0, q_2) \,|\, D^*(q_1, q_0, q_2) > 0 \text{ and } G(q_0) \leq s_1 < s_2 \leq G(q_1)\} \\
T_{326} &= \{(q_1, q_0, q_2) \,|\, D^*(q_1, q_0, q_2) > 0 \text{ and } G(q_0) \leq s_1 < G(q_1) \leq s_2\} \\
T_{327} &= \{(q_1, q_0, q_2) \,|\, D^*(q_1, q_0, q_2) > 0 \text{ and } G(q_0) < G(q_1) \leq s_1 < s_2\} \\
T_{331} &= \{(q_1, q_0, q_2) \,|\, D^*(q_1, q_0, q_2) = 0 \text{ and } s_3 \leq G(q_0) < G(q_2)\} \\
T_{332} &= \{(q_1, q_0, q_2) \,|\, D^*(q_1, q_0, q_2) = 0 \text{ and } G(q_0) < s_3 \leq G(q_2)\} \\
T_{333} &= \{(q_1, q_0, q_2) \,|\, D^*(q_1, q_0, q_2) = 0 \text{ and } G(q_0) < G(q_2) \leq s_3\}
\end{aligned} \tag{2.3.41}$$

From (2.3.30) - (2.3.36), (2.3.38) - (2.3.40), Property 6 can be altered as

Property 6*. Let $(q_1, q_0, q_2) \in T_3$. Then the membership function $\mu_{\widetilde{G(q)}}(z)$ is given by

(1°) If $(q_1, q_0, q_2) \in T_{31}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z) - q_1}{q_0 - q_1} & , \quad G(q_*) \leq z \leq G(q_0) \\ \frac{q_2 - d_2(z)}{q_2 - q_0} & , \quad G(q_0) \leq z \leq G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.42}$$

(2°) $(q_1, q_0, q_2) \in T_{32}$, then

(2° − 1) If $(q_1, q_0, q_2) \in T_{321}$, or $(q_1, q_0, q_2) \in T_{322}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q_*) \leq z \leq G(q_0) \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad G(q_0) \leq z \leq G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.43}$$

(2° − 2) If $(q_1, q_0, q_2) \in T_{323}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q_*) \leq z \leq G(q_0) \\ \frac{d_1(z)-q_1}{q_0-q_1} & , \quad G(q_0) \leq z \leq s_2 \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad s_2 \leq z \leq G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.44}$$

(2° − 3) If $(q_1, q_0, q_2) \in T_{324}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q_*) \leq z \leq G(q_0) \\ \frac{d_1(z)-q_1}{q_0-q_1} & , \quad G(q_0) \leq z \leq G(q_1) \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad G(q_1) \leq z \leq G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.45}$$

(2° − 4) If $(q_1, q_0, q_2) \in T_{325}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q_*) \leq z \leq G(q_0) \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad G(q_0) \leq z \leq s_1 \\ \frac{d_1(z)-q_1}{q_0-q_1} & , \quad s_1 \leq z \leq s_2 \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad s_2 \leq z \leq G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.46}$$

$(2^\circ - 5)$ If $(q_1, q_0, q_2) \in T_{326}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad G(q_0) \le z \le s_1 \\ \frac{d_1(z)-q_1}{q_0-q_1} & , \quad s_1 \le z \le G(q_1) \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad G(q_1) \le z \le G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.47}$$

$(2^\circ - 6)$ If $(q_1, q_0, q_2) \in T_{327}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad G(q_0) \le z \le G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.48}$$

(3°) If $(q_1, q_0, q_2) \in T_{33}$, then

$(3^\circ - 1)$ If $(q_1, q_0, q_2) \in T_{331}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{d_1(z)-q_1}{q_0-q_1} & , \quad G(q_0) \le z \le G(q_1) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.49}$$

$(3^\circ - 2)$ If $(q_1, q_0, q_2) \in T_{332}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad G(q_0) \le z \le s_3 \\ \frac{d_1(z)-q_1}{q_0-q_1} & , \quad s_3 \le z \le G(q_1) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.50}$$

$(3^\circ - 3)$ If $(q_1, q_0, q_2) \in T_{333}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{d_2(z)-q_1}{q_0-q_1} & , \quad G(q^*) \le z \le G(q_0) \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad G(q_0) \le z \le G(q_2) \\ \frac{d_1(z)-q_1}{q_0-q_1} & , \quad G(q_2) \le z \le G(q_1) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.3.51}$$

Now we proceed to find out the centroid of the membership function $\mu_{\widetilde{G(q)}}(z)$ in Property 6*. Let $I(A) = 1$, if $(q_1, q_0, q_2) \in A$; and $I(A) = 0$, if $(q_1, q_0, q_2) \notin A$. From (26) - (33), we have

Property 7. If $(q_1, q_0, q_2) \in T_3$, then the centroid of $\mu_{\widetilde{G(q)}}(z)$ in Property 6* is given by

(1°) If $(q_0, q_1, q_2) \in T_{31}$, then $E_{31}(q_1, q_0, q_2) = \frac{R_{31}}{P_{31}} I(T_{31})$ (2.3.52)

where $P_{31} = V_2(G(q_*), G(q_0)) + V_4(G(q_0), G(q_2))$

$R_{31} = V_{22}(G(q_*), G(q_0)) + V_{42}(G(q_0), G(q_2))$

(2°) If $(q_1, q_0, q_2) \in T_{32}$, then

(2° − 1) $E_{321}(q_1, q_0, q_2) = \frac{R_{31}}{P_{31}} I(T_{321})$ (2.3.53)

$E_{322}(q_1, q_0, q_2) = \frac{R_{31}}{P_{31}} I(T_{322})$ (2.3.54)

(2° − 2) $E_{323}(q_1, q_0, q_2) = \frac{R_{323}}{P_{323}} I(T_{323})$ (2.3.55)

where $P_{323} = V_2(G(q_*), G(q_0)) + V_1(G(q_0), s_2) + V_4(s_2, G(q_2))$

$R_{323} = V_{22}(G(q_*), G(q_0)) + V_{12}(G(q_0), s_2) + V_{42}(s_2, G(q_2))$

(2° − 3) $E_{324}(q_1, q_0, q_2) = \frac{R_{324}}{P_{324}} I(T_{324})$ (2.3.56)

where $P_{324} = V_2(G(q_*), G(q_0)) + V_1(G(q_0), G(q_1)) + V_4(G(q_1), G(q_2))$

$R_{324} = V_{22}(G(q_*), G(q_0)) + V_{12}(G(q_0), G(q_1)) + V_{42}(G(q_1), G(q_2))$

(2° − 4) $E_{325}(q_1, q_0, q_2) = \frac{R_{325}}{P_{325}} I(T_{325})$ (2.3.57)

where $P_{325} = V_2(G(q_*), G(q_0)) + V_4(G(q_0), s_1) + V_1(s_1, s_2) + V_4(s_2, G(q_2))$

$R_{325} = V_{22}(G(q_*), G(q_0)) + V_{42}(G(q_0), s_1) + V_{12}(s_1, s_2) + V_{42}(s_2, G(q_2))$

