



A note on cyclic-cubes

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Received 20 May 1999

Communicated by S.G. Akl and F. Dehne

Abstract

Fu and Chau have proposed a new family of Cayley graphs, named cyclic-cubes. In this note we show that cyclic-cubes are isomorphic to the well-known wrap-around butterfly graphs. © 1999 Published by Elsevier Science B.V. All rights reserved.

Keywords: Cayley graph; Cyclic-cube; Graph isomorphism; Wrap-around butterfly graph

1. Introduction

Recently, Fu and Chau [3] have proposed a new family of Cayley graphs with even fixed-degrees, named cyclic-cubes, which is a generalization of Vadapalli and Srimani's work [5]. In [2], the latter was shown isomorphic to the binary wrap-around butterfly graphs. In this note we show that the cyclic-cubes are also isomorphic to the wrap-around butterfly graphs. In the following, we review the structures of the cyclic-cubes and the wrap-around butterfly graphs.

Let t_1, t_2, \dots, t_n be n symbols with an ordering of $t_1 \succ t_2 \succ \dots \succ t_n$, where \succ represents a total ordering. Each symbol is assigned a rank from $\{1, 2, \dots, k\}$. We use t_l^j to denote a ranked symbol, indicating that t_l is assigned with rank j . The vertex set of a cyclic-cube contains all circular permutations of ranked symbols ordered by \succ . We use G_n^k to represent a cyclic-cube whose vertex set contains nk^n distinct circular permutations of $t_1^{j_1} t_2^{j_2} \dots t_n^{j_n}$, where $j_l \in \{1, 2, \dots, k\}$ for all $1 \leq l \leq n$.

The edges of G_n^k are generated by $2k$ generator functions, denoted by $g, g^{-1}, f^1, f^{-1}, f^2, f^{-2}, \dots, f^{k-1}, f^{-(k-1)}$, which are defined as follows.

$$\begin{aligned}
&g(t_{l+1}^{j_{l+1}} t_{l+2}^{j_{l+2}} \dots t_n^{j_n} t_1^{j_1} \dots t_l^{j_l}) \\
&= t_{l+2}^{j_{l+2}} \dots t_n^{j_n} t_1^{j_1} \dots t_l^{j_l} t_{l+1}^{j_{l+1}}; \\
&g^{-1}(t_{l+1}^{j_{l+1}} \dots t_n^{j_n} t_1^{j_1} \dots t_{l-1}^{j_{l-1}} t_l^{j_l}) \\
&= t_l^{j_l} t_{l+1}^{j_{l+1}} \dots t_n^{j_n} t_1^{j_1} \dots t_{l-1}^{j_{l-1}}; \\
&f^i(t_{l+1}^{j_{l+1}} t_{l+2}^{j_{l+2}} \dots t_n^{j_n} t_1^{j_1} \dots t_l^{j_l}) \\
&= t_{l+2}^{j_{l+2}} \dots t_n^{j_n} t_1^{j_1} \dots t_l^{j_l} t_{l+1}^{j_{l+1}}, \\
& \quad i = 1, 2, \dots, k - 1; \\
&f^{-i}(t_{l+1}^{j_{l+1}} \dots t_n^{j_n} t_1^{j_1} \dots t_{l-1}^{j_{l-1}} t_l^{j_l}) \\
&= t_l^{j_l} t_{l+1}^{j_{l+1}} \dots t_n^{j_n} t_1^{j_1} \dots t_{l-1}^{j_{l-1}}, \\
& \quad i = 1, 2, \dots, k - 1,
\end{aligned}$$

where $1 \leq l \leq n$,

$$t_{l+1}^{j_{l+1}+i} = t_{l+1}^{(j_{l+1}+i)-k} \quad \text{if } j_{l+1} + i > k,$$

