Bounded real lemma and H_∞ control for descriptor systems

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Abstract: In this paper, the H_∞ control problem for descriptor systems is studied. Necessary and sufficient conditions are derived for the solution to this problem, expressed in terms of two generalised algebraic Riccati equations which may be considered to be the generalisations of the Riccati equations obtained by Doyle *et al.* (1989). When these conditions hold, state space formulae for a controller solving the problem is also given. The approach used in this paper is based on a generalised version of bounded real lemma, thus the proofs given are simple.

1 Introduction

H_∞ (sub)optimal control has become one of the most important notions in the field of automatic control theory. It has drawn considerable attention from many researchers around the world. Although H_∞ control theory has been perfectly developed over the last decade, most of the results were developed based on state space equations [1–3]. State space models are very useful, but the state variables thus introduced often do not provide a physical meaning [4]. In addition, state space equations cannot represent algebraic restrictions between state variables. Besides, some physical phenomena, like impulse and hysterisis which are important in circuit theory, cannot be treated properly in the state space models [5, 6].

Descriptor system representation provides a suitable way to handle such problems, and it has been proven in the literature that descriptor systems have higher capability to describe a physical system [5–7]. Descriptor system models appear more convenient and natural than state space models in large-scale systems, economics, networks, power, neural systems others [5, 7, 8].

The control theory based on descriptor system models has been widely developed for many years: Cobb first gave a necessary and sufficient condition for the existence of an optimal solution to the linear quadratic optimisation problem [9] and also extensively studied the notions of controllability, observability and duality in descriptor systems [10]. Lewis [5], Bender *et al.* [11] and Takaba *et al.* [4] constructed different kinds of Riccati equations for solving linear quadratic regulator problems based on certain assumptions. Some excellent results on pole placement [12] and robust control [13, 14], to name only a few, were also obtained.

Recently, Copeland and Safonov used the descriptor-system-like models to solve the singular H_2 and H_∞ control problems in which the plants have pure imaginary (including infinity) poles or zeros. [15]. Solutions to the H_∞ control problem for descriptor systems were given in Takaba *et al.* [4]. They dealt with the problem using a *J*-spectral factorisation, thus their proofs were involved. Moreover, only sufficient conditions for solutions to exist were given.

Most recently, Masubuchi *et al.* [16] have considered a similar problem by using a matrix inequalities approach. They treated a more general problem with less assumptions. Their solutions were obtained by use of a version of bounded real lemma and given in terms of linear matrix inequalities (LMI) which may be solved by existing numerical tools. However, they gave a necessary and sufficient condition in terms of two generalised algebraic Riccati inequalities (GARI) involving two unknown parameters plus two to-be-determined variables.

The present paper continues this line of research to study the H_∞ control problem for descriptor systems. More precisely, we present necessary and sufficient conditions for the existence of a solution to the problem. The main contribution of this paper is to give the solution to the H_∞ control problem in terms of two generalised algebraic Riccati equations (GARE) rather than inequalities. This may be considered to be the generalisation of the condition established in the celebrated paper by Doyle *et al.* [1] for systems in a state space model. We also construct a controller to solve the problem. The resulting controller thus obtained corresponds to the central controller given in [1].

Motivaton for developing a GARE solution to the H_∞ control problem of descriptor systems stems from the following. It has been shown by Glover and Mustafa [17] that the central controller given by Doyle *et al.* [1], expressed in terms of the solutions of two algebraic Riccati equations (ARE) minimises a certain entropy integral and thus the central controller would be the preferred controller to use in practical applications. It is expected that the controller obtained here

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(displayed in Theorem 6 below) also has a similar, minimum entropy, property. A full exploration in this direction is left for future research.

To estabilish a GARE solution to the H_∞ control problem, we first develop a version of bounded real lemma for descriptor systems in terms of GARE, rather than generalised algebraic Riccati inequality (GARI) as given in [16]. For systems in a state space model, various forms of bounded real lemma have been developed in the literature (e.g. Shi *et al.* [18] and the references therein). The bounded real lemma presented here (see Lemma 5 below) can be thought of as a descriptor version of Theorem 2.1 in Petersen *et al.* [2]. Actually, the underlying idea used in this paper is essentially the same idea used in Petersen *et al.* [2] to obtain the AREs solutions to the H_∞ control problem for systems in a state space model. More precisely, building on the GARI solution given in [16], we derive our GARE solution via the bounded real lemma presented here.

2 Preliminaries

In this Section, we will review some basic notions concerning descriptor systems. Consider a descriptor system described by the state equations:

$$E\dot{x} = Ax + Bu$$
$$y = Cx \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the input and output signals, respectively. A, B and C are constant matrices with compatible dimensions and E is a square matrix of rank $r < n$. $\{E, A\}$ is assumed to be regular. It is well known that a descriptor system contains three different modes: finite dynamic modes, impulsive modes and nondynamic modes. For a detailed definition see [11]. Briefly, let $q \triangleq \deg\det(sE - A)$. Then $\{E, A\}$ has q finite dynamic modes, $r - q$ impulsive modes and $n - r$ nondynamic modes. Furthermore if $r = q$, there exist no impulsive modes and in this case the system is said to be impulse-free.

