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ORIGINAL PAPER

G.-F. Lin · C.-M. Chen

Stochastic analysis of spatial variability in unconfined groundwater flow

Abstract In this paper, spatial variability in steady one-dimensional unconfined groundwater flow in heterogeneous formations is investigated. An approach to deriving the variance of the hydraulic head is developed using the nonlinear filter theory. The nonlinear governing equation describing the one-dimensional unconfined groundwater flow is decomposed into three linear partial differential equations using the perturbation method. The linear and quadratic frequency response functions are obtained from the first- and second-order perturbation equations using the spectral method. Furthermore, under the assumption of the exponential covariance function of log hydraulic conductivity, the analytical solutions of both the spectrum and the variance of the hydraulic head produced from the linear system are derived. The results show that the variance derived herein is less than that of Gelhar (1977). The reason is that the log transmissivity is linearized in Gelhar's work. In addition, the analytical solutions of both the spectrum and the variance of the hydraulic head produced from the quadratic system are derived as well. It is found that the correlation scale and the trend in mean of log hydraulic conductivity are important to the dimensionless variance ratio.

Keywords Heterogeneous aquifer · Groundwater flow · Nonlinear filter theory

1 Introduction

For investigating the spatial variability of the unconfined heterogeneous aquifer, the log hydraulic conductivity is often regarded as the sum of a mean and a perturbation, and the one- or two-dimensional nonlinear groundwater equation is often adopted. In most of previous researches, the nonlinear governing equation is assumed to be linear or can be reduced to linear form (Gelhar, 1974, 1977; Serrano, 1995). Gelhar (1977) solved the steady unconfined-flow equation for one- and two-dimensional variations of hydraulic head with log-hydraulic transmissivity. He used the linearized equation and reduced the log transmissivity to linear form. Mizell (1980) derived the head variance for unsteady flow in a nonleaky aquifer. Serrano (1995) indicated that the linearized Boussinesq equation with the Dupuit assumptions is a reasonable approximation to the original nonlinear equation using numerical computation. In this paper, the well-known Boussinesq equation with the Dupuit assumptions is used and the log hydraulic conductivity is not reduced to linear form.

The spectral analysis of stochastic groundwater flow is based on the assumption of stationary log-hydraulic conductivity fields. It is assumed that the heterogeneities of the stationary log-hydraulic conductivity can be characterized statistically by the correlation scale of log-hydraulic conductivity. The assumption of no trend in log-hydraulic conductivity is often made, and the important results on the statistical characteristics of groundwater flow under stationary conditions can be found in the works of Bakr et al. (1978), Gelhar and Axness (1983), Dagan (1985), and Gelhar (1986). Dagan (1989) summarized the subject of stochastic analysis of the stationary groundwater flow. On the other hand, effects of the trend in the mean log hydraulic conductivity on head variance were examined by Smith and Freeze (1979), and Rejaram et al. (1990). Rehfeldt et al. (1992) analyzed the log hydraulic conductivity at the Columbus site and found that the spatial variability of log hydraulic conductivity can be explained when a spatial trend in the mean log conductivity is considered. Adams and Gelhar (1992) analyzed the motion of a tracer plume at the same site, and found that existing stochastic theories with the assumption of stationary mean log conductivity could not explain well the evolution of the spatial moments. Adams and Gelhar (1992) attributed this discrepancy to the presence of a trend. The non-stationary assumption of the log

G.-F. Lin (✉) · C.-M. Chen
Department of Civil Engineering,
National Taiwan University, Taipei 10617, Taiwan
e-mail: gflin@ccms.ntu.edu.tw

hydraulic conductivity has been introduced in recent researches (Gelhar, 1993; Loaiciga, 1993, 1994; Indelamn and Rubin, 1996). Gelhar (1993) considered the case that the trend in the mean log conductivity is parallel to the mean head gradient. Loaiciga et al. (1993, 1994) considered the case of an arbitrary angle between the head gradient and the trend in the mean log conductivity. However, the higher-order moments of the hydraulic head were not considered in the works of Gelhar (1993) and Loaiciga et al. (1993, 1994). In this paper, we introduce a trend in log hydraulic conductivity fields to approximate the nonstationary behavior.

A Gaussian process is completely characterized by its mean and spectrum, and the response from a linear system with Gaussian inputs remains Gaussian. The head variance produced by the bilinear system is zero. However, for a nonlinear filter, a Gaussian input, namely ln K, produces a non-Gaussian output. The head variance is contributed not only by the linear system but also by the bilinear system. For investigating the effects of linear and bilinear systems on head variance, one can use the nonlinear filter theory. Recent developments in the general theory of nonlinear filters (Tick, 1961; Priestley, 1988; Bendat, 1997) allow the extension of conventional spectral analysis of real physical systems to the more general case of nonlinearity. In general, a nonlinear relationship between a system input and an output may be expressed by a Volterra series of infinite terms (Priestley, 1988). The solution of a nonlinear differential equation can be expressed as the form of a Volterra series expansion (Subba Rao and Gabr, 1984). Literature reviews indicate previous researches have not dealt with the spectral and bispectral analyses of a nonlinear unconfined heterogeneous aquifer system using the nonlinear filter theory.

In this paper, an approach is developed for deriving the head variance of a nonlinear unconfined groundwater flow system using the nonlinear filter theory. The approach used in this paper is illustrated in Fig. 1. The influence of hydraulic log-conductivity variations on hydraulic head variance is investigated. The linear and quadratic frequency response functions are obtained from the first- and second-order perturbation equations, respectively. Furthermore, an analytical solution of the variance of hydraulic head is derived.

2 Volterra series representation of a general nonlinear system

In the case of a nonlinear physical system with stationary Gaussian inputs, the relationship between the input and output processes cannot be adequately described by a simple convolution integral. Likewise, the convolution linear frequency response function is not sufficient to characterize the nonlinear input/output relations. We consider a space-invariant, nonlinear physical system with a single input $I(x)$ and a single output $y(x)$.