and

$$t_l^{j_l-i} = t_l^{(j_l-i)+k} \quad \text{if } j_l - i \leq 0.$$

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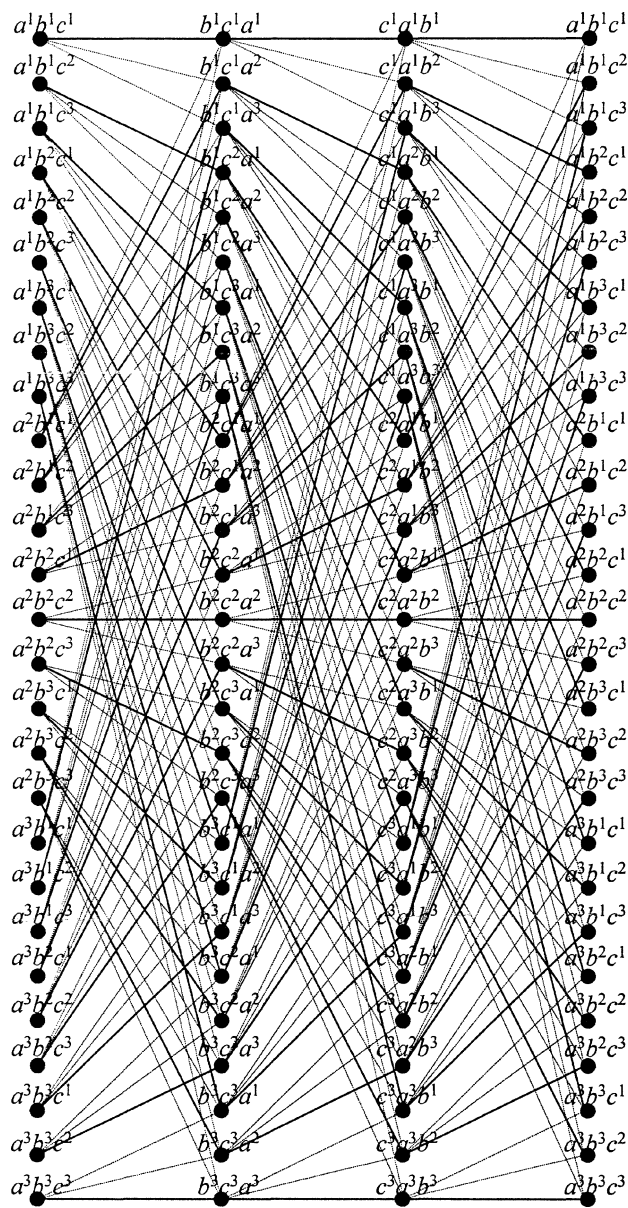


Fig. 1. A drawing of G_3^3 with the outer two columns identical.

Formally, G_n^k can be defined as follows.

if and only if $\delta(x) = y$, where $\delta \in \{g, g^{-1}, f^i, f^{-i} \mid i = 1, 2, \dots, k - 1\}$.

Definition 1. The vertex set of G_n^k is $\{t_1^{j_1} t_2^{j_2} \dots t_n^{j_n}, t_2^{j_2} t_3^{j_3} \dots t_1^{j_1}, \dots, t_n^{j_n} t_1^{j_1} \dots t_{n-1}^{j_{n-1}} \mid j_l \in \{1, 2, \dots, k\} \text{ for all } 1 \leq l \leq n\}$. Two vertices x and y of G_n^k are adjacent

Fig. 1 shows a drawing of G_3^3 , where $a > b > c$ is assumed. The vertices of G_3^3 are grouped into different columns according to their first symbol. For