$\{E, A\}$ is called stable if there exist no finite dynamic modes in $Re[s] \geq 0$. $\{E, A\}$ is admissible if $\{E, A\}$ is regular, impulse-free and stable. The triple $\{E, A, B\}$ is said to be finite dynamics stabilisable and impulse controllable if there exists a constant matrix K such that $\{E, A + BK\}$ is admissible. Similarly, $\{E, A, C\}$ is called finite dynamics detectable and impulse observable if a constant matrix L exists such that $\{E, A + LC\}$ is admissible. Without loss of generality, we can assume that the system (eqn. 1) has a Weierstrass form [19]:

$$E = \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix},\ A = \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix},$$
$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ \text{and } C = [\, C_1 \quad C_2 \,] \qquad (2)$$

where N is a nilpotent matrix (that is, $N^k = 0$ for some positive integer k).

The following proposition builds a connection between least square optimisation problems and generalised algebraic Riccati equations.

Proposition 1: Consider eqn. 1. Suppose that $\{E, A\}$ is regular, impulse-free and that $\{E, A, B\}$ is finite dynamics stabilisable and impulse controllable. Suppose that $I - B_2^T C_2^T C_2 B_2 > 0$. Furthermore, assume that the Hamiltonian system:

$$\begin{bmatrix} E & 0 \\ 0 & E_T \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & BB^T \\ -C^T C & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \qquad (3)$$

is regular, impulse-free and has no finite dynamic modes on the imaginary axis. Then there exists an admissible solution X to the GARE:

$$\begin{cases} A^T X + X^T A + C^T C + X^T B B^T X = 0 \\ E^T X = X^T E \end{cases}$$

Remark: Here a solution X to the GARE is called an admissible solution if $\{E, A + B B^T X\}$ is admissible. It is noted that X might not be unique, but $E^T X = X^T E$ is unique (for details, see [4, 14, 20] and the references quoted therein).

Proof: consider a linear dynamical system of the form,

$$\dot{x} = -x + B_2 u$$
$$y = C_2 x$$

It is easy to verify that $I - B_2^T C_2^T C_2 B_2 > 0$ if and only if the system is strictly bounded real with an upper bound 1. Then, using a standard result from algebraic Riccati equations (ARE) ([21], Theorem 2.3.1), a stabilising solution $X_{22} = X_{22}^T \geq 0$ exists satisfying the ARE:

$$(-X) + (-X) + C_2^T C_2 + X B_2 B_2^T X = 0$$

(Note that a solution to ARE $A^T X + XA + Q + X RX = 0$ is said to be stabilising if the matrix $(A + RX)$ is stable.) Set

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

compatible with eqn. 2. Then, eqn. 3 can be rearranged in the following form:

$$\left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \dot{x}_1 \\ \dot{\lambda}_1 \\ \dot{x}_2 \\ \dot{\lambda}_2 \end{bmatrix}$$
$$= \left[\begin{array}{cc|cc} A_1 & B_1 B_1^T & B_1 B_2^T & 0 \\ -C_1^T C_1 & -A_1^T & 0 & -C_1^T C_2 \\ \hline 0 & B_2 B_1^T & B_2 B_2^T & I \\ C_2^T C_1 & 0 & I & C_2^T C_2 \end{array}\right] \begin{bmatrix} x_1 \\ \lambda_1 \\ x_2 \\ \lambda_2 \end{bmatrix}$$
$$\triangleq \begin{bmatrix} \mathcal{I} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\xi} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_2 \\ \mathcal{T}_3 & \mathcal{T}_4 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}$$

Since, by hypothesis $I - B_2^T C_2^T C_2 B_2 > 0$, this implies that $\mathcal{T}_4$ is nonsingular. Hence, the above system can be simplified as:

$$\dot{\xi} = [\mathcal{T}_1 - \mathcal{T}_2 \mathcal{T}_4^{-1} \mathcal{T}_3]\, \xi \triangleq \dot{\xi} = \begin{bmatrix} \mathcal{A}_0 & \mathcal{R}_0 \\ -\mathcal{Q}_0 & -\mathcal{A}_0^T \end{bmatrix} \xi \quad (4)$$

It is easy to verify that $\mathcal{R}_0 \geq 0$ (since $I - B_2^T C_2^T C_2 B_2 > 0$) and $\{\mathcal{A}_0, \mathcal{R}_0\}$ is stabilisable (since $\{E, A, B\}$ is finite dynamic stabilisable). Then, by a standard result from ARE ([21], Lemma A.2.3), this implies that a stabilising solution $X_0 = X_0^T \geq 0$ to the ARE $S(X_0) \triangleq \mathcal{A}_0^T X_0 + X_0 \mathcal{A}_0 + \mathcal{Q}_0 + X_0 \mathcal{R}_0 X_0 = 0$ exists. Set:

$$X_{21} = L_2 - X_{22} L_1 \text{ and } X_E = \begin{bmatrix} X_0 & 0 \\ X_{21} & X_{22} \end{bmatrix}$$

where

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \triangleq -\mathcal{T}_4^{-1} \mathcal{T}_3 \begin{bmatrix} I \\ X_0 \end{bmatrix}$$

It now follows that the GARE $A^T X_E + X_E^T A + C^T C + X_E^T B B^T X_E = 0$, $E^T X_E = X_E^T E \geq 0$ with $\{E, A + B B^T X_E\}$ admissible. This completes the proof. Q.E.D.

The proof is constructive (i.e. a procedure is presented for obtaining an admissible solution to a GARE). Some relevant results can also be found in [14, 20, 22].