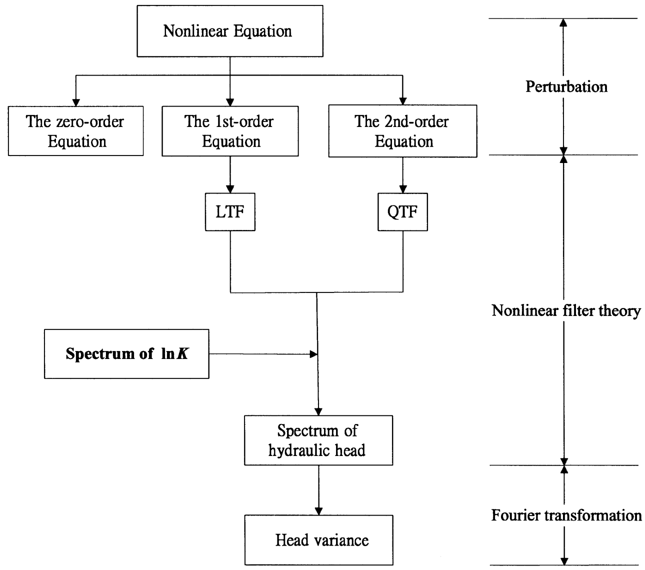

Fig. 1 The illustration of methodology used in this paper

A general relationship between $I(x)$ and $y(x)$ can be expressed in terms of a Volterra series of the form

$$\begin{aligned} y(x) = & \int_0^\infty \mathrm{Kn}_1(u) I(x-u) \mathrm{d}u \\ & + \int_0^\infty \int_0^\infty \mathrm{Kn}_2(u,v) I(x-u) I(x-v) \mathrm{d}u\,\mathrm{d}v \\ & + \int_0^\infty \int_0^\infty \int_0^\infty \mathrm{Kn}_3(u,v,w) I(x-u) I(t-v) I \\ & \times (x-w) \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}w + \cdots \end{aligned} \tag{1}$$

where $\mathrm{Kn}_1(u)$, $\mathrm{Kn}_2(u,v)$ and $\mathrm{Kn}_3(u,v,w)$ are the linear, bilinear and cubic impulse response functions (IRF), respectively. They can also be called the first-, second-, and third-order time domain kernels (Bendat, 1997) or functional response functions (Amorocho, 1963, 1973). A system that can be represented by a single term in the series expansion is homogeneous (Amorocho, 1973; Subba Rao and Gabr, 1984). For example, a first-order homogeneous system is called the conventional linear system or convolution integral, and a second-order homogeneous system is a bilinear system (Bendat, 1997). A system that can be represented by the first two terms is called a quadratic system (Priestley, 1988).

A system of the finite-order successive linear differential equation is often used to approximate a nonlinear input/output relationship in many physical systems. The solution of the equation can be represented by the linear, bilinear, cubic,..., terms in Eq. (1). Such a manner which was introduced by Wiener (1942, 1958) had also been explored in the hydrologic field by Jacby (1966), Amorocho (1963, 1973), Singh (1964), Napiorkowski and Strupczewski (1979, 1981), and Xia (1991). The temporally variable subsurface flow for an integral balance model has been investigated by Jin and Duffy (1994). In this paper, the spatially variable subsurface flow is studied.

Provided that both $I(x)$ and $y(x)$ are zero-mean stationary processes and only the linear and quadratic terms in Eq. (1) are dominant; then Eq. (1) becomes (Tick, 1961; Subba Rao and Gabr, 1984)

$$y(x) = \int_0^\infty \mathrm{Kn}_1(\alpha) I(x-\alpha)\mathrm{d}\alpha + \int_0^\infty\int_0^\infty \mathrm{Kn}_2(\alpha,\beta) \times \{I(x-\alpha)I(x-\beta) - R_{\mathrm{II}}(\alpha-\beta)\}\mathrm{d}\alpha\,\mathrm{d}\beta \quad (2)$$

The introduction of $R_{\mathrm{II}}(\alpha-\beta)$ in the second term on the right-hand side of Eq. (2) is to ensure $E[y(x)] = 0$.

3 Linear filter

3.1 Spectral representation of autocovariance functions for stationary processes

For a linear physical system with a random input $I(x)$ and a random output $y(x)$, if we assume that the system is causal without feedback, then the general input-output relationship can be expressed by the convolution integral (Priestley, 1981)

$$y_{1,\mathrm{I}}(x) = \int_0^\infty I(x-\tau)\mathrm{Kn}_1(\tau)\mathrm{d}\tau \quad (3)$$

If $I(x)$ is a zero-mean continuous parameter stationary process, then there exists a complex orthogonal process $Z_{\mathrm{I}}(f)$ such that any realization $I(x)$ can be expressed as a Fourier-Stieltjes transform of the form (Lumley and Panofsky, 1964)

$$I(x) = \int_{-\infty}^\infty e^{\mathrm{j}\alpha f}\,\mathrm{d}Z_{\mathrm{I}}(f) \quad (4)$$

where

$$E[\mathrm{d}Z_{\mathrm{I}}(f)] = 0 \quad (5)$$

and

$$E[\mathrm{d}Z_{\mathrm{I}}(f_1)\mathrm{d}Z_{\mathrm{I}}^*(f_2)] = \begin{cases} 0 & f_1 \neq f_2 \\ S_{\mathrm{II}}(f)\mathrm{d}f & f_1 = f_2 = f \end{cases} \quad (6)$$

where $\mathrm{d}Z_{\mathrm{I}}^*(f)$ denotes the complex conjugate of $\mathrm{d}Z_{\mathrm{I}}(f)$. Equations (5) and (6) state that any zero-mean stationary process can be represented by the sum of sine and cosine functions with random amplitude $|Z_{\mathrm{I}}(f)|$ and random phase arg $Z_{\mathrm{I}}(f)$.