(2° − 5) $E_{326}(q_1, q_0, q_2) = \frac{R_{326}}{P_{326}} I(T_{326})$ (2.3.58)

where $P_{326} = V_2(G(q_*), G(q_0)) + V_4(G(q_0), s_1) + V_1(s_1, G(q_1)) + V_4(G(q_1), G(q_2))$

$$R_{326} = V_{22}(G(q_*), G(q_0)) + V_{42}(G(q_0), s_1) + V_{12}(s_1, G(q_1)) + V_{42}(G(q_1), G(q_2))$$

$$(2^\circ - 6) \quad E_{327}(q_1, q_0, q_2) = \frac{R_{327}}{P_{327}} I(T_{327}) \tag{2.3.59}$$

where $P_{327} = V_2(G(q_*), G(q_0)) + V_4(G(q_0), G(q_2))$

$$R_{327} = V_{22}(G(q_*), G(q_0)) + V_{42}(G(q_0), G(q_2))$$

(3°) If $(q_1, q_0, q_2) \in T_{33}$, then

$$(3^\circ - 1) \quad E_{331}(q_1, q_0, q_2) = \frac{R_{331}}{P_{331}} I(T_{331}) \tag{2.3.60}$$

where $P_{331} = V_2(G(q_*), G(q_0)) + V_1(G(q_0), G(q_1))$

$$R_{331} = V_{22}(G(q_*), G(q_0)) + V_{12}(G(q_0), G(q_1))$$

$$(3^\circ - 2) \quad E_{332}(q_1, q_0, q_2) = \frac{R_{332}}{P_{332}} I(T_{332}) \tag{2.3.61}$$

where $P_{332} = V_2(G(q_*), G(q_0)) + V_4(G(q_0), s_3) + V_1(s_3, G(q_1))$

$$R_{332} = V_{22}(G(q_*), G(q_0)) + V_{42}(G(q_0), s_3) + V_{12}(s_3, G(q_1))$$

$$(3^\circ - 3) \quad E_{333}(q_1, q_0, q_2) = \frac{R_{333}}{P_{333}} I(T_{333}) \tag{2.3.62}$$

where $P_{333} = V_2(G(q_*), G(q_0)) + V_4(G(q_0), G(q_2)) + V_1(G(q_2), G(q_1))$

$$R_{333} = V_{22}(G(q_*), G(q_0)) + V_{42}(G(q_0), G(q_2)) + V_{12}(G(q_2), G(q_1))$$

§2.4. Find $\mu_{\underset{\sim}{G(q)}}(z)$ and its centroid under the condition $0 < q_1 < q_0 < q_* < q_2$

$$0 < q_1 < q_0 < q_* < q_2 \tag{2.4.1}$$

From (15), (17) and (2.4.1), we have

$$G(q_*) < G(q_0) < G(q_1) \tag{2.4.2}$$

Under the condition (2.4.2), by (15) - (17), all the permutations of $G(q_1), G(q_0), G(q_2)$ are

$$G(q_0) < G(q_1) < G(q_2), \quad \text{if} \quad q_2q_1 > q_*^2 \tag{2.4.3}$$

$$G(q_0) < G(q_2) < G(q_1), \quad \text{if} \quad q_1q_2 < q_*^2 \text{ and } \quad q_2q_0 > q_*^2 \tag{2.4.4}$$

$$G(q_2) < G(q_0) < G(q_1), \quad \text{if} \quad q_0q_2 < q_*^2 \tag{2.4.5}$$

Then by Table 1, (18) - (25) and (2.4.1) - (2.4.5), we have

(i) For case $1, 4, 10,$ $\mu_{\tilde{G(q)}}(z) = 0$ (2.4.6)

(ii) For case $2, 5,$ $d_2(z) \leq q_0$ and $q_0 \leq q_*$ By (24), there is no solution

(2.4.7)

(iii) For case 3, $d_1(z) \leq q_1,$ $q_0 \leq d_2(z) \leq q_2$. From (20), (22), (25), each we have

$G(q_1) \leq z, \quad G(q_*) \leq z, \quad G(q_*) \leq z \leq G(q_2).$ By (2.4.3)

$$\text{when } q_2q_1 > q_*^2, \quad \mu_{\tilde{G(q)}}(z) = \frac{q_2 - d_2(z)}{q_2 - q_0}, \quad G(q_1) \leq z \leq G(q_2) \tag{2.4.8}$$

(iv) For case 6, $q_1 \leq d_1(z) \leq q_0 \leq d_2(z) \leq q_2$. From (18), (20), (22), (25), each we have

$G(q_*) \leq z \leq G(q_1), \quad G(q_0) \leq z, \quad G(q_*) \leq z, \quad G(q_*) \leq z \leq G(q_2).$

By (2.4.3), (2.4.4), we have

when $q_2q_1 > q_*^2,$

$$\mu_{\tilde{G(q)}}(z) = \max\left[\frac{d_1(z) - q_1}{q_0 - q_1}, \frac{q_2 - d_2(z)}{q_2 - q_0}\right], \quad G(q_0) \leq z \leq G(q_1) \tag{2.4.9}$$

when $q_1q_2 < q_*^2,$ and $q_2q_0 > q_*^2,$

$$\mu_{\tilde{G(q)}}(z) = \max\left[\frac{d_1(z) - q_1}{q_0 - q_1}, \frac{q_2 - d_2(z)}{q_2 - q_0}\right], \quad G(q_0) \leq z \leq G(q_2) \tag{2.4.10}$$

(v) For case 7, $q_1 \leq d_1(z) \leq q_0, q_2 \leq d_2(z)$. From (18), (20), (23), each we have

$G(q_*) \leq z \leq G(q_1), G(q_0) \leq z, G(q_2) \leq z.$ By (2.4.4) and (2.4.5),

when $q_1q_2 < q_*^2$ and $q_2q_0 > q_*^2,$

$$\mu_{\tilde{G(q)}}(z) = \frac{d_1(z) - q_1}{q_0 - q_1}, \quad G(q_2) \leq z \leq G(q_1) \tag{2.4.11}$$

$$\text{when } q_0q_2 < q_*^2, \quad \mu_{\underset{\sim}{G(q)}}(z) = \frac{d_1(z)-q_1}{q_0-q_1}, \quad G(q_0) \le z \le G(q_1) \tag{2.4.12}$$

(vi) For case 8, $q_0 \le d_1(z), d_2(z) \le q_2$. From $(18), (25)$, each we have

$G(q_*) \le z \le G(q_0), G(q_*) \le z \le G(q_2)$. By (2.4.3) – (2.4.5), we have

$$\text{when } q_2q_1 > q_*^2, \mu_{\underset{\sim}{G(q)}}(z) = \frac{q_2-d_1(z)}{q_2-q_0}, \quad G(q_*) \le z \le G(q_0) \tag{2.4.13}$$

when $q_1q_2 < q_*^2$ and $q_2q_0 > q_*^2$,

$$\mu_{\underset{\sim}{G(q)}} = \frac{q_2-d_1(z)}{q_2-q_0}, \quad G(q_*) \le z \le G(q_0) \tag{2.4.14}$$

$$\text{when } q_0q_2 < q_*^2, \quad \mu_{\underset{\sim}{G(q)}}(z) = \frac{q_2-d_1(z)}{q_2-q_0}, \quad G(q_*) \le z \le G(q_2) \tag{2.4.15}$$

(vii) For case 9, $q_0 \le d_1(z) \le q_2 \le d_2(z)$. From $(18), (21), (23)$, each we have

$G(q_*) \le z \le G(q_0)$, $G(q_*) \le z$, $G(q_2) \le z$ By (2.4.5), we have

$$\text{when } q_0q_2 < q_*^2, \mu_{\underset{\sim}{G(q)}}(z) = \frac{q_2-d_1(z)}{q_2-q_0}, \ G(q_2) \le z \le G(q_0) \tag{2.4.16}$$

Let $T_4 = \{\, (q_1, q_0, q_2) \,|\, 0 < q_1 < q_0 < q_*^2 < q_2 \,\}$. Then by (2.4.6) - (2.4.16), we have

Property 8. Let $(q_1, q_0, q_2) \in T_4$. Then the membership function $\mu_{\underset{\sim}{G(q)}}(z)$ is given by

(1°) If $q_2q_1 > q_*^2$, then

$$\mu_{\underset{\sim}{G(q)}}(z) = \begin{cases} \frac{q_2-d_1(z)}{q_2-q_0} & , \quad G(q_*) \le z \le G(q_0) \quad [(2.4.13)] \\ \max[\frac{d_1(z)-q_1}{q_0-q_1}, \frac{q_2-d_2(z)}{q_2-q_0}] & , \quad G(q_0) \le z \le G(q_1) \quad [(2.4.9)] \\ \frac{q_2-d_2(z)}{q_2-q_0} & , \quad G(q_1) \le z \le G(q_2) \quad [(2.4.8)] \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.17}$$