The following result, essentially taken from [16, 23], is a version of Lyapunov stability theorem for descriptor systems.

Proposition 2: Consider the descriptor system (eqn. 1). Suppose that $\{E, A\}$ is regular. Then we have the following.

(i) Suppose that $\{E, A, C\}$ is finite dynamics detectable and impulse observable. Then $\{E, A\}$ is stable and impulse-free if and only if there exists a matrix X satisfying the generalised Lyapunov inequality:

$$A^T X + X^T A + C^T C \leq 0, \quad E^T X = X^T E \geq 0$$

(ii) Suppose there exists a matrix P satisfying the generalised Lyapunov inequality:

$$A^T P + P^T A < 0, \quad E^T P = P^T E \geq 0$$

Then $\{E, A\}$ is stable and impulse-free.

The following two lemmas summarise some properties relevent to the GARE, GARI, and the Hamiltonian pencil.

Lemma 3: Suppose that $\{E, A\}$ is impulse-free. Suppose that $Q = Q^T$, $R \geq 0$ and the pencil $P_2(s) \triangleq s[-E\ 0] + [A\ R]$ has full row rank on the imaginary axis. Furthermore, suppose that GARI:

$$\begin{cases} A^T P + P^T A + Q + P^T R P < 0 \\ E^T P = P^T E \end{cases} \tag{5}$$

has a solution P with $E^T P = P^T E \geq 0$. Under these conditions, the Hamiltonian pencil:

$$s\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} - \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} \triangleq s\bar{E} - \bar{H} \tag{6}$$

has no pure imaginary zeros and $\{\bar{E}, \bar{H}\}$ is impulse-free.

Proof: Without loss of generality, we take $\{E, A\}$ in the following form,

$$\{E, A\} = \left\{ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_1 & 0 \\ 0 & I \end{bmatrix} \right\} \tag{7}$$

and

$$P = \begin{bmatrix} P_{11} & 0 \\ P_{21} & -P_{22} \end{bmatrix} \text{ with } P_{11} > 0, P_{22} > 0 \tag{8}$$

See also [16]. Set

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}, \ B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

and

$$R = BB^T = \begin{bmatrix} B_1 B_1^T & B_1 B_2^T \\ B_2 B_1^T & B_2 B_2^T \end{bmatrix}$$

where the partition is compatible with eqn. 7, and set:

$$S = A^T P + P^T A + Q + P^T B B^T P = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

Then $S < 0$ by hypothesis. we first show that the Hamiltonian pencil is column-reduced. This is equivalent to showing that:

$$H_{22} \triangleq \begin{bmatrix} -I & B_2 B_2^T \\ -Q_{22} & I \end{bmatrix} \tag{9}$$

is nonsingular. Observe now that:

$$\begin{aligned} S_{22} &= -P_{22} - P_{22} + Q_{22} + P_{22} B_2 B_2^T P_{22} \\ &= (-I)^T P_{22} + P_{22}(-I) + Q_{22} + P_{22} B_2 B_2^T P_{22} \\ &< 0 \end{aligned}$$

and $[I - jwI\ R_{22}]$ has full row rank for all $w \in \mathbb{R}$. Then, by standard results of algebraic Riccati inequality (ARI), this implies that eqn. 9 has no eigenvalues on the jw-axis (see [21], Lemma A.2.4) (i.e. H_{22} is nonsingular).

We now show that the Hamiltonian pencil has no pure imaginary zeros. We assume, for convenience, that all signals may be complex (i.e. C^n) at this time. Observe the following identity:

$$\begin{aligned} x^*(A^T P + P^T A)x &+ x^* Q x - u^* u \\ &\quad + x^* P^T B u + u^* B^T P x \\ &= x^*(A^T P + P^T A + Q + P^T B B^T P)x \\ &\quad - (u - B^T P)^*(u - B^T P) \\ &= x^* S x - (u - B^T P)^*(u - B^T P) \leq x^* S x \end{aligned} \tag{10}$$

Consider the descriptor system:

$$E\dot{x} = Ax + Bu \tag{11}$$

Choose an input $u(\bullet)$, an initial condition $Ex(0) \in C^n$, and let $x(\bullet)$ denote the corresponding solution. Observing that:

$$\begin{aligned} &\frac{d\left(x^*(t) P^T E x(t)\right)}{dt} \\ &= x^*(A^T P + P^T A)x + x^* P^T B u + u^* B^T P x \end{aligned}$$

and using eqn. 10 yields

$$\frac{d\left(x^*(t) P^T E x(t)\right)}{dt} + x^* Q x - u^* u \leq x^* S x \tag{12}$$

Suppose, by contradiction, the pencil has a zero on the imaginary axis. By definition, there exist vectors $x_0 \in C^n$, $p_0 \in C^n$ and a number $w_0 \in \mathbb{R}$ such that:

$$\begin{bmatrix} A - j\omega_0 E & R \\ -Q & -A^T - j\omega_0 E^T \end{bmatrix} \begin{bmatrix} x_0 \\ p_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{13}$$

