CHINESE JOURNAL OF PHYSICS VOL. 40, NO. 5 OCTOBER 2002

The Addition Theorem for Spherical Harmonics and Monopole Harmonics

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(Received March 1, 2002)

The addition theorem for spherical harmonics and monopole harmonics is studied in a unified approach. The given theorem is valid for all gauges.

PACS. 03.65.–w – Quantum mechanics.
PACS. 14.80.Hv – Magnetic monopoles.

I. Introduction

Spherical harmonics are the angular eigenfunctions of the rotationally invariant Laplacian in spherical co-ordinates. They were well studied long ago. With the advent of quantum mechanics, it was realised that they are quantum mechanical wavefunctions for the angular momentum operators. Dirac [1] introduced the concept of a nonintegrable phase factor into quantum mechanics with the consideration of the problem of an electron in a monopole field. The motion of an electron in a monopole field is rotationally invariant classically. Quantum mechanically the system is still rotationally invariant modulo a gauge factor. Tamm [2] first solved the Schrödinger equation of an electron in a monopole field and obtained the monopole harmonics. Without doubt, the monopole harmonics are the wavefunctions for the angular momentum operators in this problem. It was realised that angular momentum operators for the monopole problem have additional parts due to the contribution from the magnetic field, but the algebra is still the same. Wu and Yang [3] re-studied the monopole harmonics in the conceptual context of sections of the monopole line bundle.

In [4] we found a unified way of treating the spherical harmonics and monopole harmonics. The key point is to observe that the angular momentum operators in both cases satisfy the same $SU(2)$ algebra. The only difference is that, for ordinary angular momentum operators $\mathbf{L}$, they satisfy $\hat{\mathbf{r}}.\mathbf{L} = 0$, where $\hat{\mathbf{r}}$ is the radial unit vector. For the monopole angular momentum operators $\mathbf{J}$, we have the condition $\hat{\mathbf{r}}.\mathbf{J} = -eg = -q$, where e and g are the electron and monopole charge, respectively. Dirac [1] had shown that their product q must be an integer or a half-integer. We employed [4] the holomorphic representation of the angular momentum operators, which was first studied by Bargmann [5]. Indeed, this is just a concrete representation of the angular momentum operators by bosonic oscillators, due to Schwinger [6]. The up and down spins are represented by the two complex variables, u and v, respectively. We obtained a duet of rotated states u' and v' such that they are eigenstates of $\hat{\mathbf{r}}.\mathbf{J}$ with the eigenvalues $-\frac{1}{2}$ and $\frac{1}{2}$, respectively. The gauge degree of freedom appears in the ambiguity of defining u' and v', as a result of the Hopf mapping [7]. The states u' and v' come with reciprocal phase factors. The spherical harmonics and monopole harmonics can be built out of suitable combinations of the duet of the rotated states, u' and v'. The gauge factors just cancel out in the case of the spherical harmonics. This was the main content of [4].

Following their previous paper [3], Wu and Yang studied the addition theorem of the monople harmonics in [8]. The addition theorem is just a manifestation of the composition of two rotations for the representation theory and is certainly an important theorem. Therefore we have chosen to study the addition theorem for spherical harmonics and monopole harmonics here using our unified approach. Our method has the advantage of keeping track of all the gauge factors throughout. The theorem can be given for all gauges. This paper is organised as follows. In section 2 we directly apply our formula to obtain the addition theorem for the spherical harmonics. In this case we do not need to worry too much about the gauge factors, since they just cancel out. In section 3 we study the addition theorem for the monopole harmonics. It can be observed that in our approach, the gauge factors emerge quite naturally. We conclude in the final section with a simple computation. We explicitly multiply the $SU(2)$ matrices for the group composition formula. After some manipulations we do recover some cases of the addition theorem for the spherical harmonics and monopole harmonics. This tests the correctness of our approach. It also reveals the underlying principle of the physics of the addition theorem, the manifestation of the group composition law, with appropriate phases in the ensuing representation.