According to the Wiener-Khintchine theorem (Priestley, 1981), the autocovariance function $R_{\mathrm{II}}(\alpha)$ of a continuous stationary random process can be expressed as a Fourier-Stieltjes transform

$$R_{\mathrm{II}}(\alpha) = E[I(x)I(x+\alpha)] = \int_{-\infty}^\infty S_{\mathrm{II}}(f)e^{\mathrm{j}\alpha f}\,\mathrm{d}f \quad (7)$$

where $\mathrm{j} = (-1)^{1/2}$; the α is the angular frequency, and the $S_{\mathrm{II}}(f)$ is the spectrum of $I(x)$ or the first-order autospectral density function of $I(x)$. The spectrum function is uniquely determined by the following transformation

$$S_{\mathrm{II}}(f) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{-\mathrm{j}\alpha f} R_{\mathrm{II}}(\alpha)\mathrm{d}\alpha \quad (8)$$

When $\alpha = 0$, the $S_{\mathrm{II}}(f)\mathrm{d}f$ is a frequency decomposition of variance because $R_{\mathrm{II}}(\alpha)$ is even and $S_{\mathrm{II}}(f)$ is real.

3.2 Linear filters and spectral analysis

Assuming that $I(x)$ is stationary and using the spectral representations for $I(x)$ and $y(x)$, one can obtain the following relationship

$$\mathrm{d}Z_{y_1}(f) = \mathrm{P}_{y_1,\mathrm{I}}(f)\mathrm{d}Z_{\mathrm{I}}(f) \quad (9)$$

where

$$\mathrm{P}_{y_1,\mathrm{I}}(f) = \int_0^\infty e^{\mathrm{j}f\tau}\mathrm{Kn}_1(\tau)\mathrm{d}\tau \quad (10)$$

The $\mathrm{P}_{y_1,\mathrm{I}}(f)$ is the linear frequency response function of the linear system. Using Eqs. (5) and (9), one can find the input–output spectral relationship

$$S_{yy}(f) = E\left[\mathrm{d}Z_{y_1}^*(f)\mathrm{d}Z_{y_1}(f)\right] = \left|\mathrm{P}_{y_1,\mathrm{I}}(f)\right|^2 S_{\mathrm{II}}(f) \quad (11)$$

where $\left|\mathrm{P}_{y_1,\mathrm{I}}(f)\right|^2$ is the transfer function (TF) of the linear system. The cross-spectrum for a linear system are given by

$$S_{\mathrm{I}y}(f) = \int_0^\infty e^{-\mathrm{j}f\tau}R_{\mathrm{I}y}(\tau)\mathrm{d}\tau \quad (12)$$

The cross spectrum is related to the complex processes $\mathrm{d}Z_{\mathrm{I}}(f)$ and $\mathrm{d}Z_y(f)$ by

$$E\left[\mathrm{d}Z_y(f_1)\mathrm{d}Z_{\mathrm{I}}^*(f_2)\right] = \begin{cases} S_{\mathrm{I}y}(f)\mathrm{d}f, & f_1 = f_2 = f \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

Thus, from Eqs. (13) and (9) one can obtain another important relation

$$S_{\mathrm{I}y}(f) = E\left[\mathrm{d}Z_{\mathrm{I}}^*(f)\mathrm{d}Z_{y_1}(f)\right] = \mathrm{P}_{y_1,\mathrm{I}}(f)S_{\mathrm{II}}(f) \quad (14)$$

Since $x(t)$ and $y(t)$ are real, the following symmetry relations hold:

$$\begin{aligned} S_{\mathrm{II}}(f) &= S_{\mathrm{II}}(-f) \\ S_{yy}(f) &= S_{yy}(-f) \\ S_{\mathrm{I}y}(f) &= S_{\mathrm{I}y}^*(-f) = S_{y\mathrm{I}}^*(f) \\ \mathrm{P}_{y_1,\mathrm{I}}(f) &= \mathrm{P}_{y_1,\mathrm{I}}(-f) \end{aligned} \quad (15)$$

where the asterisk denotes the complex conjugate.

4 Quadratic filter

4.1 An input/output quadratic system

Using the spectral representation for $I(x)$ and $y(x)$, one can rewrite Eq. (2) as

$$\mathrm{d}Z_y(f_1) = \mathrm{P}_{y_1,\mathrm{I}}(f_1)\mathrm{d}Z_\mathrm{I}(f_1) + \int_{\alpha=-\infty}^{\infty} \mathrm{P}_{y_2,\mathrm{I},\mathrm{I}}(f_1-\alpha,\alpha)\mathrm{d}Z_\mathrm{I}(f_1-\alpha)\mathrm{d}Z_\mathrm{I}(\alpha) - \mathrm{I}_{\mathrm{I}y}\delta(f_1)\mathrm{d}f_1 \quad (16)$$

where $\delta[f]$ is the Direct δ function,

$$\mathrm{Kn}_{Iy} = \int_{\alpha=-\infty}^{\infty} \mathrm{P}_{y_2,\mathrm{I},\mathrm{I}}(-\alpha,\alpha)S_{\mathrm{II}}(\alpha)\mathrm{d}\alpha \quad (17)$$

$$\mathrm{P}_{y_1,\mathrm{I}}(f_1) = \int_0^{\infty} \mathrm{e}^{-\mathrm{j}f_1\alpha}\mathrm{Kn}_1(\alpha)\mathrm{d}\alpha \quad (18)$$

and

$$\mathrm{P}_{y_2,\mathrm{I},\mathrm{I}}(f_1,f_2) = \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-\mathrm{j}(f_1\alpha+f_2\beta)}\mathrm{Kn}_2(\alpha,\beta)\mathrm{d}\alpha\,\mathrm{d}\beta \quad (19)$$