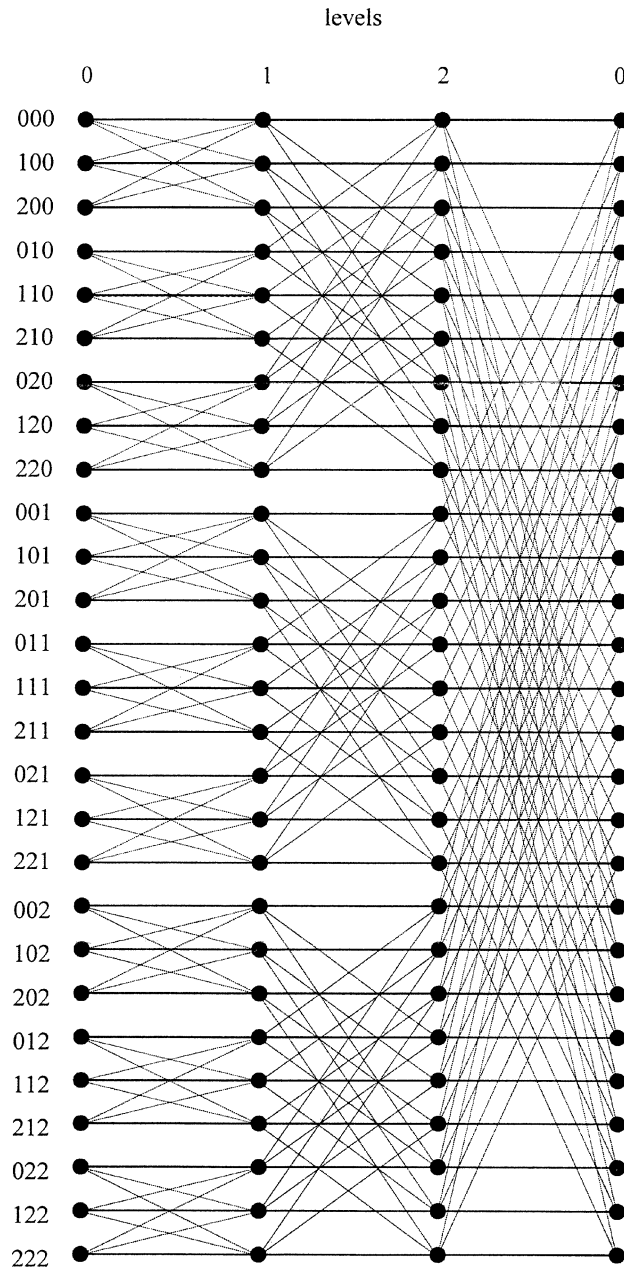


Fig. 2. A drawing of $BF(3, 3)$ with level 0 replicated.

the convenience of drawing, the column with the first symbol a is duplicated.

We use $BF(k, n)$ to denote an n -dimensional k -ary wrap-around butterfly graph. $BF(k, n)$ contains

n levels, numbered $0, 1, \dots, n - 1$, each of k^n vertices. Each vertex v is represented by a two-tuple $\langle l, \beta_0\beta_1 \dots \beta_{n-1} \rangle$, where $0 \leq l \leq n - 1$ is the level of v and $\beta_0\beta_1 \dots \beta_{n-1}$ is a k -ary sequence that is used to

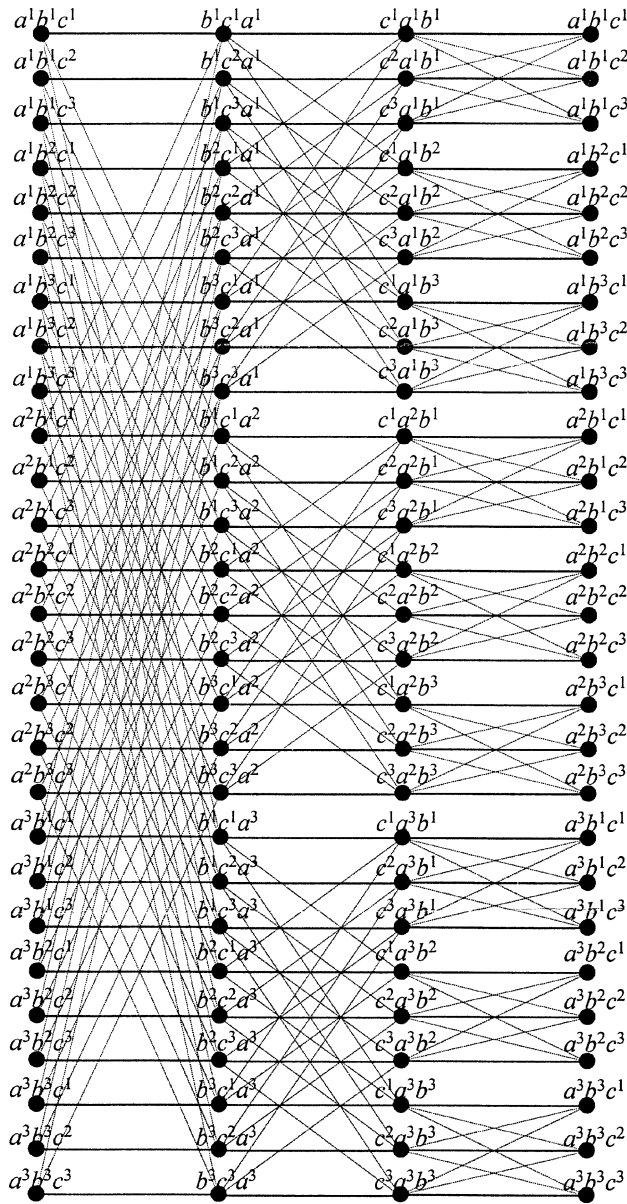


Fig. 3. Redrawing of G_3^3 .

distinguish v from the others with the same level as v . The edges exist between adjacent levels. Formally, $BF(k, n)$ can be defined as follows.