II. Addition theorem for spherical harmonics

One of the key results of [4] is that the spherical harmonics Y_{lm}, can be expressed as

$$Y_{lm}(\theta,\phi) = \int \frac{\bar{u}^{l+m}}{\sqrt{(l+m)!}} \frac{\bar{v}^{l-m}}{\sqrt{(l-m)!}} \frac{(Ru)^l (Rv)^l}{l!} [du][dv] \,, \tag{1}$$

where u, v are two complex variables [6, 5] which denote the up and down spins, respectively. The measures $[du]$ and $[dv]$ are $ie^{-\bar{u}u}/2\pi$ and $ie^{-\bar{v}v}/2\pi$, respectively, so that

$$\int \bar{u}^h u^{h'} [du] = \int \bar{v}^h v^{h'} [dv] = h! \delta_{hh'} \,. \tag{2}$$

The rotation R rotates the z-axis *directly* into the (θ,ϕ) direction. The rotated states Ru and Rv are then denoted by

$$\begin{pmatrix} Ru \\ Rv \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} e^{-i\phi} \\ -\sin\frac{\theta}{2} e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} . \tag{3}$$

Indeed, there is a $U(1)$ gauge degree of freedom in the definition of R corresponding to a further rotation of angle χ around the (θ,ϕ) direction.

$$R^\chi = \begin{pmatrix} e^{i\chi/2} & 0 \\ 0 & e^{-i\chi/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} e^{-i\phi} \\ -\sin\frac{\theta}{2} e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix} . \tag{4}$$

However, for the spherical harmonic case the gauge factors just cancel out, and do not contribute to the expression for the spherical harmonics. From the above expression it can be easily seen that

$$Y_{lm}(0,\theta) = \delta_{m0}\,. \tag{5}$$

Equivalently, we can sum up the spherical harmonics as

$$\sum_{m=-l}^{l} Y_{lm}(\theta,\phi)\frac{u^{l+m}}{\sqrt{(l+m)!}}\frac{v^{l-m}}{\sqrt{(l-m)!}} = \frac{(Ru)^l(Rv)^l}{l!}\,. \tag{6}$$

If we now consider the composition of two rotations as

$$R'^{\chi'} = R^{\chi}Q\,, \tag{7}$$

where we retain the gauge factors for ease of manipulation. The rotation Q is parameterised by the three Euler angles α, β, and γ,

$$Q = \begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\beta}{2} & \sin\frac{\beta}{2} \\ -\sin\frac{\beta}{2} & \cos\frac{\beta}{2} \end{pmatrix}\begin{pmatrix} e^{i\frac{\gamma}{2}} & 0 \\ 0 & e^{-i\frac{\gamma}{2}} \end{pmatrix}. \tag{8}$$

We immediately get the formula

$$Y_{lm}(\theta,\phi) = \sum_{m'} D^l_{mm'}(Q^{-1})Y_{lm'}(\theta',\phi')\,, \tag{9}$$

where the function $D^l_{mm'}$ is given by

$$D^l_{mm'}(Q) = \int \frac{\bar{u}^{l+m}}{\sqrt{(l+m)!}}\frac{\bar{v}^{l-m}}{\sqrt{(l-m)!}}\frac{(Qu)^{l+m'}}{\sqrt{(l+m')!}}\frac{(Qv)^{l-m'}}{\sqrt{(l-m')!}}[du][dv]\,. \tag{10}$$

As an application of this formula we observe that

$$\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} = \begin{pmatrix} e^{i\frac{\phi}{2}} & 0 \\ 0 & e^{-i\frac{\phi}{2}} \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2}e^{-i\phi} \\ -\sin\frac{\theta}{2}e^{i\phi} & \cos\frac{\theta}{2} \end{pmatrix}\begin{pmatrix} e^{-i\frac{\phi}{2}} & 0 \\ 0 & e^{i\frac{\phi}{2}} \end{pmatrix}, \tag{11}$$

and we get

$$Y_{lm}(\theta,\phi) = e^{im\phi}Y_{lm}(\theta,0)\,. \tag{12}$$

The addition theorem, here, can be expressed as

$$Y_{l0}(\theta,\phi) = \sum_{m'} Y^*_{lm'}(\beta,\gamma)Y_{lm'}(\theta',\phi')\,, \tag{13}$$

since

$$\begin{aligned} D^l_{0m'}(R^{-1}) &= Y_{l,-m'}(-\theta,\phi) \\ &= Y^*_{l,m'}(\theta,\phi)\,. \end{aligned} \tag{14}$$