$\mathrm{P}_{y_1,\mathrm{I}}(f_1)$ and $\mathrm{P}_{y_2,\mathrm{I},\mathrm{I}}(f_1,f_2)$ are called the linear and quadratic frequency response functions, respectively. The $\mathrm{P}_{y_1,\mathrm{I}}(f_1)$ of Eq. (18) has the same form as the $\mathrm{P}_{y_1,\mathrm{I}}(f)$ of Eq. (10). The $\mathrm{P}_{y_2,\mathrm{I},\mathrm{I}}(f_1,f_2)\mathrm{d}Z_\mathrm{I}(f_1)\mathrm{d}Z_\mathrm{I}(f_2)$ represents the contribution of the components with frequencies f_1 and f_2 in $I(t)$ to the component with frequency f_1+f_2 in $y(x)$. Without loss of generality, one may assume that $\mathrm{P}_{y_2,\mathrm{I},\mathrm{I}}(f_1,f_2)$ is symmetric (Priestley, 1988). The symmetric relations are

$$\mathrm{P}_{y_2,\mathrm{I},\mathrm{I}}(f_1,f_2) = \mathrm{P}_{y_2,\mathrm{I},\mathrm{I}}(f_2,f_1) = \mathrm{P}^*_{y_2,\mathrm{I},\mathrm{I}}(-f_1,-f_2) \quad (20)$$

where the asterisk denotes the complex conjugate. According to Eq. (20), the $\mathrm{P}_{y_2,\mathrm{I},\mathrm{I}}(f_1,f_2)$ has to be evaluated within the domain $0 \le f_1 \le \infty$, $-f_1 \le f_2 \le f_1$. Such a quadratic system is characterized by the linear and quadratic FRF (or linear and quadratic IRF).

It is assumed that $I(x)$ is Gaussian with autocovariance function $R_{\mathrm{II}}(\tau)$ in order to obtain analytical expressions for the linear and quadratic FRF in terms of polyspectra and cross polyspectra. Under this assumption, the linear and quadratic terms in Eq. (2) or Eq. (16) are orthogonal or uncorrelated (Tick, 1961; Subba Rao and Gabr, 1984). Multiplying both sides of Eq. (9) by $\mathrm{d}Z_\mathrm{I}(-f)$, taking expectations, and then cancelling the common term $\mathrm{d}f$, one can obtain

$$S_{\mathrm{I}y}(f) = \mathrm{P}_{y_1,\mathrm{I}}(f)S_{\mathrm{II}}(f) \quad (21)$$

In a like manner, we use Eq. (21) and the orthogonal properties stated earlier. Then, multiplying both sides of Eq. (16) by $\mathrm{d}Z_\mathrm{I}(-f)\mathrm{d}Z_\mathrm{I}(-g)$, taking expectations, and cancelling the common term $\mathrm{d}f\ \mathrm{d}g$ on both sides, we can obtain

$$S_{\mathrm{II}y}(f,g) = 2\mathrm{P}_{y_2,\mathrm{I},\mathrm{I}}(f,g)S_{\mathrm{II}}(f)S_{\mathrm{II}}(g) \quad (22)$$

Using the symmetric relation in Eq. (20) and assuming that $x(t)$ is Gaussian, one can obtain following quadruple product

$$\begin{aligned} E[\mathrm{d}Z_\mathrm{I}(f_1)\mathrm{d}Z_\mathrm{I}(f_2)\mathrm{d}Z_\mathrm{I}(f_3)\mathrm{d}Z_\mathrm{I}(f_4)] &= E[\mathrm{d}Z_\mathrm{I}(f_1)\mathrm{d}Z_\mathrm{I}(f_2)]E[\mathrm{d}Z_\mathrm{I}(f_3)\mathrm{d}Z_\mathrm{I}(f_4)] \\ &+ E[\mathrm{d}Z_\mathrm{I}(f_1)\mathrm{d}Z_\mathrm{I}(f_3)]E[\mathrm{d}Z_\mathrm{I}(f_2)\mathrm{d}Z_\mathrm{I}(f_4)] \\ &+ E[\mathrm{d}Z_\mathrm{I}(f_1)\mathrm{d}Z_\mathrm{I}(f_4)]E[\mathrm{d}Z_\mathrm{I}(f_2)\mathrm{d}Z_\mathrm{I}(f_3)] \end{aligned} \quad (23)$$

Thus, one can estimate the linear and quadratic FRF from the input spectrum and the second- and third-order cross spectra.

Making use of orthogonality of the linear and quadratic terms in Eq. (2), along with expressions of Eqs. (21) and (22), one can obtain the output spectrum (Tick, 1961)

$$S_{yy}(f) = \left|\mathrm{P}_{y_1,\mathrm{I}}(f)\right|^2 S_{\mathrm{II}}(f) + 2\int_{\alpha=-\infty}^{\infty} \left|\mathrm{P}_{y_2,\mathrm{I},\mathrm{I}}(f-\alpha,\alpha)\right|^2 \times S_{\mathrm{II}}(f-\alpha)S_{\mathrm{II}}(\alpha)\mathrm{d}\alpha \quad (24)$$

Once the linear and quadratic frequency response functions are derived and the adequate input spectrum $S_{\mathrm{II}}(f)$ is used, the output spectrum $S_{yy}(f)$ can be obtained. Furthermore, the variance of output can be derived as well.