(2°) If $q_1q_2 < q_*^2$, and $q_2q_0 > q_*^2$, then

$$\mu_{\underset{\sim}{G(q)}}(z) = \begin{cases} \frac{q_2-d_1(z)}{q_2-q_0} & , \quad G(q_*) \le z \le G(q_0) \quad [(2.4.14)] \\ \max[\frac{d_1(z)-q_1}{q_0-q_1}, \frac{q_2-d_2(z)}{q_2-q_0}] & , \quad G(q_0) \le z \le G(q_2) \quad [(2.4.10)] \\ \frac{d_1(z)-q_1}{q_0-q_1} & , \quad G(q_2) \le z \le G(q_1) \quad [(2.4.11)] \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.18}$$

(3°) If $q_0 q_2 < q_*^2$, then

$$\mu_{\underset{G(q)}{\sim}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_*) \le z \le G(q_0) \quad [(2.4.15),\ (2.4.16)] \\ \frac{d_1(z) - q_1}{q_0 - q_1} & , \quad G(q_0) \le z \le G(q_1) \quad [(2.4.12)] \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.19}$$

The value $\max[\frac{d_1(z)-q_1}{q_0-q_1}, \frac{q_2-d_2(z)}{q_2-q_0}]$ can be altered by the following process.

[C]. The region $G(q_0) \le z \le G(q_1)$ of the max in (2.4.17) coincides with the region $G(q_0) \le z \le G(q_1)$ of the max in (2.3.18) in section 2.3 [A]. Therefore we can use (2.3.30) - (2.3.36). However, the first item $\frac{d_2(z)-q_1}{q_0-q_1}$ in Property 6 (2.3.18) is different from the first item $\frac{q_2-d_1(z)}{q_2-q_0}$ in Property 8 (2.4.17), but have the same region $G(q_*) \le z \le G(q_0)$. The rest two terms are identical the same. Hence by (2.3.43) - (2.3.48) in Property 6*, we shall have (2.4.20) - (2.4.25) of Property 8* in the following.

[D]. The region $G(q_0) \le z \le G(q_2)$ of the max in (2.4.18) coincides with the region $G(q_0) \le z \le G(q_2)$ of the max in (2.3.19) in section 2.3 [B]. Therefore we can use (2.3.38) - (2.3.40). Similar to [C], by (2.3.49) - (2.3.51) in Property 6', we shall have(2.4.26) - (2.4.28) of Property 8' in the following.

$$\begin{aligned} \text{Let } T_{41} &= \{\, (q_1, q_0, q_2) \mid q_2 q_1 > q_*^2 \} \\ T_{42} &= \{\, (q_1, q_0, q_2) \mid q_1 q_2 < q_*^2 \quad \text{and} \quad q_2 q_0 > q_*^2 \} \\ T_{43} &= \{\, (q_1, q_0, q_2) \mid q_0 q_2 < q_*^2 \} \end{aligned}$$

Let $T_{41j} = T_{32j}, j = 1, \cdots, 7$; $T_{42j} = T_{33j}, j = 1, 2, 3$.

Then by (2.3.30) - (2.3.36), (2.3.38) - (2.3.40) and Property 6*, (2.3.43) - (2.3.48), (2.3.49) - (2.3.51), we have

Property 8*. Let $(q_1, q_0, q_2) \in T_4$. Then the membership function $\mu_{\underset{G(q)}{\sim}}(z)$ is given by

(1°) If $(q_1, q_0, q_2) \in T_{41}$, then

$(1^\circ - 1)$ If $(q_1, q_0, q_2) \in T_{411}$ or $(q_1, q_0, q_2) \in T_{412}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{q_2 - d_2(z)}{q_2 - q_0} & , \quad G(q_0) \le z \le G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.20}$$

$(1^\circ - 2)$ If $(q_1, q_0, q_2) \in T_{413}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{d_1(z) - q_1}{q_0 - q_1} & , \quad G(q_0) \le z \le s_2 \\ \frac{q_2 - d_2(z)}{q_2 - q_0} & , \quad s_2 \le z \le G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.21}$$

$(1^\circ - 3)$ If $(q_1, q_0, q_2) \in T_{414}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{d_1(z) - q_1}{q_0 - q_1} & , \quad G(q_0) \le z \le G(q_1) \\ \frac{q_2 - d_2(z)}{q_2 - q_0} & , \quad G(q_1) \le z \le G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.22}$$

$(1^\circ - 4)$ If $(q_1, q_0, q_2) \in T_{415}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{q_2 - d_2(z)}{q_2 - q_0} & , \quad G(q_0) \le z \le s_1 \\ \frac{d_1(z) - q_1}{q_0 - q_1} & , \quad s_1 \le z \le s_2 \\ \frac{q_2 - d_2(z)}{q_2 - q_0} & , \quad s_2 \le z \le G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.23}$$

$(1^\circ - 5)$ If $(q_1, q_0, q_2) \in T_{416}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{q_2 - d_2(z)}{q_2 - q_0} & , \quad G(q_0) \le z \le s_1 \\ \frac{d_1(z) - q_1}{q_0 - q_1} & , \quad s_1 \le z \le G(q_1) \\ \frac{q_2 - d_2(z)}{q_2 - q_0} & , \quad G(q_1) \le z \le G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.24}$$

$(1^\circ - 6)$ If $(q_1, q_0, q_2) \in T_{417}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{q_2 - d_2(z)}{q_2 - q_0} & , \quad G(q_0) \le z \le G(q_2) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.25}$$

(2°) If $(q_1, q_0, q_2) \in T_{42}$, then

$(2^\circ - 1)$ If $(q_1, q_0, q_2) \in T_{421}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{d_1(z) - q_1}{q_0 - q_1} & , \quad G(q_0) \le z \le G(q_1) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.26}$$

$(2^\circ - 2)$ If $(q_1, q_0, q_2) \in T_{422}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{q_2 - d_2(z)}{q_2 - q_0} & , \quad G(q_0) \le z \le s_3 \\ \frac{d_1(z) - q_1}{q_0 - q_1} & , \quad s_3 \le z \le G(q_1) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.27}$$

$(2^\circ - 3)$ If $(q_1, q_0, q_2) \in T_{423}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{q_2 - d_2(z)}{q_2 - q_0} & , \quad G(q_0) \le z \le G(q_2) \\ \frac{d_1(z) - q_1}{q_0 - q_1} & , \quad G(q_2) \le z \le G(q_1) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.28}$$

(3°) If $(q_1, q_0, q_2) \in T_{43}$, then

$$\mu_{\widetilde{G(q)}}(z) = \begin{cases} \frac{q_2 - d_1(z)}{q_2 - q_0} & , \quad G(q_*) \le z \le G(q_0) \\ \frac{d_1(z) - q_1}{q_0 - q_1} & , \quad G(q_0) \le z \le G(q_1) \\ 0 & , \quad \text{otherwise} \end{cases} \tag{2.4.29}$$

Now we proceed to find the centroid of the membership function $\mu_{\underset{\sim}{Gq)}}(z)$ of Property 8^*. From (26) - (33), we have

Property 9. If $(q_1, q_0, q_2) \in T_4$, then the centroid of $\mu_{\underset{\sim}{Gq)}}(z)$ in Property 8^* is given by