III. Addition theorem for monopole harmonics

The analogous formula for the monopole harmonics is

$$Y^{\chi}_{q,j,m}(\theta,\phi)=\int\frac{\bar{u}^{j+m}}{\sqrt{(j+m)!}}\frac{\bar{v}^{j-m}}{\sqrt{(j-m)!}}\frac{(R^{\chi}u)^{j-q}}{\sqrt{(j-q)!}}\frac{(R^{\chi}v)^{j+q}}{\sqrt{(j+q)!}}[du][dv]\,. \tag{15}$$

By this definition we have

$$e^{iq\chi}Y^{\chi}_{q,j,m}(0,\phi)=\delta_{m,-q}\,. \tag{16}$$

There is a symmetry which is not obvious in the spherical harmonic case. This is a kind of charge conjugate transformation,

$$\begin{aligned}\bar{U}&=R^{\chi}v \qquad \bar{V}=R^{\chi}u\\ \bar{u}&=\tilde{R}^{\chi}V \qquad \bar{v}=\tilde{R}^{\chi}U\,,\end{aligned} \tag{17}$$

where

$$\tilde{R}^{\chi}=\begin{pmatrix} e^{-i\frac{\chi}{2}} & 0\\ 0 & e^{i\frac{\chi}{2}}\end{pmatrix}R^{\chi}\begin{pmatrix} e^{-i\frac{\chi}{2}} & 0\\ 0 & e^{i\frac{\chi}{2}}\end{pmatrix}\,. \tag{18}$$

Thus we have

$$e^{iq\chi}Y^{\chi}_{q,j,m}=e^{im\chi}Y^{\chi}_{m,j,q}\,. \tag{19}$$

This symmetry was first observed in [8]. We now have the equivalent expansion formula

$$\sum_{m=-j}^{j}Y^{\chi}_{q,j,m}(\theta,\phi)\frac{u^{j+m}}{\sqrt{(j+m)!}}\frac{v^{j-m}}{\sqrt{(j-m)!}}=\frac{(R^{\chi}u)^{j-q}}{\sqrt{(j-q)!}}\frac{(R^{\chi}v)^{j+q}}{\sqrt{(j+q)!}}\,. \tag{20}$$

The analogous composition formula is

$$Y^{\chi}_{q,j,m}(\theta,\phi)=\sum_{m'}D^{j}_{mm'}(Q^{-1})Y^{\chi'}_{q,j,m'}(\theta',\phi')\,, \tag{21}$$

where the function $D^{j}_{mm'}$ is given by

$$D^{j}_{mm'}(Q)=\int\frac{\bar{u}^{j+m}}{\sqrt{(j+m)!}}\frac{\bar{v}^{j-m}}{\sqrt{(j-m)!}}\frac{(Qu)^{j+m'}}{\sqrt{(j+m')!}}\frac{(Qv)^{j-m'}}{\sqrt{(j-m')!}}[du][dv]\,. \tag{22}$$

We can also easily derive the following relations

$$Y^{\chi+\phi}_{q,j,m}(\theta,\phi)=e^{im\phi}Y^{\chi}_{q,j,m}(\theta,0)\,, \tag{23}$$

or

$$Y^{\chi}_{q,j,m}(\theta,\phi) = e^{i(m+q)\phi}\, Y^{\chi}_{q,j,m}(\theta,0)\,. \tag{24}$$

For the addition theorem for monopole harmonics we have to be a little bit careful about the additional phase factor. We should take

$$R'^{\chi'} = R^{(\chi+\ \ )}Q\,, \tag{25}$$

where   is the area of the geodesic triangle formed by the rotations R', R^{-1}, and Q^{-1}. This ensures that the successive rotations will give the correct non-integrable phase factor [9]. Hence we have

$$Y^{\chi+\ }_{q,j,-q}(\theta,\phi) = \sum_{m'} D^{j}_{-q,m'}(Q^{-1})\, Y^{\chi'}_{q,j,m'}(\theta',\phi')\,, \tag{26}$$

but,

$$D^{j}_{-q,m'}(Q^{-1}) = Y^{-(\alpha+\gamma)}_{-m',j,-q}(-\beta,-\alpha)\,. \tag{27}$$