5 Stochastic one-dimensional groundwater flow equation

In general, the equation describing the unconfined-flow is nonlinear. In most of previous researches, the governing equation is assumed to be linear or can be reduced to linear form (Gelhar, 1974; Serrano, 1995). Serrano (1995) indicated that the linearized Boussinesq equation with Dupuit assumptions is a reasonable approximation to the exact solution for hydraulic head and the regional flow velocity. Using the Dupuit assumptions, one can express the steady one-dimensional flow equation for the unconfined aquifer without recharge as

$$\frac{\partial}{\partial x}\left[K(x)h(x)\frac{\partial h(x)}{\partial x}\right] = 0 \quad (25)$$

where $K(x)$ is the hydraulic conductivity, and $h(x)$ is the hydraulic head. We consider the flow of incompressible fluid in heterogeneous media with random conductivity. When the non-stationary hydraulic conductivity $K(x)$ has a trend, it can be written as (Rajaram, 1990; Loaiciga, 1993; Indelamn and Rubin 1996)

$$K(x) = \exp[Y(x)], \quad Y(x) = A\cdot x + y_K(x), \quad E[y_K(x)] = 0 \quad (26)$$

where A is a linear trend in mean log conductivity, and $y_k(x)$ is the fluctuation of the stationary log hydraulic conductivity. Substituting Eq. (26) into Eq. (25) yields

$$Ah\frac{\partial h}{\partial x} + \left(\frac{\partial h}{\partial x}\right)^2 + h\frac{\partial y_K}{\partial x}\frac{\partial h}{\partial x} + h\frac{\partial^2 h}{\partial x^2} = 0 \quad (27)$$

where $\ln K(x) = Y(x)$ is the log hydraulic conductivity. Using the perturbation method, one can express the hydraulic head in terms of a series (Indelamn and Rubin, 1996)

$$h(x) = h_0(x) + h_1(x) + h_2(x) + \cdots + h_n(x) + h_{n+1}(x) + \cdots, \quad \left\|\frac{h_n}{h_{n+1}}\right\| = O(\sigma_Y) \tag{28}$$

where σ_Y^2 is the variance of log hydraulic conductivity, and $E[h_1(x)] = 0$. We assume that the head is locally stationary, so that it has a spectral representation. The A and $-\partial h_0/\partial x$ are regarded as varying slowly in space, that is, the local homogeneity assumption are used in A and $-\partial h_0/\partial x$. Therefore, the mean of hydraulic head has the property: $E[h(x)] = E[h_0(x)] + E[h_2(x)]$. Substituting Eq. (28) into Eq. (27) and then ignoring the third-order term, one can obtain the zero-, first-, and second-order perturbation equations, respectively, as

$$Ah_0\frac{\partial h_0}{\partial x} + h_0\frac{\partial^2 h_0}{\partial x^2} = -\frac{\partial h_0}{\partial x}\frac{\partial h_0}{\partial x} \tag{29}$$

$$\left(A\frac{\partial h_0}{\partial x} + \frac{\partial^2 h_0}{\partial x^2}\right)h_1 + \left(Ah_0 + 2\frac{\partial h_0}{\partial x}\right)\frac{\partial h_1}{\partial x} + h_0\frac{\partial h_0}{\partial x}\frac{\partial y_K}{\partial x} + h_0\frac{\partial^2 h_1}{\partial x^2} = 0 \tag{30}$$

and

$$-AJh_2 + \left(\frac{\partial h_1}{\partial x}\right)^2 - 2J\frac{\partial h_2}{\partial x} = 0 \tag{31}$$

where $J = -\partial h_0/\partial x$. It should be noted that the σ_Y has to be less than unity, because of the perturbation approach. If the log hydraulic conductivity is stationary, then A is equal to zero in Eqs. (29)–(31).

6 Derivation of linear and quadratic FRF for a steady groundwater flow system

6.1 Linear frequency response function

Dividing Eq. (29) by h_0, substituting Eq. (29) into Eq. (30), and then using Eqs. (4) and (9), one can obtain the linear frequency response function:

$$P_1(f) = \frac{ifJh_0}{[-AJ - J' - i(fJ + Afh_0)] - f^2h_0 - ifJ} \tag{32}$$

where f is frequency, and $J' = -\partial^2 h_0/\partial x^2$. Gelhar (1977) obtained different expression of $P_1(f)$, because he linearized the log transmissivity and assumed it stationary. In his case, the terms in the square bracket of the denominator of Eq. (32) are neglected. Hence, ignoring the terms in the square bracket of Eq. (32) yields Gelhar's (1977) expression. Furthermore, substituting Eq. (32) into Eq. (11) with the spectrum of the log hydraulic conductivity, one can obtain the spectrum of hydraulic head.

6.2 Quadratic frequency response function

The spectral representation of Eq. (31) can be expressed as

$$-AJdZ_{h_2}(f_1) - f_1^2 dZ_{h_1}^2(f_1) + 2jJf_1 dZ_{h_2}(f_1) = 0 \tag{33}$$

Multiplying both sides of Eq. (33) by $\mathrm{d}Z_Y(f)\mathrm{d}Z_Y(g)$, taking expectations and then using the properties of Eqs. (20), (22), and (23), one can obtain

$$\mathrm{P}_2(f,g) = \frac{fg\mathrm{P}_1(f)\mathrm{P}_1(g)}{(2jJ(f+g) - AJ)} \tag{34}$$

7 Output spectral analysis

7.1 Variance produced from the linear system

If the spectrum of the log hydraulic conductivity is known, the spectrum of the hydraulic head can be obtained from Eq. (11). The exponential $\ln K$ covariance function is often assumed. It can be expressed as (Gelhar, 1993)

$$R_{YY}(\tau) = \sigma_Y^2 e^{-\frac{\tau}{\lambda}} \tag{35}$$

where λ is the correlation scale. The corresponding spectrum can be given as

$$S_{YY}(f) = \frac{\sigma_Y^2\lambda}{\pi(1+\lambda^2 f^2)} \tag{36}$$

where $S_{YY}(f)$ is the spectrum of the log hydraulic conductivity. According to the spectral theory, one can obtain the spectrum of hydraulic head as

$$S_{h_1h_1}(f) = f^2h_0^4J^4\sigma_Y^2\lambda/[\pi(f^4h_0^4 + 2f^2h_0^2J^2 + J^4 + f^2h_0^2J^2\mu^2) \times (J^2 + f^2h_0^2v^2)] \tag{37}$$