(1°) If $(q_1, q_0, q_2) \in T_{41}$

$$(1° - 1) \quad E_{411}(q_1, q_0, q_2) = \frac{R_{411}}{P_{411}} I(T_{411}) \tag{2.4.30}$$

where $P_{411} = V_3(G(q_*), G(q_0)) + V_4(G(q_0), G(q_2))$

$R_{411} = V_{32}(G(q_*), G(q_0)) + V_{42}(G(q_0), G(q_2))$

and $$E_{412}(q_1, q_0, q_2) = \frac{R_{411}}{P_{411}} I(T_{412}) \tag{2.4.31}$$

$$(1° - 2) \quad E_{413}(q_1, q_0, q_2) = \frac{R_{413}}{P_{413}} I(T_{413}) \tag{2.4.32}$$

where $P_{413} = V_3(G(q_*), G(q_0)) + V_1(G(q_0), s_2) + V_4(s_2, G(q_2))$

$R_{413} = V_{32}(G(q_*), G(q_0)) + V_{12}(G(q_0), s_2) + V_{42}(s_2, G(q_2))$

$$(1° - 3) \quad E_{414}(q_1, q_0, q_2) = \frac{R_{414}}{P_{414}} I(T_{414}) \tag{2.4.33}$$

where $P_{414} = V_3(G(q_*), G(q_0)) + V_1(G(q_0), G(q_1)) + V_4(G(q_1), G(q_2))$

$R_{414} = V_{32}(G(q_*), G(q_0)) + V_{12}(G(q_0), G(q_1)) + V_{42}(G(q_1), G(q_2))$

$$(1° - 4) \quad E_{415}(q_1, q_0, q_2) = \frac{R_{415}}{P_{415}} I(T_{415}) \tag{2.4.34}$$

where $P_{415} = V_3(G(q_*), G(q_0)) + V_4(G(q_0), s_1)) + V_1(s_1, s_2) + V_4(s_2, G(q_2))$

$R_{415} = V_{32}(G(q_*), G(q_0)) + V_{42}(G(q_0), s_1) + V_{12}(s_1, s_2) + V_{42}(s_2, G(q_2))$

$$(1° - 5) \quad E_{416}(q_1, q_0, q_2) = \frac{R_{416}}{P_{416}} I(T_{416}) \tag{2.4.35}$$

where $P_{416} = V_3(G(q_*), G(q_0)) + V_4(G(q_0), s_1)) + V_1(s_1, G(q_1)) + V_4(G(q_1), G(q_2))$

$R_{416} = V_{32}(G(q_*), G(q_0)) + V_{42}(G(q_0), s_1) + V_{12}(s_1, G(q_1)) + V_{42}(G(q_1), G(q_2))$

$$(1° - 6) \quad E_{417}(q_1, q_0, q_2) = \frac{R_{417}}{P_{417}} I(T_{417}) \tag{2.4.36}$$

where $\quad P_{417} = V_3(G(q_*), G(q_0)) + V_4(G(q_0), G(q_2))$

$$R_{417} = V_{32}(G(q_*), G(q_0)) + V_{42}(G(q_0), G(q_2))$$

(2°) If $(q_1, q_0, q_2) \in T_{42}$, then

$$(2^\circ - 1) \quad E_{421}(q_1, q_0, q_2) = \frac{R_{421}}{P_{421}} I(T_{421}) \tag{2.4.37}$$

where $\quad P_{421} = V_3(G(q_*), G(q_0)) + V_1(G(q_0), G(q_1))$

$$R_{421} = V_{32}(G(q_*), G(q_0)) + V_{12}(G(q_0), G(q_1))$$

$$(2^\circ - 2) \quad E_{422}(q_1, q_0, q_2) = \frac{R_{422}}{P_{422}} I(T_{422}) \tag{2.4.38}$$

where $\quad P_{422} = V_3(G(q_*), G(q_0)) + V_4(G(q_0), s_3) + V_1(s_3, G(q_1))$

$$R_{422} = V_{32}(G(q_*), G(q_0)) + V_{42}(G(q_0), s_3) + V_{12}(s_3, G(q_1))$$

$$(2^\circ - 3) \quad E_{423}(q_1, q_0, q_2) = \frac{R_{423}}{P_{423}} I(T_{423}) \tag{2.4.39}$$

where $\quad P_{423} = V_3(G(q_*), G(q_0)) + V_4(G(q_0), G(q_2)) + V_1(G(q_2), G(q_1))$

$$R_{423} = V_{32}(G(q_*), G(q_0)) + V_{42}(G(q_0), G(q_2)) + V_{12}(G(q_2), G(q_1))$$

(3°) If $(q_1, q_0, q_2) \in T_{43}$, then

$$E_{43}(q_1, q_0, q_2) = \frac{R_{43}}{P_{43}} I(T_{43}) \tag{2.4.40}$$

where $\quad P_{43} = V_3(G(q_*), G(q_0)) + V_1(G(q_0), G(q_1))$

$$R_{423} = V_{32}(G(q_*), G(q_0)) + V_{12}(G(q_0), G(q_1))$$

3. Solution in the fuzzy sense and example.

From Properties 3, 5, 7, 9, we have the centroid of $\mu_{\tilde{G(q)}}(z)$ as follows:

Theorem 1. If $0 < q_1 < q_0 < q_2$, then the centroid of $\mu_{\underset{\sim}{G(q)}}(z)$ is given by

$$\begin{aligned} E(q_1, q_0, q_2) = &E_1(q_1, q_0, q_2)I(T_1) + E_2(q_1, q_0, q_2)I(T_2) + E_{31}(q_1, q_0, q_2)I(T_3)I(T_{31}) \\ &+ \sum_{j=1}^{7} E_{32j}(q_1, q_0, q_2)I(T_3)I(T_{32})I(T_{32j}) \\ &+ \sum_{j=1}^{3} E_{33j}(q_1, q_0, q_2)I(T_3)I(T_{33})I(T_{33j}) \\ &+ \sum_{j=1}^{7} E_{41j}(q_1, q_0, q_2)I(T_4)I(T_{41})I(T_{41j}) \\ &+ \sum_{j=1}^{3} E_{42j}(q_1, q_0, q_2)I(T_4)I(T_{42})I(T_{42j}) + E_{43}(q_1, q_0, q_2)I(T_4)I(T_{43}) \end{aligned}$$

Here $E(q_1, q_0, q_2)$ denotes the estimated value of the total cost in the fuzzy sense when (q_1, q_0, q_2) is given, and the order quantity can be found from (10) and the shortage quantity can be found from (11).

Example 1. In (1), let $a = 10, b = 20, c = 200, R = 2000, T = 12$. Then we can have the crisp optimal solution: the optimal order quantity $q_* = 100$, the optimal shortage quantity $s_* = 33.3333$ and the minimal total cost $F(q_*, s_*) = 8000$. From (10), the centroid of the fuzzy number $\widetilde{q} = (q_1, q_0, q_2)$ is $C(\widetilde{q})$. From (11), we have $C(\widetilde{s})$, total cost $G(C(\widetilde{q}))$ and also from (5) and Theorem 1. we have the ratio

$$r(\widetilde{q}) = \frac{E(q_1, q_0, q_2) - G(C(\widetilde{q}))}{G(C(\widetilde{q}))} \times 100$$

We shall calculate this $r(\widetilde{q})$ for the four cases:

$100 \leq q_1 < q_0 < q_2$; $0 < q_1 \leq 100 < q_0 < q_2$;

$0 < q_1 < q_0 \leq 100 < q_2$; $0 < q_1 < q_0 < q_2 < 100$

Table 2. For the case of the fuzzy number $\widetilde{q} = (q_1, q_0, q_2), 100 < q_1 < q_0 < q_2$

q_1	q_0	q_2	$C(\widetilde{q})$	$S(\widetilde{q})$	$E(q_1, q_0, q_2)$	$G(C(\widetilde{q}))$	$r(\widetilde{q})(\%)$
102	104	106	104.00	34.67	8006.86	8006.15	0.01
102	104	110	105.33	35.11	8013.86	8010.80	0.04
102	108	112	107.33	35.78	8024.09	8020.04	0.05
102	108	116	108.67	36.22	8035.40	8027.65	0.10
104	108	112	108.00	36.00	8026.24	8023.70	0.03
104	108	116	109.33	36.44	8037.66	8031.87	0.07
104	112	114	110.00	36.67	8040.50	8036.36	0.05
104	112	120	112.00	37.33	8060.54	8051.43	0.11
106	110	112	109.33	36.44	8033.29	8031.87	0.02
106	110	116	110.67	36.89	8044.88	8041.12	0.05
106	112	120	112.67	37.56	8063.89	8056.96	0.09
106	114	116	112.00	37.33	8055.37	8051.43	0.05
108	112	116	112.00	37.33	8053.71	8051.43	0.03
108	112	118	112.67	37.56	8060.52	8056.96	0.04
108	116	120	114.67	38.22	8079.97	8075.04	0.06
108	116	124	116.00	38.67	8096.48	8088.28	0.10
110	114	122	115.33	38.44	8086.43	8051.54	0.06
110	116	118	114.67	38.22	8077.33	8075.04	0.03
110	118	122	116.67	38.89	8099.93	8095.24	0.06
110	118	126	118.00	39.33	8117.63	8109.83	0.10