After some straightforward manipulations, we get the relation

$$D^{j}_{-q,m'}(Q^{-1}) = e^{iq(\alpha+\gamma)} e^{-iq\chi}\, Y^{\chi *}_{q,j,m'}(\beta,\gamma)\,. \tag{28}$$

Hence, the addition theorem for the monopole harmonics in its final form is

$$e^{-iq\ \ } Y^{\chi}_{q,j,-q}(\theta,\phi) = \sum_{m'} e^{iq(\alpha+\gamma)} e^{-iq\chi}\, Y^{\chi *}_{q,j,m'}(\beta,\gamma)\, Y^{\chi'}_{q,j,m'}(\theta',\phi')\,. \tag{29}$$

In [8], Wu and Yang gave a generalised addition theorem, which in our formulation reads

$$Y^{\chi}_{q,j,-q'}(\theta,\phi) = e^{i(q\ \ -q'\chi)} \sum_{m'} e^{iq'(\alpha+\gamma)}\, Y^{\chi *}_{q',j,m'}(\beta,\gamma)\, Y^{\chi'}_{q,j,m'}(\theta',\phi')\,. \tag{30}$$

To compare our results with those of Wu and Yang [8] we have to take $\chi = 0$ and $\chi' = -(\alpha+\gamma)$.

IV. Addition theorem and the composition of two $SU(2)$ rotations

The crucial piece of physics in the addition theorem is sinply the composition of two rotations. We consider here the $SU(2)$ matrices, since we can discuss the role of the gauge factors more directly in this representation. We consider the case in [8]; the composition law is just

$$R' = R^{\ \ +\alpha+\gamma}Q\,. \tag{31}$$

Working out the explicit expression for R', $R^{-\ }$, and Q, we get

$$\cos\frac{\theta}{2} = e^{-i\frac{\ }{2}} e^{-i(\alpha+\gamma)} \left(\cos\frac{\beta}{2}\cos\frac{\theta'}{2} + \sin\frac{\beta}{2}\sin\frac{\theta'}{2} e^{i(\gamma-\phi')}\right)\,, \tag{32}$$

$$\sin\frac{\theta}{2}e^{i\phi} = e^{i\frac{}{2}}\left(\cos\frac{\beta}{2}\sin\frac{\theta'}{2}e^{i\phi'} - \sin\frac{\beta}{2}\cos\frac{\theta'}{2}e^{i\gamma}\right) . \tag{33}$$

On closer scrutiny, these two relations are simply two cases of the monopole harmonics addition theorem.

The above relations are equations for the half-angles of θ, β, and θ'. By algebraic manipulations we can get equations for the whole angles:

$$\cos\theta = \cos\beta\cos\theta' + \sin\beta\sin\theta'\cos(\gamma - \phi') . \tag{34}$$

This is an example of the addition theorem for spherical harmonics. Indeed, this is the well-known cosine law for the geodesic triangle, which implies

$$\cos T = \cos(\gamma - \phi') , \tag{35}$$

where we have defined the three angles of the geodesic triangle as B, T, and T', the angles opposite the sides β, θ, and θ', respectively. We also have the equation for $\sin\theta$

$$\sin\theta e^{i\phi} = e^{-i(\alpha+\gamma)}\left(\cos^2\frac{\beta}{2}\sin\theta' e^{i\phi'} - 2\cos\frac{\beta}{2}\sin\frac{\beta}{2}\cos\theta' e^{i\gamma} - \sin^2\frac{\beta}{2}\sin\theta' e^{i(2\gamma-\phi')}\right) . \tag{36}$$

It is readily realised that this is just one case of the generalised addition theorem for monopole harmonics.

Indeed, we should first work out $\cos\theta$ from the cosine law. Then we use the cosine law to get the angles B and T'. From the three calculated angles the area of the geodesic triangle can be readily obtained from the well-known formula

$$T + B + T' = \pi - \quad . \tag{37}$$

Finally the quantity $\alpha + \gamma$ can be calculated from the defining composition law.

Acknowledgement

This work is supported in part by the National Science Council of the R.O.C. under the contract number NSC-90-2112-M-003-011.

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