Definition 2. The vertex set of $BF(k, n)$ is $\{ \langle l, \beta_0 \beta_1 \dots \beta_{n-1} \rangle \mid l \in \{0, 1, \dots, n-1\} \text{ and } \beta_i \in \{0, 1, \dots, k-$

$1\}$ for all $0 \leq i \leq n-1\}$. Two vertices $\langle l, \beta_0 \beta_1 \dots \beta_{n-1} \rangle$ and $\langle l', \beta'_0 \beta'_1 \dots \beta'_{n-1} \rangle$ are adjacent if and only if $l' \equiv l + 1 \pmod n$ and $\beta_i = \beta'_i$ for all $0 \leq i \leq n-1$ and $i \neq l$.

Fig. 2 shows a drawing of $BF(3, 3)$. It was shown in [1] that $BF(k, n)$ is an instance of the Cayley graphs.

2. The isomorphism

Two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a one-to-one correspondence ϕ from V_1 to V_2 so that for all $a, b \in V_1$, $(a, b) \in E_1$ if and only if $(\phi(a), \phi(b)) \in E_2$. For example, it is easy to see that G_3^3 is isomorphic to $BF(3, 3)$ if we redraw G_3^3 as shown in Fig. 3. The following theorem shows that G_n^k is isomorphic to $BF(k, n)$.

Theorem 1. G_n^k is isomorphic to $BF(k, n)$.

Proof. For an arbitrary vertex $x = t_{l+1}^{j_{l+1}} t_{l+2}^{j_{l+2}} \dots t_n^{j_n} t_1^{j_1} \dots t_l^{j_l}$ of G_n^k , define $y = \phi(x) = \langle l, (j_1 - 1)(j_2 - 1) \dots (j_n - 1) \rangle$. Clearly, ϕ is a one-to-one correspondence. Since G_n^k and $BF(k, n)$ have the same number of edges, they are isomorphic if and only if for each edge (x_1, x_2) of G_n^k , $(\phi(x_1), \phi(x_2))$ is an edge of $BF(k, n)$.

Since the Cayley graphs are vertex symmetric, we assume $x_1 = t_{l+1}^{j_{l+1}} t_{l+2}^{j_{l+2}} \dots t_n^{j_n} t_1^{j_1} \dots t_l^{j_l}$. If (x_1, x_2) is induced by g , then $x_2 = t_{l+2}^{j_{l+2}} \dots t_n^{j_n} t_1^{j_1} \dots t_l^{j_l} t_{l+1}^{j_{l+1}}$, and $(\phi(x_1), \phi(x_2)) = (\langle l, (j_1 - 1)(j_2 - 1) \dots (j_n - 1) \rangle, \langle l + 1, (j_1 - 1)(j_2 - 1) \dots (j_n - 1) \rangle)$, which is an edge of $BF(k, n)$. Similarly, $(\phi(x_1), \phi(x_2))$ is an edge of $BF(k, n)$ if (x_1, x_2) is induced by g^{-1} .

If (x_1, x_2) is induced by f^i , then $x_2 = t_{l+2}^{j_{l+2}} t_{l+3}^{j_{l+3}} \dots t_n^{j_n} t_1^{j_1} \dots t_l^{j_l} t_{l+1}^{j_{l+1}+i}$, and $(\phi(x_1), \phi(x_2)) = (\langle l, (j_1 -$

$1)(j_2 - 1) \dots (j_n - 1) \rangle, \langle l + 1, (j_1 - 1)(j_2 - 1) \dots (j_l - 1)(j_{l+1} - 1 + i)(j_{l+2} - 1) \dots (j_n - 1) \rangle)$, which is an edge of $BF(k, n)$. Similarly, $(\phi(x_1), \phi(x_2))$ is an edge of $BF(k, n)$ if (x_1, x_2) is induced by f^{-i} . \square

In [3], a shortest-path routing algorithm for G_n^k was proposed. According to Theorem 1 the algorithm can be adapted to $BF(k, n)$. A shortest-path routing algorithm for $BF(k, n)$ was also proposed in [4], independently.

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