Table 3. For the case of the fuzzy number $\widetilde{q} = (q_1, q_0, q_2)$, $0 < q_1 \leq 100 < q_0 < q_2$

q_1	q_0	q_2	$C(\widetilde{q})$	$S(\widetilde{q})$	$E(q_1, q_0, q_2)$	$G(C(\widetilde{q}))$	$r(\widetilde{q})(\%)$
98	102	103	101.00	33.67	8001.40	8000.40	0.01
98	102	108	102.67	34.22	8007.44	8002.77	0.06
98	106	114	106.00	35.33	8024.19	8013.58	0.13
98	108	118	108.00	36.00	8039.45	8023.70	0.20
94	104	112	103.33	34.44	8016.61	8004.30	0.15
94	106	107	102.33	34.11	8008.67	8002.13	0.08
94	106	109	103.00	34.33	8011.95	8003.50	0.11
94	106	116	105.33	35.11	8029.42	8010.80	0.23
90	104	112	102.00	34.00	8016.47	8001.57	0.19
90	106	114	103.33	34.44	8023.24	8004.30	0.24
90	108	116	104.67	34.89	8031.43	8008.32	0.29
90	110	117	105.67	35.22	8037.71	8012.16	0.32
86	110	117	104.33	34.78	8037.25	8007.20	0.38
86	116	119	107.00	35.67	8057.43	8018.32	0.49
86	116	120	107.33	35.78	8060.88	8020.04	0.51
86	116	126	109.33	36.44	8084.74	8031.87	0.66
84	110	120	104.67	34.89	8047.30	8008.32	0.49
84	116	120	106.67	35.56	8060.51	8016.67	0.55
84	116	122	107.33	35.78	8067.85	8020.04	0.60
84	118	128	110.00	36.67	8098.91	8036.36	0.78

Table 4. For the case of the fuzzy number $\widetilde{q} = (q_1, q_0, q_2)$, $0 < q_1 < q_0 < 100 \leq q_2$

q_1	q_0	q_2	$C(\widetilde{q})$	$S(\widetilde{q})$	$E(q_1, q_0, q_2)$	$G(C(\widetilde{q}))$	$r(\widetilde{q})(\%)$
95	97	103	98.33	32.78	8003.95	8001.13	0.04
95	97	110	100.67	33.56	8010.09	8000.18	0.12
95	97	120	104.00	34.67	8041.37	8006.15	0.44
95	97	130	107.33	35.78	8086.36	8020.04	0.83
90	92	102	94.67	31.56	8020.74	8012.02	0.11
90	92	104	95.33	31.78	8020.23	8009.14	0.14
90	92	106	96.00	32.00	8019.90	8006.67	0.17
90	92	108	96.67	32.22	8019.65	8004.60	0.19
84	86	102	90.67	30.22	8061.81	8034.43	0.29
84	86	104	91.33	30.44	8060.63	8032.90	0.35
84	88	106	92.67	30.87	8052.51	8023.31	0.37
84	88	112	94.67	31.56	8051.08	8012.02	0.49
76	80	102	86.00	28.67	8146.55	8091.16	0.68
76	82	104	87.33	29.11	8132.29	8073.49	0.73
76	84	108	89.33	29.78	8119.36	8050.95	0.85
76	80	116	90.67	30.22	8137.46	8038.42	1.23
70	80	110	86.67	28.89	8200.00	8082.05	1.46
70	80	112	87.33	29.11	8198.78	8073.49	1.55
70	80	114	88.00	29.33	8197.72	8065.45	1.64
70	80	116	88.67	29.56	8196.78	8057.94	1.72

Table 5. For the case of the fuzzy number $\widetilde{q}=(q_1,q_0,q_2)$, $0<q_1<q_0<q_2<100$

q_1	q_0	q_2	$C(\widetilde{q})$	$S(\widetilde{q})$	$E(q_1,q_0,q_2)$	$G(C(\widetilde{q}))$	$r(\widetilde{q})(\%)$
95	97	99	97.00	32.33	8004.59	8003.71	0.01
92	94	98	94.67	31.56	8014.16	8012.02	0.03
89	93	95	92.33	30.78	8027.90	8025.46	0.03
87	93	95	91.67	30.56	8035.01	8030.30	0.06
86	90	96	90.67	30.22	8045.11	8038.43	0.08
85	89	91	88.33	29.44	8064.40	8061.64	0.03
84	90	96	90.00	30.00	8054.39	8044.44	0.12
83	91	97	90.33	30.11	8055.18	8041.38	0.17
83	87	93	87.67	29.22	8076.82	9069.40	0.09
83	87	91	87.00	29.00	8082.57	8077.70	0.06
82	86	94	87.33	29.11	8084.36	8073.49	0.13
82	84	92	86.00	28.67	8099.60	8091.16	0.10
80	88	96	88.00	29.33	8084.46	8065.45	0.24
80	86	94	86.67	28.89	8097.06	8082.05	0.19
78	88	96	87.33	29.11	8098.73	8073.49	0.31
78	88	92	86.00	28.67	8108.49	8091.16	0.21
77	85	93	85.00	28.33	8126.96	8105.88	0.26
77	83	91	83.67	27.89	8144.25	8127.54	0.21
76	86	94	85.33	28.44	8127.85	8100.83	0.33
74	84	92	83.33	27.78	8162.32	8133.33	0.36

4. Conclusion.

[A] For $\widetilde{q}=(q_1,q_0,q_2)$, compare $E(q_1,q_0,q_2)$ with $G(C(\widetilde{q}))$

From (14), $G(q_j)=\frac{abT}{2(a+b)}q_j+\frac{cR}{q_j}$, $\quad j=0,1,2$

From (10), $C(\widetilde{q})=\frac{1}{3}(q_1+q_0+q_2)$

From (5), $G(C(\widetilde{q}))=\frac{abT}{2(a+b)}C(\widetilde{q})+\frac{cR}{C(\widetilde{q})}$

Let $q_2-q_0=\Delta_{20}\,(>0)$, $q_0-q_1=\Delta_{01}\,(>0)$, we have

$C(\widetilde{q})=q_1+\frac{1}{3}(2\Delta_{01}+\Delta_{20})$, then

$$G(C(\widetilde{q}))-G(q_1)=[\frac{abT}{2(a+b)}-\frac{cR}{q_1[q_1+\frac{1}{3}(2\Delta_{01}+\Delta_{20})]}]\frac{1}{3}(2\Delta_{01}+\Delta_{20}) \qquad (*_1)$$

$C(\widetilde{q}) = q_0 + \frac{1}{3}(\Delta_{20} - \Delta_{01})$, then

$$G(C(\widetilde{q})) - G(q_0) = [\frac{abT}{2(a+b)} - \frac{cR}{q_0[q_0 + \frac{1}{3}(\Delta_{20} - \Delta_{01})]}]\frac{1}{3}(\Delta_{20} - \Delta_{01}) \qquad (*_2)$$

$C(\widetilde{q}) = q_2 - \frac{1}{3}(2\Delta_{20} + \Delta_{01})$, then

$$G(C(\widetilde{q})) - G(q_2) = [\frac{abT}{2(a+b)} + \frac{cR}{q_2[q_2 - \frac{1}{3}(\Delta_{20} + \Delta_{01})]}]\frac{1}{3}(2\Delta_{20} + \Delta_{01}) \qquad (*_3)$$