Note that $x_0 \neq 0$. Otherwise,

$$Rp_0 = 0$$

$$\left(A^T + j\omega_0 E^T\right) p_0 = 0$$

This leads to a contradiction, because, by hypothesis, $P_2(s)$ has full row rank. From eqn. 13, we have:

$$p_0^* (A - j\omega_0 E) x_0 + p_0^* R p_0 = 0$$

$$x_0^* Q x_0 + x_0^* (A - j\omega_0 E)^* p_0 = 0$$

and therefore:

$$x_0^* Q x_0 - p_0^* R p_0 = 0$$

Set $u(t) = B^T p_0 e^{jw_0 t}$ and note that $Ex(t) = Ex_0 e^{jw_0 t}$ is the solution of [21] satisfying $Ex(0) = Ex_0$. Then:

$$x^*(t) P^T E x(t) = x_0^* P^T E x_0$$

$$x^*(t) Q x(t) - u^*(t) u(t) = x_0^* Q x_0 - p_0^* R p_0 = 0$$

$$x^*(t) S x(t) = x_0^* S x_0$$

The inequality (eqn. 12) yields $x_0^* S x_0 \geq 0$ which is a contradiction, because S is negative definite and $x_0 \neq 0$. Q.E.D.

Lemma 4: Suppose that $\{E, A\}$ is impulse-free. Suppose that $Q \geq 0$, $R = R^T$ and the pencil

$$P_1(s) \triangleq s\begin{bmatrix} -E \\ 0 \end{bmatrix} + \begin{bmatrix} A \\ Q \end{bmatrix}$$

has full column rank on the imaginary axis. Furthermore, suppose that GARI (eqn. 5) has a solution P with $E^T P = P^T E \geq 0$. Under these conditions, there exists an admissible solution X_E to the GARE:

$$\begin{cases} A^T X + X^T A + Q + X^T R X = 0 \\ E^T X = X^T E \end{cases} \tag{14}$$

with $E^T X_E - X_E^T E \geq 0$ and having the property that $\{E, A + RX_E\}$ is admissible. Moreover, $\rho(P) > \rho(X_E)$.

Proof: The hypothesis on GARI (eqn. 5) implies that:

$$s\begin{bmatrix} E^T & 0 \\ 0 & E \end{bmatrix} - \begin{bmatrix} A^T & Q \\ -R & -A \end{bmatrix} = 0$$

has no zeros on the jw-axis and is column-reduced. It follows that:

$$s\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} - \begin{bmatrix} A & R \\ -Q & -A^T \end{bmatrix} = 0 \tag{15}$$

has no zeros on the jw-axis and is column-reduced. Set:

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix} \text{ and } R = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{bmatrix}$$

where the partition is compatible with eqn. 7. From eqn. 5, there exists a positive definite matrix P_{22} such that:

$$(-I)P_{22} + P_{22}(-I) + Q_{22} + P_{22}^T R_{22} P_{22} < 0 \tag{16}$$

$[I - jwI\ Q_{22}]^T$ has full column rank for all $w \in \mathbb{R}$. This, again from the standard results of ARI (see [21], Lemma A.2.5), implies that there exists a matrix $X_{22} = X_{22}^T \geq 0$ satisfying the ARE:

$$-X_{22} - X_{22} + Q_{22} + X_{22} R_{22} X_{22} = 0$$

with $\rho(P_{22}) > \rho(X_{22})$ [2, 21]. Rewrite the pencil (eqn. 15) as:

$$s\left[\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] - \left[\begin{array}{cc|cc} A_1 & R_{11} & R_{12} & 0 \\ -Q_{11} & -A_1^T & 0 & -Q_{12} \\ \hline 0 & R_{12}^T & R_{22} & I \\ Q_{12}^T & 0 & I & Q_{22} \end{array}\right]$$

$$\triangleq s\begin{bmatrix} \mathcal{I} & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathcal{T}_1 & \mathcal{T}_2 \\ \mathcal{T}_3 & \mathcal{T}_4 \end{bmatrix}$$

From ARI (eqn. 16), the above pencil can be simplified as:

$$s\mathcal{I} - [\mathcal{T}_1 - \mathcal{T}_2\mathcal{T}_4^{-1}\mathcal{T}_3] \triangleq s\mathcal{I} - \begin{bmatrix} \mathcal{A}_0 & \mathcal{R}_0 \\ -\mathcal{Q}_0 & -\mathcal{A}_0^T \end{bmatrix}$$

The existence of P_{22} to ARI (eqn. 16) implies that $\mathcal{Q}_0 \geq 0$ (See Willems [24], Lemma 1). Moreover, GARI (eqn. 5) and ARI (eqn. 16), together, imply that there exists a positive definite matrix P_0 $(= P_{11})$ satisfying:

$$S(P_0) \triangleq \mathcal{A}_0^T P_0 + P_0\mathcal{A}_0 + \mathcal{Q}_0 + P_0\mathcal{R}_0 P_0 < 0 \tag{17}$$

This, together with the hypothesis on pencil $P_1(s)$, implies that a stabilising solution $X_0 \geq 0$ exists satisfying the ARE $S(X_0) = 0$ since one can deduce that $[\mathcal{A}_0^T - jwI\ \mathcal{Q}_0]^T$ has full column rank for all $w \in \mathbb{R}$ from the assumption that $[A^T - jwE^T\ Q]^T$ has full column rank on the jw-axis. Note that $\rho(P_0) > \rho(X_0)$. Set:

$$X_{21} = L_2 - X_{22}L_1 \text{ and } X_E = \begin{bmatrix} X_0 & 0 \\ X_{21} & X_{22} \end{bmatrix}$$

where

$$\begin{bmatrix} L_1 \\ L_2 \end{bmatrix} \triangleq -\mathcal{T}_4^{-1}\mathcal{T}_3 \begin{bmatrix} I \\ X_0 \end{bmatrix}$$

It is easy to see that X_E satisfies the GARE (eqn. 14) with $\{E, A + RX_E\}$ admissible and $\rho(P) > \rho(X_E)$. This completes the proof. Q.E.D.