where $S_{h_1h_1}(f)$ is the spectrum of the hydraulic head, $\chi = Ah_0/J, |\mu| = |(Ah_0/J) - 2| = |\chi - 2|$ and $v = \lambda J/h_0$. Furthermore, the variance of hydraulic head, namely the Fourier transform of Eq. (37), can be obtained as

$$\sigma_{h_1}^2 = -\frac{h_0J\lambda\left(\sqrt{2+\mu^2-\mu\sqrt{4+\mu^4}} - \sqrt{2+\mu^2+\mu\sqrt{4+\mu^4}}\right)^3\sigma_Y^2}{\sqrt{2}\mu^3\sqrt{4+\mu^2}(2+\sqrt{2}\sqrt{2+\mu^2-\mu\sqrt{4+\mu^4}}v + \sqrt{2}\sqrt{2+\mu^2+\mu\sqrt{4+\mu^4}}v + 2v^2)} \tag{38}$$

where $\nu = \lambda J / h_0 \geq 0$ and $J \neq 0$. The ν and μ are dimensionless factors and they are proportional to λ and A, respectively. In Eq. (38), there exits a singularity of head variance at $\mu = 0$, i.e. $\chi = 2$. The head variance approaches zero as $\nu \to 0$. Figure 2 shows that the dimensionless head variance versus the variance of log hydraulic conductivity for various values of ν as $\chi = 0$, i.e. $A = 0$. The dimensionless head variance increases with increasing ν when σ_Y^2 is fixed. Figure 3 shows that dimensionless variance ratio versus ν with various values of χ. The result shows that the dimensionless variance ratio increases as ν increases. Ratio of the head variance to the variance of log hydraulic conductivity versus ν and χ is presented in Fig. 4. In Fig. 4, the dimensionless variance ratio increases with increasing χ for $\chi < 2$, but it decreases with increasing χ for $\chi > 2$. The maximum of dimensionless variance ratio occurs as $\chi \to 2$ when ν is fixed. The result shows that the effects of ν and χ on the dimensionless variance ratio are insignificant. That is, both the correlation scale and the trend in mean of log hydraulic conductivity are important to the dimensionless head variance.

If the log hydraulic conductivity is stationary (i.e. $A = 0$) and $h_0 \succ\succ J\lambda$, which is the typical situation in the field (Gelhar, 1993), then for the case of the exponential ln K covariance function, the variance of hydraulic head becomes

$$\sigma_h^2 = 0.3535\sigma_Y^2 J\lambda h_0 \tag{39}$$

When the exponential ln K covariance function is used, the variance of hydraulic head is proportional to the gradient of hydraulic head, the mean hydraulic head and the correlation scale as shown in Eq. (39). For the same case, the variance of hydraulic head derived by Gelhar (1977) is of the same form as Eq. (39) except with a constant coefficient of one. One can obtain the same equation derived by Gelhar (1977) if the terms in the bracket of the denominator of Eq. (32) is ignored. If the hydraulic head process is locally stationary, a general form propose by Gelhar (1993) is

$$\sigma_h^2 = C\sigma_Y^2 J^2 \lambda^2 \tag{40}$$

where C is a coefficient. The relationships for C are summarized in Table 1. Figure 5 gives the variances of hydraulic head obtained from Table 1. As indicated in Fig. 5, the variance obtained from Eq. (39) is 0. 3535 times that of Gelhar (1977).

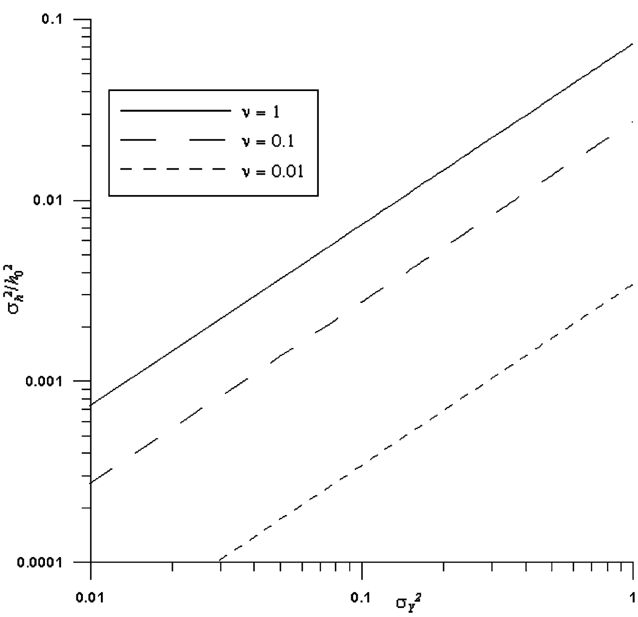

Fig. 2 Dimensionless head variance versus variance of log hydraulic conductivity for various ν as $\chi = 0$

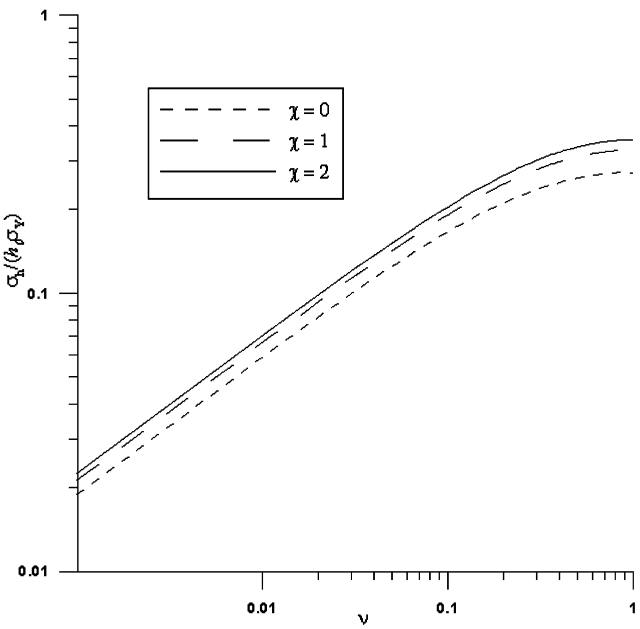

Fig. 3 Dimensionless variance ratio of hydraulic head to log hydraulic conductivity versus ν for various χ