When Δ_{20}, Δ_{01} are small , $G(q_0), G(q_1), G(q_2)$ are close to $G(C(\widetilde{q}))$;

$(*_4)$

and when Δ_{20}, Δ_{01} getting larger , $G(q_j), j = 0, 1, 2$ are away from $G(C(\widetilde{q}))$

From Property 2 (2.1.7), we know that $\mu_{G(\widetilde{q})}(z)$ is defined and continuous in $[G(q_1), G(q_2)]$. Therefore its centroid (2.1.8) $E_1(q_1, q_0, q_2) \in [G(q_1), G(q_2)]$. And from $(*1) - (*4)$, when $q_2 - q_0 = \Delta_{20}$, $q_0 - q_1 = \Delta_{01}$ are small. $E_1(q_1, q_0, q_2)$ are close to $G(C(\widetilde{q}))$. When Δ_{20}, Δ_{01} are larger, $E_1(q_1, q_0, q_2)$ are away from $G(C(\widetilde{q}))$. From Table 2, we shall see this result. Also from 4, $6^*, 8^*$ and Table 3, 4, 5 we can see this too.

[B] Comparison of the estimate $E(q_1, q_0, q_2)$ of the total cost in the fuzzy sense of Theorem 1 with the crisp minimal total cost $G(q_*)$.

From (13), the domain of z in $\mu_{G(\widetilde{q})}(z)$ of the membership function of the fuzzy total cost $G(\widetilde{q})$ is $z \geq G(q_*)$, and by Properties $2, 4, 6^*, 8^*$, for each different cases, the domain of z in $\mu_{G(\widetilde{q})}(z)$ is continuous. Hence the centroid $E(q_1, q_0, q_2)$ of $\mu_{G(\widetilde{q})}(z)$ satisfies $E(q_1, q_0, q_2) \geq G(q_*)$. i.e., the estimate of the total cost in the fuzzy sense is larger than the crisp minimal total cost $G(q_*)$. This is for sure. As in the illustration in section 2, formula (1) is obtained by assuming the time needed from the ordering point to the delivering point are fixed, and then get the minimal total cost $G(q_*)$. But in the reality, usually the time cost from the ordering point to the delivering point are not fixed and will vary a little. Therefore we should not use the crisp minimal total cost $G(q_*)$, in stead, should consider the fuzzy case and employing the larger the estimate of the total cost $E(q_1, q_0, q_2)$ in the fuzzy sense to suit the real situation.

[C] Comparison our article and Chang, Yao and Lee[1], and Yao, Lee [13], using section 3, Example 1:

The total cost in Chang, Yao and Lee [1] as our paper is defined by (1), $F(q,s) = \frac{aT(q-s)^2}{2q} + \frac{bTs^2}{2q} + \frac{cR}{q}$. For each $q(>0)$, let $G_q(s) = F(q,s) = \frac{(a+b)T}{2q}s^2 - aTs + \frac{aT}{2}q + \frac{cR}{q}, 0 < s < q$. Fuzzify s into a triangular fuzzy number $\widetilde{S} = (s_1, s_0, s_2), 0 \leq s_1, s_0, s_2$, then we have the fuzzy total cost $G_q(\widetilde{S})$. Then by the Extension Principle and centroid, we have $M(s_1, s_0, s_2, q)$. This is the estimate of the total cost in the fuzzy sense when given (s_1, s_0, s_2, q). Let $M_1 = M(s_1, s_0, s_2, q), s = \frac{a}{a+b}q$

The total cost in Yao and Lee [13] is defined as $F^*(q,s) = \frac{aTs^2}{2q} + \frac{bT(q-s)^2}{2q} + \frac{cR}{q}$. Exchange s into $q-s$ and vice versa, we will have the $F(q,s)$ in our paper. For each $s > 0$, let $G_s(q) = F^*(q,s), 0 < s < q$. Fuzzify q into a triangular fuzzy number $\widetilde{Q} = (q_1, q_0, q_2), 0 < q_1 < q_0 < q_2$, we get the fuzzy total cost $G_s(\widetilde{Q})$. By the Extension Principle and centroid, we have the estimate of the total cost in the fuzzy sense $M(q_1, q_0, q_2, s)$ when given (q_1, q_0, q_2, s). Let $M_2 = M(q_1, q_0, q_2.s), s = \frac{a}{a+b}q$.

Here, in our paper, $G(q) = F(q, \frac{a}{a+b}q), s = \frac{a}{a+b}q$. Fuzzify s, q both at the same time into the triangular numbers, $\widetilde{q} = (q_1, q_0, q_2)$, $\widetilde{s} = \frac{a}{a+b}\widetilde{q}$, $0 < q_1 < q_0 < q_2$. By Theorem 1, we have the estimate of the total cost in the fuzzy sense $E(q_1, q_0, q_2)$ when given q_1, q_0, q_2.

Let $r_j = \frac{E(q_1,q_0,q_2) - M_j}{M_j} \times 100, j = 1, 2$.

Use section 3, Example 1, $a = 10, b = 20, c = 200, R = 2000, T = 12$ to form the following tables: In the following Table $2^* - 5^*$, the values of q_1, q_0, q_2 are the same as in the Table 2 - 5 respectively.

Table 2*. For the case of the fuzzy number $\widetilde{q} = (q_1, q_0, q_2), 100 < q_1 < q_0 < q_2$

q_1	q_0	q_2	$E(q_1, q_0, q_2)$	M_1	M_2	$r_1(\%)$	$r_2(\%)$
102	104	106	8006.865	8006.385	8008.399	0.006	-0.019
102	104	110	8013.861	8011.954	8021.508	0.024	-0.095
102	108	112	8024.088	8021.883	8032.816	0.027	-0.109
102	108	116	8035.397	8030.609	8054.319	0.060	-0.235
104	108	112	8026.243	8024.593	8032.138	0.021	-0.073
104	108	116	8037.664	8034.239	8052.537	0.043	-0.185
104	112	114	8040,496	8040.232	8049.365	0.003	-0.110
104	112	120	8060.539	8054.857	8083.113	0.071	-0.279
106	110	112	8033.288	8032.651	8036.530	0.008	-0.040
106	110	116	8044.884	8042.655	8054.385	0.028	-0.118
106	112	120	8063.891	8059.818	8081.889	0.051	-0.223
106	114	116	8055.367	8055.228	8064.328	0.002	-0.111
108	112	116	8053.706	8052.286	8059.404	0.018	-0.071
108	112	118	8060.520	8058.466	8069.792	0.025	-0.115
108	116	120	8079.677	8078.019	8092.169	0.024	-0.151
108	116	124	8096.483	8091.586	8118.376	0.061	-0.270
110	114	122	8086.433	8083.788	8100.027	0.033	-0.168
110	116	118	8077.327	8076.906	8083.034	0.005	-0.071
110	118	122	8099.927	8098.168	8112.039	0.022	-0.149
110	118	126	8117.629	8113.085	8139.182	0.056	-0.265