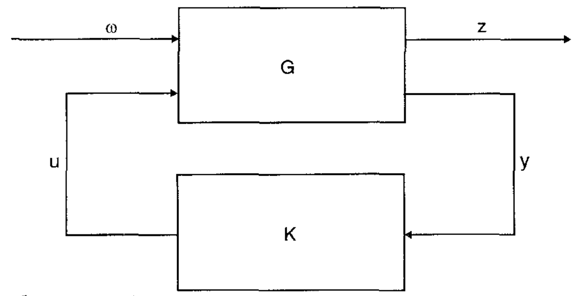

Fig. 1 *Standard block diagram*

3 Problem formulation

Consider the standard feedback configuration shown in Fig. 1. Let the plant G be described by:

$$\begin{aligned} E\dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned} \tag{18}$$

where $x \in \mathbb{R}^n$ is the state, and $w \in \mathbb{R}^m$ represents a set of exogenous inputs which includes disturbances to be rejected and/or reference commands to be tracked. $z \in \mathbb{R}^p$ is the output to be controlled and $y \in \mathbb{R}^q$ is the measured output. $u \in \mathbb{R}^l$ is the control input. A, B_1, B_2, C_1, C_2, D_{12}, and D_{21} are constant matrices with compatible dimensions. $E \in \mathbb{R}^{n \times n}$ and rank$E = r < n$.

The standard H_∞ control problem for descriptor systems consists of finding a controller K of the form:

$$\begin{aligned} \hat{E}\dot{\xi} &= \hat{A}\xi + \hat{B}y \\ u &= \hat{C}\xi \end{aligned} \tag{19}$$

where $\hat{E}$, $\hat{A} \in \mathbb{R}^{n \times n}$, $\hat{B} \in \mathbb{R}^{n \times q}$ and $\hat{C} \in \mathbb{R}^{l \times n}$, such that the resulting closed-loop system is internally stable and T_{zw}, the closed-loop system from w to z, has H_∞ norm strictly less than a prescribed positive number γ. Here closed-loop internal stability means that the closed-loop system is regular and impulse-free, and that the states of G and K go to zero from all initial values when $w = 0$. Note that we do not assume *a priori* structure of the matrix $\hat{E}$; it may be singular or nonsingular, equal to E or not.

The system (eqn. 18) is assumed to satisfy the following assumptions, see also [4].

(A1) $\{E, A\}$ is regular.

(A2) $\{E, A, B_2\}$ is finite dynamics stabilisable and impulse controllable.

(A3) $\{E, A, C_2\}$ is finite dynamics detectable and impulse observable.

(A4)

$$\begin{bmatrix} A - j\omega E & B_1 \\ C_2 & D_{21} \end{bmatrix}$$

has full row rank for all $w \in \mathbb{R}$ and is row reduced.

(A5)

$$\begin{bmatrix} A - j\omega E & B_2 \\ C_1 & D_{12} \end{bmatrix}$$

has full column rank for all $w \in \mathbb{R}$ and is column reduced.

(A6) $R_1 \triangleq D_{12}^T D_{12} > 0$.

(A7) $R_2 \triangleq D_{21} D_{21}^T > 0$.

4 Main results

In this Section, we first give a version of bounded real lemma for descriptor systems in terms of GARE. This forms the basis of our solutions to the H_∞ control problem.

Lemma 5 (bounded real lemma): Consider eqn. 1. The following statements are equivalent.

(i) $\{E, A\}$ is stable, impulse-free amd $\|G(s)\|_\infty < \gamma$, where:

$$G(s) \triangleq C(sE - A)^{-1}B$$

(ii) $\{E, A\}$ is stable, impulse-free and $\gamma^2 I - B_2^T C_2^T C_2 B_2 > 0$. Furthermore, the Hamiltonian system

$$\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & \frac{1}{\gamma^2}BB^T \\ -C^TC & -A^T \end{bmatrix}\begin{bmatrix} x \\ \lambda \end{bmatrix}$$

is regular, impulse-free and has no finite dynamic modes on the imaginary axis.

(iii) There exists an admissible solution to the GARE

$$\begin{cases} A^TX + X^TA + C^TC + \frac{1}{\gamma^2}X^TBB^TX = 0 \\ E^TX = X^TE \geq 0 \end{cases} \quad (20)$$

Proof:

(i) $\Rightarrow$ (ii).