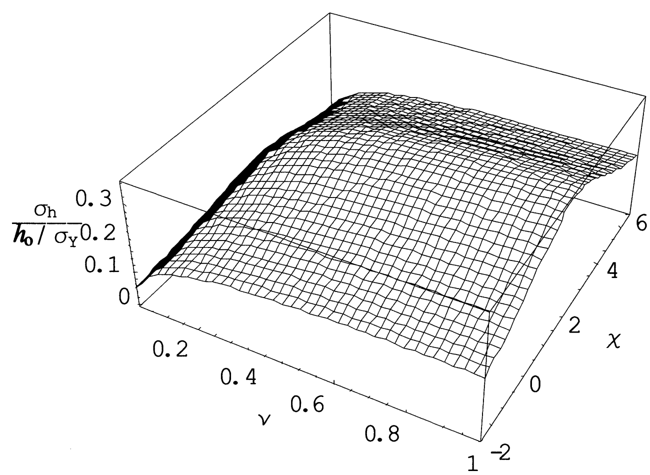

Fig. 4 Ratio of head variance to variance of log hydraulic conductivity versus ν and χ

7.2 Variance produced from the quadratic system

The first term of Eq. (24) has been derived in Sect. 7. 1, that is, Eq. (38). Furthermore, the second term of Eq. (24) can be derived with Eqs. (34) and (36), i. e. the quadratic response function and the spectrum of exponential $\ln K$ covariance function. Hence, the spectrum of hydraulic head produced from the quadratic system can be obtained by summing the first and second terms of Eq. (24). Consequently, using the Fourier transform, the variance of hydraulic head produced from the quadratic system can be derived as

Table 1 The relationship for coefficient C in Eq. (40)

Flow configuration		C	Note
Type of aquifer	Type of ln K covariance function		
Unconfined	Exponential	$0.3535\, h_0 J\lambda$	See (41)
Unconfined	Exponential	$B/[J\lambda(1+J\lambda/B)]$	See (4.3.4) of Gelhar (1993)

$$
\begin{aligned}
\sigma_{h_2}^2 = & -\frac{h_0 J\lambda\left(\sqrt{2+\mu^2-\mu\sqrt{4+\mu^4}}-\sqrt{2+\mu^2+\mu\sqrt{4+\mu^4}}\right)^3\sigma_Y^2}{\sqrt{2}\mu^3\sqrt{4+\mu^2}(2+\sqrt{2}\sqrt{2+\mu^2-\mu\sqrt{4+\mu^4}}v+\sqrt{2}\sqrt{2+\mu^2+\mu\sqrt{4+\mu^4}}v+2v^2)} \\
& +\frac{3h_0^2v^2\left(\sqrt{2+\mu^2}-\mu\right)^3\left(\sqrt{2+\mu^2}+\mu\right)^3\sigma_Y^4}{\left(\left(-4+v^2\left(\sqrt{2+\mu^2}+\mu\right)^2\right)(2+\mu^2)\mu^2\left(-4+v^2\left(\sqrt{2+\mu^2}-\mu\right)^2\right)(4(2+\mu^2)-\chi^2)\right)} \\
& -\frac{3h_0^2v^2\left(\sqrt{2+\mu^2}-\mu\right)^3\left(\sqrt{2+\mu^2}+\mu\right)^3\sigma_Y^4}{\left(\left(-4+v^2\left(\sqrt{2+\mu^2}+\mu\right)^2\right)(2+\mu^2)\mu^2\left(-4+v^2\left(\sqrt{2+\mu^2}-\mu\right)^2\right)(4(\mu^2)-\chi^2)\right)} \\
& -\frac{3h_0^2v^2\left(\sqrt{2+\mu^2}+\mu\right)^6\sigma_Y^4}{\left(\left(-4+v^2\left(\sqrt{2+\mu^2}+\mu\right)^2\right)^2(2+\mu^2)\mu^2\left(4\left(\sqrt{2+\mu^2}+\mu\right)^2-\chi^2\right)\right)} \\
& -6h_0^2v^2\left(\sqrt{2+\mu^2}+\mu\right)^3\left(\sqrt{2+\mu^2}+\mu-\chi\right)^4\sigma_Y^4\Big/\left(\left(-4+v^2\left(\sqrt{2+\mu^2}+\mu\right)^2\right)\right. \\
& \times\sqrt{(2+\mu^2)}\sqrt{\mu^2}\left(2\left(\sqrt{2+\mu^2}\right)-\chi\right)\left(-2\left(\sqrt{\mu^2}\right)+\chi\right)\left(2\left(\sqrt{2+\mu^2}\right)+2\sqrt{\mu^2}-\chi\right)\chi \\
& \times\left.\left(-4+\left(v\left(\sqrt{2+\mu^2}+\sqrt{\mu}-\chi\right)\right)^2\right)|\chi|\right)-6h_0^2v^2\left(\sqrt{2+\mu^2}+\mu\right)^3\left(\sqrt{2+\mu^2}+\mu+\chi\right)^4\sigma_Y^4\Big/ \\
& \left(\left(-4+v^2\left(\sqrt{2+\mu^2}+\mu\right)^2\right)\sqrt{(2+\mu^2)}\sqrt{\mu^2}\left(2\left(\sqrt{2+\mu^2}\right)+\chi\right)\left(2\left(\sqrt{\mu^2}\right)+\chi\right)\right. \\
& \times\left.\left(2\left(\sqrt{2+\mu^2}\right)+2\sqrt{\mu^2}+\chi\right)\chi\left(-4+\left(v\left(\sqrt{2+\mu^2}+\sqrt{\mu}+\chi\right)\right)^2\right)|\chi|\right)
\end{aligned}
\tag{41}
$$