Table 3*. For the case of the fuzzy number $\tilde{q} = (q_1, q_0, q_2)$, $0 < q_1 \leq 100 < q_0 < q_2$

q_1	q_0	q_2	$E(q_1, q_0, q_2)$	M_1	M_2	$r_1(\%)$	$r_2(\%)$
98	102	103	8001.396	8001.449	8002.240	-0.001	-0.011
98	102	108	8007.440	8004.420	8019.216	0.038	-0.147
98	106	114	8024.194	8017.208	8047.664	0.087	-0.292
98	108	118	8039.453	8029.259	8075.236	0.127	-0.443
94	104	112	8016.611	8010.125	8040.854	0.081	-0.301
94	106	107	8008.668	8015.827	8005.447	-0.089	0.040
94	106	109	8011.953	8012.791	8016.036	-0.010	-0.051
94	106	116	8029.421	8019.275	8068.033	0.127	-0.479
90	104	112	8016.470	8011.802	7974.265	0.058	0.529
90	106	114	8023.239	8017.531	8017.754	0.071	0.068
90	108	116	8031.426	8025.179	8040.932	0.078	-0.118
90	110	117	8037.710	8034.273	8050.247	0.043	-0.156
86	110	117	8037.251	8041.733	7922.730	-0.056	1.445
86	116	119	8057.421	8096.040	8005.991	-0.477	0.642
86	116	120	8060.879	8090.693	8017.712	-0.369	0.538
86	116	126	8084.743	8080.816	8093.700	0.049	-0.111
84	110	120	8047.298	8044.891	7837.114	0.030	2.682
84	116	120	8060.514	8099.332	7925.797	-0.479	1.700
84	116	122	8067.854	8091.069	7962.055	-0.287	1.329
84	118	128	8098.915	8101.887	8066.963	-0.037	0.396

Table 4*. For the case of the fuzzy number $\widetilde{q} = (q_1, q_0, q_2)$, $0 < q_1 < q_0 < 100 \leq q_2$

q_1	q_0	q_2	$E(q_1, q_0, q_2)$	M_1	M_2	$r_1(\%)$	$r_2(\%)$
95	97	103	8003.949	8002.364	8007.698	0.020	-0.047
95	97	110	8010.088	8004.668	8386.511	0.068	-4.488
95	97	120	8041.371	8018.636	8224.418	0.284	-2.226
95	97	130	8086.359	8044.150	8385.432	0.525	-3.567
90	92	102	8020.740	8015.023	8032.834	0.071	-0.151
90	92	104	8020.233	8013.248	8034.826	0.087	-0.182
90	92	106	8019.896	8012.050	8035.931	0.098	-0.2
90	92	108	8019.655	8011.420	8034.943	0.103	-0.19
84	86	102	8061.813	8045.705	8092.198	0.2	-0.375
84	86	104	8060.237	8041.871	8096.001	0.233	-0.437
84	88	106	8052.506	8033.452	8091.534	0.237	-0.482
84	88	112	8051.077	8028.575	8098.993	0.280	-0.592
76	80	102	8146.553	8106.787	8214.203	0.491	-0.824
76	82	104	8132.286	8090.811	8207.310	0.513	-0.914
76	84	108	8119.355	8072.677	8212.511	0.578	-1.134
76	80	116	8137.458	8074.599	8262.232	0.778	-1.51
70	80	110	8200.004	8117.052	8354.827	1.022	-1.853
70	80	112	8198.780	8112.015	8365.235	1.070	-1.99
70	80	114	8197.717	8107.646	8375.852	1.111	-2.127
70	80	116	8196.785	8103.933	8386.563	1.146	-2.263

Table 5*. For the case of the fuzzy number $\widetilde{q} = (q_1, q_0, q_2)$, $0 < q_1 < q_0 < q_2 < 100$

q_1	q_0	q_2	$E(q_1, q_0, q_2)$	M_1	M_2	$r_1(\%)$	$r_2(\%)$
95	97	99	8004.589	8003.959	8006.247	0.008	-0.021
92	94	98	8014.156	8012.703	8017.724	0.018	-0.045
89	93	95	8027.797	8026.389	8032.261	0.019	-0.054
87	93	95	8035.014	8032.638	8043.623	0.030	-0.107
86	90	96	8045.113	8040.300	8055.714	0.060	-0.132
85	89	91	8069.399	8062.603	8068.863	0.084	-0.055
84	90	96	8054.389	8046.884	8070.657	0.094	-0.202
83	91	97	8055.181	8045.475	8078.883	0.121	-0.293
83	87	93	8076.816	8071.336	8087.906	0.068	-0.137
83	87	91	8082.575	8078.875	8089.978	0.047	-0.092
82	86	94	8084.364	8076.452	8100.071	0.098	-0.194
82	84	92	8099.600	8093.422	8111.183	0.076	-0.143
80	88	96	8084.456	8069.818	8114.317	0.181	-0.368
80	86	94	8097.062	8085.763	8119.178	0.140	-0.272
78	88	96	8098.728	8080.377	8139.040	0.227	-0.495
78	88	92	8108.489	8097.669	8136.295	0.134	-0.342
77	85	93	8126.960	8110.400	8157.795	0.204	-0.378
77	83	91	8144.250	8131.388	8167.273	0.158	-0.282
76	86	94	8127.854	8107.886	8168.841	0.246	-0.502
74	84	92	8164.321	8140.555	8204.089	0.292	-0.509

From Tables $2^*, 3^*, 5^*$, we see that the total cost in the fuzzy sense obtained from our formula is close to the total costs in the fuzzy sense $M_1 = M(s_1, s_0, s_2, q)$ in [1], and $M_2 = M(q_1, q_), q_2, s)$ in [13].But as mentioned in section 2, to fuzzify both s, q, fits better in the real situation than either just fuzzify s, leave q as a positive real variable in [1] or just fuzzify q leave s as a positive real variable [13].

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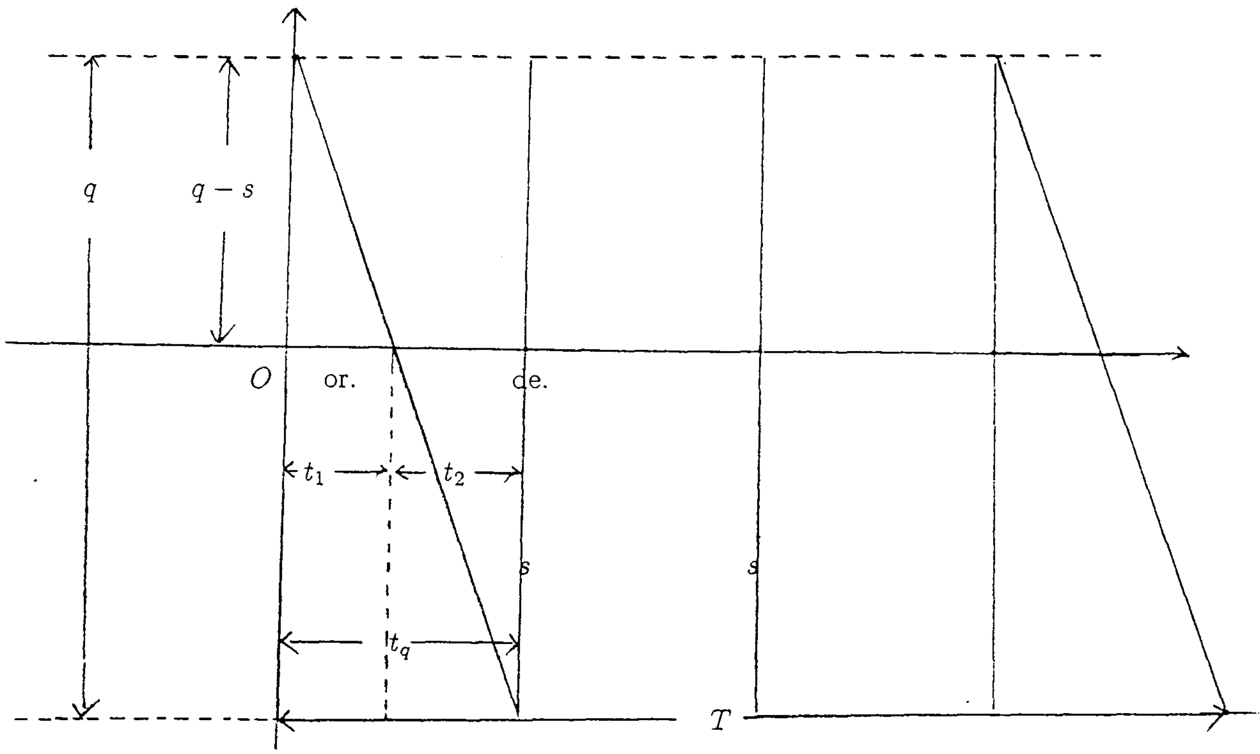