Without loss of generality, we can assume that the system (eqn. 1) has a Weierstrass form (eqn. 2). Since $\{E, A\}$ is impulse-free, $N = 0$. Writing

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ and } \lambda = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

compatible with eqn. 2, the Hamiltonian system can be put in the form:

$$\begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \dot{x}_1 \\ \dot{\lambda}_1 \\ \dot{x}_2 \\ \dot{\lambda}_2 \end{bmatrix} =$$

$$\begin{bmatrix} A_1 & \frac{1}{\gamma^2}B_1B_1^T & 0 & \frac{1}{\gamma^2}B_1B_2^T \\ -C_1^TC_1 & -A_1^T & -C_1^TC_2 & 0 \\ 0 & \frac{1}{\gamma^2}B_2B_1^T & I & \frac{1}{\gamma^2}B_2B_2^T \\ -C_2^TC_1 & 0 & -C_2^TC_2 & -I \end{bmatrix}\begin{bmatrix} x_1 \\ \lambda_1 \\ x_2 \\ \lambda_2 \end{bmatrix}$$

This system is regular and impulse-free since $\|G(s)\|_\infty < \gamma$ implies $\gamma^2 I - B_2^T C_2^T C_2 B_2 > 0$. This implies that:

$$\begin{bmatrix} I & \frac{1}{\gamma^2}B_2B_2^T \\ -C_2^TC_2 & -I \end{bmatrix}$$

is nonsingular. Furthermore, it is easy to see that:

$$det\left(s\begin{bmatrix} E & 0 \\ 0 & E^T \end{bmatrix} - \begin{bmatrix} A & \frac{1}{\gamma^2}BB^T \\ -C^TC & -A^T \end{bmatrix}\right) =$$

$$det(sE - A)det(sE^T + A^T)det(\gamma^2 I - G^*(s)G(s))$$

It follows that the characteristic polynomial has no pure imaginary roots.

(ii) $\Rightarrow$ (iii).

This part follows immediately by Proposition 1.

(iii) $\Rightarrow$ (i).

Suppose, by hypothesis, that there exists an admissible solution to the GARE:

$$\begin{cases} A^TX + X^TA + C^TC + \frac{1}{\gamma^2}X^TBB^TX = 0 \\ E^TX = X^TE \geq 0 \end{cases} \quad (21)$$

Set $\tilde{C} = 1/\gamma B^T X$. Since $\{E, A + 1/\gamma^2 B B^T X\}$ is admissible, $\{E, A, \tilde{C}\}$ is finite dynamics detectable and impulse observable. Rearrange eqn. 21 as:

$$\begin{cases} A^TX + X^TA + \tilde{C}^T\tilde{C} = -C^TC \leq 0 \\ E^TX = X^TE \geq 0 \end{cases}$$

which is a generalised Lyapunov inequality. Then, by Proposition 2, $\{E, A\}$ is stable and impulse-free. Next, we show that $\|C(sE - A)^{-1} B\|_\infty < \gamma$. Observe that eqn. 20 implies that:

$$\frac{1}{\gamma^2}B^T\left(-j\omega E - A^T\right)^{-1}C^TC\left(j\omega E - A\right)^{-1}B$$
$$= I - \left[I - \frac{1}{\gamma^2}B^TX\left(-j\omega E - A\right)^{-1}B\right]^T$$
$$\times \left[I - \frac{1}{\gamma^2}B^TX\left(j\omega E - A\right)^{-1}B\right] \quad (22)$$

for all $w \geq 0$. It follows that $\|C(sE - A)^{-1} B\|_\infty \leq \gamma$. To complete the proof, we need to show that the strict inequality holds. Suppose, by contradiction, that there exists an $\bar{w} \geq 0$ such that $\|C(j\bar{w}E - A)^{-1} B\|_\infty = \gamma$. Then eqn. 22 implies that there exists a vector p_0 such that $[I - B^T X(j\bar{w}E - A)^{-1} B]p_0 = 0$. This gives $\det[I - B^T X(j\bar{w}E - A)^{-1} B] = 0$. Now, by a standard result on determinants, we have:

$$\det\left[j\bar{\omega}E - A - \frac{1}{\gamma^2}BB^TX\right]$$
$$= \det\left[j\bar{\omega}E - A\right]\det\left[I - \frac{1}{\gamma^2}B^TX\left(j\bar{\omega}E - A\right)^{-1}B\right]$$

Thus $\det[j\bar{w}E - A - 1/\gamma^2 B B^T X] = 0$. This contradicts the fact that X is an admissible solution to GARE (eqn. 20). Hence, we can conclude that $\|C(sE - A)^{-1} B\|_\infty < \gamma$. This completes the proof. Q.E.D.

We are now in a position to give our main result, which is summarised in the following statements.

Theorem 6: Consider eqn. 18. Suppose that assumptions (A1)–(A7) hold. Then there exists a controller of the form (eqn. 19) that internally stabilises eqn. 18 and render $\|T_{zw}\|_\infty < \gamma$ if and only if the following conditions are satisfied.

(i) There exists an admissible solution X_∞ to the GARE:

$$\begin{aligned} R_1(x) = &\left(A - B_2R_1^{-1}D_{12}^TC_1\right)^TX \\ &+ X^T\left(A - B_2R_1^{-1}D_{12}^TC_1\right) \\ &+ C_1^T\left(I - D_{12}R_1^{-1}D_{12}^T\right)C_1 \\ &+ X^T\left(\frac{1}{\gamma^2}B_1B_1^T - B_2R_1^{-1}B_2^T\right)X = 0 \\ &E^TX = X^TE \end{aligned}$$

with $E^T X_\infty = X_\infty^T E \geq 0$.