Equation (41) shows that there exist singularities at $\chi = 0, 2$, and 4. The χ cannot be zero in Eq. (41), that is, A cannot be zero in Eq. (41). It should be noted that $A\lambda$ should be small relative to 1 (Gelhar, 1993), that is, $v\chi < 1$ in Eq. (41). Equation (41) can be rewritten as

$$\sigma_{h_2}^2 = \sigma_{h_1}^2\left(1-\Theta\sigma_Y^2\right) \tag{42}$$

From Eq. (42), we know that $\Theta\sigma_Y^2 \prec 1$ because of Eq. (28). For the case as $v = 0.01$ and $\chi = 5$, the variance of hydraulic head can be expressed as

$$\sigma_{h_2}^2 = 0.3535\,\sigma_Y^2 h_0^2(1-0.167\sigma_Y^2) \tag{43}$$

The dimensionless head variances produced by the linear system and the quadratic system are compared in Fig. 6. Under the conditions of $v = 0.01$ and $\chi = 5$, the difference between head variances produced respectively from the linear system and the quadratic systems are obvious when $J\lambda/h_0$ is lager than 0.1.

8 Summary and conclusions

In this paper, an approach to deriving the head variance for a nonlinear unconfined groundwater system is developed using the nonlinear filter theory. Most of previous researches did not derive the analytical solution up to the second-order perturbation on this problem. The linear and quadratic frequency functions of a phreatic aquifer are obtained. Furthermore, the

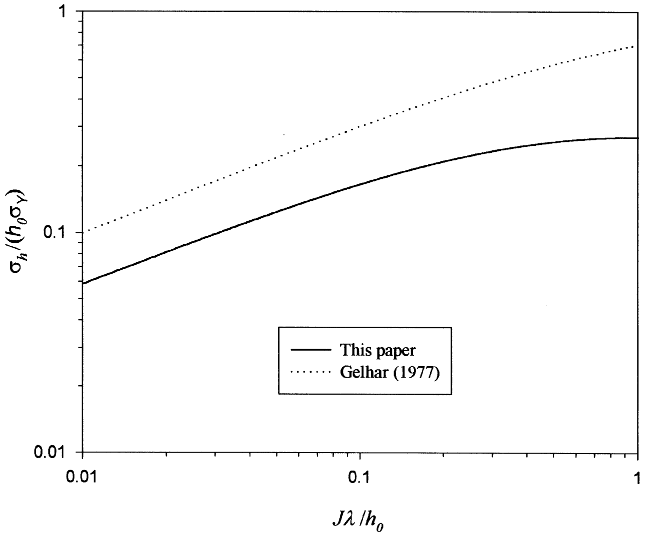

Fig. 5 Comparison of head-variance

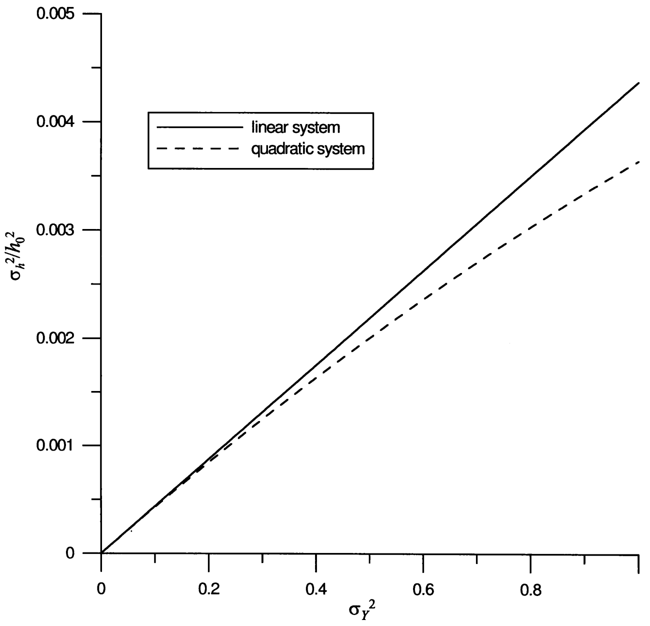

Fig. 6 Comparison of head variances produced by the linear system and the quadratic system for $v = 0.01$ and $\chi = 5$

analytical solutions of the head variance are respectively derived for the linear and quadratic systems with nonstationary log hydraulic conductivity. The major conclusions are summarized as follows:

A. For the linear system

1. Under the assumption of the exponential ln K covariance function, the head variance derived by Gelhar (1977) is of the same form as Eq. (14) except with a constant coefficient of one. The head variance derived in this paper is 0.3535 times than that of Gelhar (1977) as $h_0 \succ\succ J\lambda$. The difference is because the log transmissivity is linearized in Gelhar's work.
2. The linear frequency response function derived in this paper, i.e. Eq. (32), can be reduced to that of Gelhar (1977) if the terms in the square bracket of the denominator of Eq. (32) are ignored.
3. The head variance produced from the linear system exits a singularity at $\mu = 0$, i.e., $\chi = 2$. The head variance approaches zero as $v \to 0$.
4. The maximum of dimensionless variance ratio occurs as $\chi \to 2$ when v is fixed.

B. For the quadratic system

1. The head variance exists singularities at $\chi = 0, 2,$ and 4. The χ cannot be zero in Eq. (41), that is, the trend of log hydraulic conductivity (i.e. A) cannot be zero in Eq. (41).
2. The constraints for head variance are $\Theta\sigma_Y^2 \prec 1$ and $v\chi < 1$.

The effects of v and χ on the dimensionless variance ratio are insignificant, that is, both the correlation scale and the trend in mean of log hydraulic conductivity are important to the dimensionless variance ratio. The approach developed in this paper can be applied to the more complex problems.

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