or: ordering point de: delivering point

Fig. 1. Inventory with backorder model

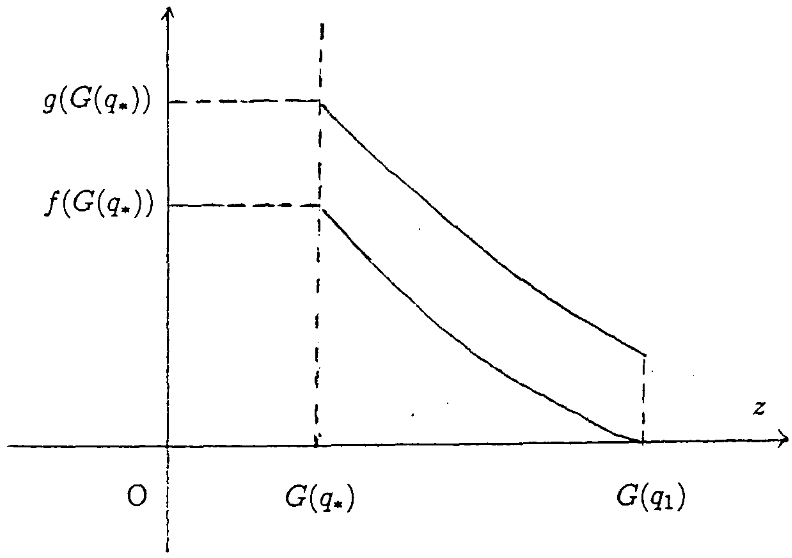

Fig. 2. $D^*(q_1, q_0, q_2) < 0$, $f(z)$ and $g(z)$ have no point of intersection

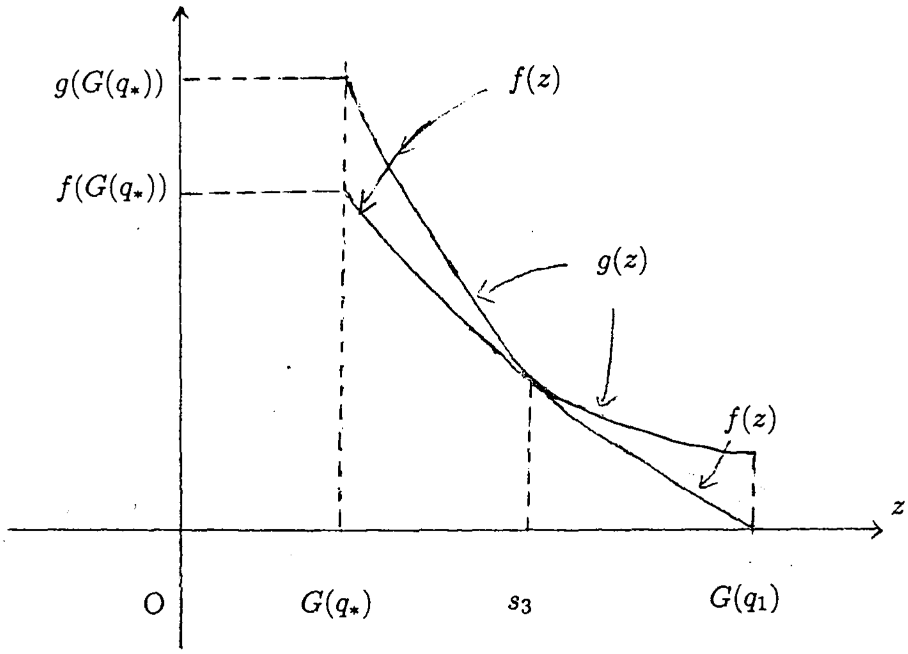

Fig. 3. $D^*(q_1, q_0, q_2) = 0$, $f(z)$ and $g(z)$ have one point of intersection

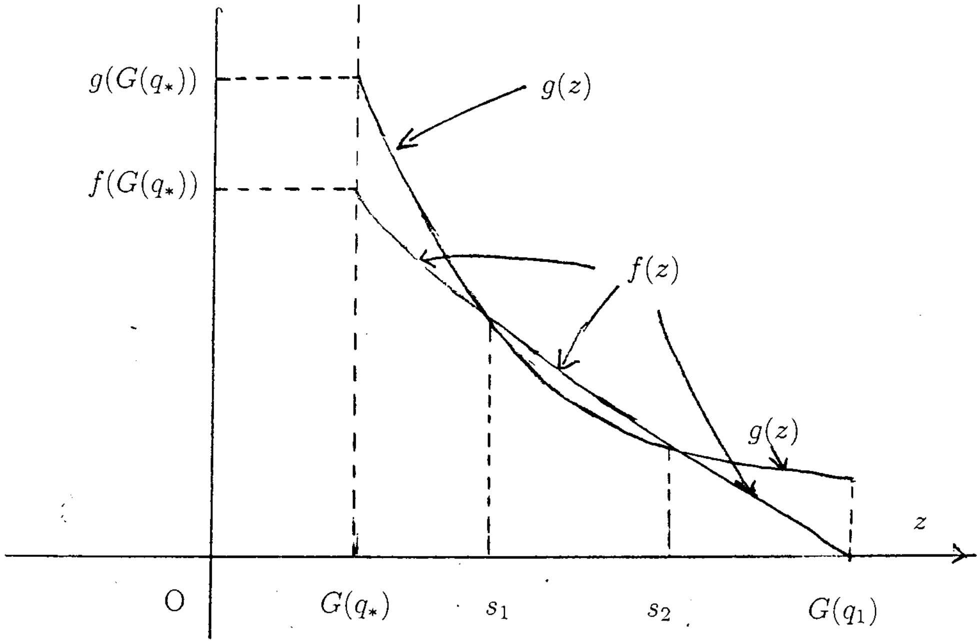

Fig. 4. $D^*(q_1, q_0, q_2) > 0$, $f(z)$ and $g(z)$ have two points of intersection

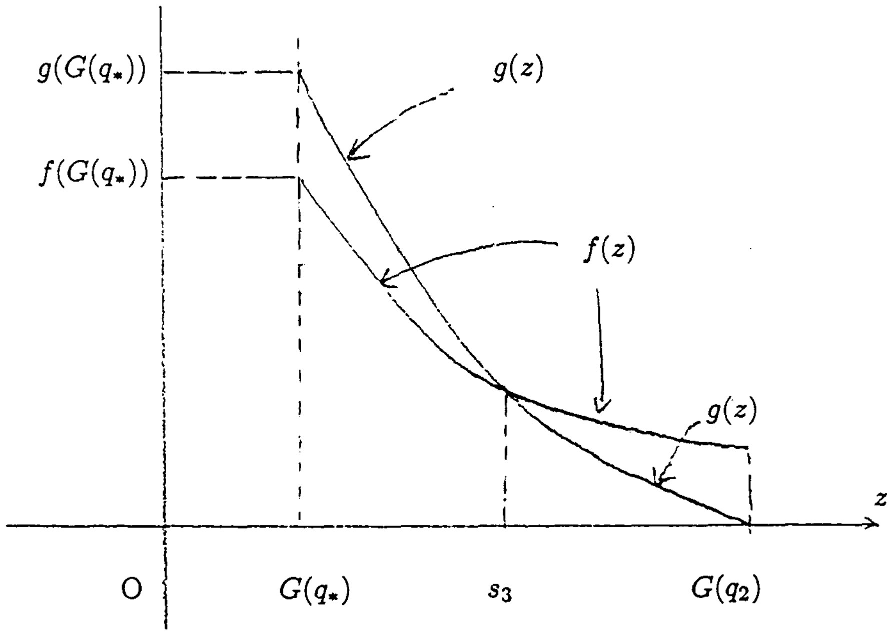

Fig. 5. $D^*(q_1, q_0, q_2) = 0$, $f(z)$ and $g(z)$ have one point of intersection