(ii) There exists an admissible solution Z_∞ to the GARE:

$$\begin{aligned} R_3(Z) = &\left(\tilde{A} - B_1D_{21}^TR_2^{-1}\tilde{C}_2\right)Z \\ &+ Z^T\left(\tilde{A} - B_1D_{21}^TR_2^{-1}\tilde{C}_2\right)^T \end{aligned}$$

$$- Z^T\left(\tilde{C}_2^T R_2^{-1}\tilde{C}_2 - \frac{1}{\gamma^2}F_\infty^T R_1 F_\infty\right)Z$$
$$+ \tilde{B}_1\tilde{B}_1^T = 0$$
$$EZ = Z^T E^T \tag{23}$$

with $EZ_\infty = Z_\infty^T E^T \geq 0$, where:

$$\tilde{A} = A + \frac{1}{\gamma^2}B_1 B_1^T X_\infty$$
$$\tilde{C}_2 = C_2 + \frac{1}{\gamma^2}D_{21}B_1^T X_\infty$$
$$\tilde{B}_1 = B_1\left(I - D_{21}^T R_2^{-1}D_{21}\right)$$

When these conditions hold, one such controller is given by:

$$\hat{E} = E$$
$$\hat{A} = A_k$$
$$\triangleq A + B_2\hat{C} - \hat{B}C_2 + \frac{1}{\gamma^2}\left(B_1 - \hat{B}D_{21}\right)B_1^T X_\infty$$
$$\hat{B} = B_k$$
$$\triangleq \left(Z_\infty^T C_2^T + \left(I + \frac{1}{\gamma^2}X_\infty^T Z_\infty\right)^T B_1 D_{21}^T\right)R_2^{-1}$$
$$\hat{C} = C_k$$
$$\triangleq F_\infty = -R_1^{-1}\left(B_2^T X_\infty + D_{21}^T C_1\right) \tag{24}$$

Proof

Sufficiency: We will show that the controller defined in eqn. 24 both stabilises the system and makes $\|T_{zw}\|_\infty < \gamma$. Observe that the resulting closed-loop system can be written as:

$$\underbrace{\begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}}_{E_c}\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A + B_2C_k & -B_2C_k \\ A - A_k + B_2C_k - B_kC_2 & A_k - B_2C_k \end{bmatrix}}_{A_c}\begin{bmatrix} x \\ e \end{bmatrix} + \underbrace{\begin{bmatrix} B_1 \\ B_1 - B_kD_{21} \end{bmatrix}}_{B_c} w$$

$$z = \underbrace{\begin{bmatrix} C_1 + D_{21}C_k & -D_{21}C_k \end{bmatrix}}_{C_c}\begin{bmatrix} x \\ e \end{bmatrix}$$

where $e \triangleq x - \xi$.

The GARE $R_3(Z)$ can be rewritten as:

$$R_3(Z) = A_0Z + Z^T A_0^T + \frac{1}{\gamma^2}Z^T C_0^T C_0 Z + B_0 B_0^T = 0$$
$$EZ = Z^T E^T \tag{25}$$

where

$$A_0 = \tilde{A} - B_1 D_{21}^T R_2^{-1}\tilde{C}_2 - Z_\infty^T \tilde{C}_2^T R_2^{-1}\tilde{C}_2$$
$$B_0 = \tilde{B}_1 - Z_\infty^T \tilde{C}_2^T R_2^{-1} D_{21}$$
$$C_0 = -R_1^{-\frac{1}{2}}\left(B_2^T X_\infty + D_{21}^T C_1\right) = R_1^{\frac{1}{2}}F_\infty$$

Note that Z_∞ is also an admissible solution to eqn. 25.

By Lemma 5, we can conclude that $\{E^T, A_0^T\}$ is stable, impulse-free and $\|B_0^T(sE^T - A_0^T)^{-1}C_0^T\|_\infty < \gamma$. This, in turn, implies that $\{E, A_0\}$ is stable, impulse-free and $\|C_0(sE - A_0)B_0\|_\infty < \gamma$.

Again by Lemma 5, the GARE:

$$R_4(W) = A_0^T W + W^T A_0 + \frac{1}{\gamma^2}W^T B_0 B_0^T W + C_0^T C_0 = 0$$
$$W^T E = E^T W \tag{26}$$

has an admissible solution W_∞ with $W_\infty^T E = E^T W_\infty^T \geq 0$.

Now set:

$$P_c \triangleq \begin{bmatrix} X_\infty & 0 \\ 0 & W_\infty \end{bmatrix},\ E_c^T P_c = P_c^T E_c$$

A lengthy but otherwise routine calculation shows that P_c is an admissible solution to the GARE:

$$A_c^T P + P^T A_c + C_c^T C_c + \frac{1}{\gamma^2}P^T B_c B_c^T P = 0$$
$$E_c^T P = P^T E_c$$

with $E_c^T P_c = P_c^T E_c \geq 0$. It follows, again from Lemma 5, that eqn. 19 is an admissible controller such that $\|T_{zw}\|_\infty < \gamma$. This completes the proof of sufficiency.

Necessity: To prove the necessity part, we need the following lemma. See [16] for proof.

Lemma 7: Consider the standard system diagram (Fig. 1). Suppose that a controller of the form (eqn. 19) exists that internally stabilises (eqn. 18) and renders $\|T_{zw}\|_\infty < \gamma$. Then the following conditons hold:

(i) A state feedback matrix K and a matrix P exist such that:

$$\begin{cases} (A + B_2K)^T P + P^T(A + B_2K) + PB_1B_1^T P \\ \quad + (C_1 D_{12}K)^T(C_1 + D_{12}K) < 0 \\ E^T P = P^T E \geq 0 \end{cases} \tag{27}$$

(ii) An output injection matrix L and a matrix Σ exist such that:

$$\begin{cases} (A + LC_2)\Sigma + \Sigma^T(A + LC_2)^T + \Sigma C_1^T C_1 \Sigma \\ \quad + (B_1 + LD_{21})(B_1 + LD_{21})^T < 0 \\ E\Sigma = \Sigma^T E^T \geq 0 \end{cases} \tag{28}$$

(iii) $\rho(P\Sigma) < \gamma^2$, where $\rho(\bullet)$ denotes the spectral radius.

We can now prove the necessity part. Suppose that the H_∞ control problem is solvable. Then, from Lemma 7, inequality (eqn. 27) has a solution P. Inequality (eqn. 27) can be rewritten as:

$$R_1(P) = \left(A - B_2R_1^{-1}D_{12}^T C_1\right)^T P + P^T\left(A - B_2R_1^{-1}D_{12}^T C_1\right) + C_1^T\left(I - D_{12}R_1^{-1}D_{12}^T\right)C_1 + P^T\left(\frac{1}{\gamma^2}B_1B_1^T - B_2R_1^{-1}B_2^T\right)P < 0 \tag{29}$$

Now, assumption (A4) implies that:

$$\begin{bmatrix} -j\omega E + \left(A - B_2 R_1^{-1} D_{12}^T C_1\right) \\ C_1^T \left(I - D_{12} R_1^{-1} D_{12}^T\right) C_1 \end{bmatrix}$$

has full column rank for all $w \in \mathbb{R}$. Furthermore, we can show that $\{E, (A - B_2 R_1^{-1} D_{12}^T C_1)\}$ is impulse-free. To see this, observe that:

$$\begin{bmatrix} I & -B_2 R_1^{-1} D_{12}^T \\ 0 & I \end{bmatrix} \begin{bmatrix} -sE + A & B_2 \\ C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} -sE + \left(A - B_2 R_1^{-1} D_{12}^T C_1\right) & 0 \\ C_1 & D_{12} \end{bmatrix}$$

Applying Lemma 4 shows that an admissible solution X_∞ exists satisfying $R_1(X_\infty) = 0$ with $E^T X_\infty = X_\infty^T E \geq 0$. Similarly, the inequality (eqn. 28) implies that an admissible solution Y_∞ exists satisfying the GARE:

$$\begin{aligned} R_2(Y) = {} & \left(A - B_1 D_{21}^T R_2^{-1} C_2\right) Y \\ & + Y^T \left(A - B_1 D_{21}^T R_2^{-1} C_2\right)^T \\ & + Y^T \left(\frac{1}{\gamma^2} C_1^T C_1 - C_2^T R_2^{-1} C_2\right) Y \\ & + B_1 \left(I - D_{21}^T R_2^{-1} D_{21}\right) B_1^T = 0 \\ EY = {} & Y^T E^T \end{aligned}$$

with $EY_\infty = Y_\infty^T E^T \geq 0$. Moreover, the spectral radius $\rho(Y_\infty X_\infty) < \gamma^2$. Define $Z_\infty = (I - 1/\gamma^2\ Y_\infty X_\infty)^{-1} Y_\infty = Y_\infty (I - 1/\gamma^2\ X_\infty Y_\infty)^{-1}$. A little bit of algebra shows that Z_∞ is an admissible solution for eqn. 23 with $EZ_\infty = Z_\infty^T E^T \geq 0$. Q.E.D.

Remark: For the proof of necessity, the use of GARI in the intermediate stage is needed. To avoid the former use of GARI, one might adopt the method proposed in [1] where the operator theory was used and thus the proofs given there were more involved.

We have given the necessary and sufficient conditions in the above theorem in terms of two coupled GAREs. It is possible to give an alternative set of necessary and sufficient conditions involving two uncoupled GAREs and a spectral radius condition, as given in [1] for systems in the state space model. The proof is quite standard and actually given in the necessity proof of the Theorem 6. This is summarised in the following statement.

Corollary 8: Suppose that GARE $R_1(X) = 0$ has an admissible X_∞ with $E^T X_\infty = X_\infty^T E \geq 0$. Then the condition (ii) of Theorem 6 holds, if and only if, the following conditions hold.

(i) the GARE:

$$\begin{aligned} R_2(Y) = {} & \left(A - B_1 D_{21}^T R_2^{-1} C_2\right) Y \\ & + Y^T \left(A - B_1 D_{21}^T R_2^{-1} C_2\right)^T \\ & + Y^T \left(\frac{1}{\gamma^2} C_1^T C_1 - C_2^T R_2^{-1} C_2\right) Y \\ & + B_1 \left(I - D_{21}^T R_2^{-1} D_{21}\right) B_1^T = 0 \\ EY = {} & Y^T E^T \end{aligned}$$

has an admissible solution Y_∞ with $EY_\infty = Y_\infty^T E^T \geq 0$

(ii) the spectral radius $\rho(Y_\infty X_\infty) < \gamma^2$ Moreover, when these conditions are satisfied, the matrices X_∞, Y_∞ and Z_∞ have the following relationship:

$$\begin{aligned} Z_\infty &= \left(I - \frac{1}{\gamma^2} Y_\infty X_\infty\right)^{-1} Y_\infty \\ &= Y_\infty \left(I - \frac{1}{\gamma^2} X_\infty Y_\infty\right)^{-1} \end{aligned}$$

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