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# The Iris-Loaded Wave Guide as a Boundary Value Problem

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We formulate the problem of the corrugated waveguide with infinitely thin irises in the form of integral equations and obtain a solution for small separation of the irises, which improves the solution of Chu and Hansen, because we do not have to make an assumption about the magnitude of the propagation constant $h$. Our result indicates that the solution given by Walkinshaw has serious defects. For it becomes clear from our computation that any further generalization of the result of Chu and Hansen must take into consideration at least the next higher mode in the region between the irises.

## THE PROBLEM

WE have to find a solution of Maxwell's equations, which satisfies the boundary conditions, that on the surface of the cylinder of radius $b$ the z-component and on the irises for values of $r$ between a and $b$ the r-component of the electric field vanish. Here we take our coordinate system such that the z-axis coincides with the axis of the wave guide. The infinitely thin irises have the internal radius a and their separation we denote by 1.

The Maxwell equations for the time dependence exp $(-i\omega t)$ of the electromagnetic field for an *E-wave* take with $k=\frac{\omega}{c}$ the form

$$ikE_z=-\frac{1}{r}\frac{\partial}{\partial r}(rH_\varphi)$$

$$ikE_r=\frac{\partial H_\varphi}{\partial Z}$$

$$ikH_\varphi=\frac{\partial E_r}{\partial Z}-\frac{\partial E_z}{\partial r},\quad H_z=H_r=E_\varphi=0.$$

Thus $H_\varphi$ satisfies

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H_\varphi}{\partial r}\right)+\frac{\partial^2 H_\varphi}{\partial Z^2}+\left(k^2-\frac{1}{r^2}\right)H_\varphi=0.$$

With the usual notation for the Bessel functions we obtain then the elementary solutions in the n-th corrugation

$$H_\varphi=J_1(\gamma_\nu r)\sinh\sqrt{\gamma_\nu^2-k^2}z'$$

$$H_\varphi=J_1(\gamma_\nu r)\sinh\sqrt{\gamma_\nu^2-k^2}(l-z'),$$

where $z=nl+z'$ and $J_0(\gamma_\nu b)=0$, so that $E_z$ vanishes for $r=b$. In the zeroth corrugation ($n=0$) we denote by $F(r)$ the value of $H_\varphi$ on the plane $z=0$, which contains the zeroth iris, and by $G(r)$ its value on the first iris ($z'=z=l, r>a$). Then we find as the solution with $h$ as the propagation constant.

$$H_\varphi=e^{inhl}\sum\frac{J_1(\gamma_\nu r)}{\sinh\sqrt{(\ )}\,l}\Big[\Big(e^{ihl}\int_o^a tF(t)\,J_1(\gamma_\nu t)\,dt+\int_b^a t\,G(t)\,J_1(\gamma_\nu t)\,dt\Big)\times$$
$$\sinh\sqrt{(\ )}\,z'+\int_o^b tF(t)\,J_1(\gamma_\nu t)\,dt\,\sinh\sqrt{(\ )}\,(l-z')\Big]$$

Here we have abbreviated

$$\sqrt{\gamma_\nu^2-k^2}=\sqrt{(\ )}.$$

$E_r$ is now obtained by differentiation of $H_\varphi$ with respect to $z$,

$$ikE_r=e^{inhl}\sum\frac{\sqrt{(\ )}\,J_1(\gamma_\nu r)}{\sinh\sqrt{(\ )}\,l}\Big[\Big(e^{ihl}\int_o^a tF(t)\,J_1(\gamma_\nu t)\,dt+\int_b^a tG(t)\,J_1(\gamma_\nu t)\,dt\Big)\times$$
$$\cosh\sqrt{(\ )}\,z'-\int_o^b t\,F(t)\,J_1(\gamma_\nu t)\,dt\,\cosh\sqrt{(\ )}\,(l+z')\Big]$$

Because $E_r$ must be continuous in the iris-holes and vanishes on the irises, we find the relations

$$\Big(e^{ihl}\int_o^a t\,F(t)\,J_1(\gamma_\nu t)\,dt+\int_a^b t\,G(t)\,J_1(\gamma_\nu t)\,dt\Big)\Big(1-e^{-ihl}\cosh\sqrt{(\ )}\,l\Big)$$
$$=\int_o^b t\,F(t)\,J_1(\gamma_\nu t)\,dt\,\Big(\cosh\sqrt{(\ )}\,l-e^{-ihl}\Big),$$

and
$$e^{ihl}\int_o^a t\,F(t)\,J_1(\gamma_\nu t)\,dt=\frac{e^{ihl}\cosh\sqrt{(\ )}\,l-1}{\sqrt{(\ )}\,\sinh\sqrt{(\ )}\,l}\int_o^a tK(t)\,J_1(\gamma_\nu t)\,dt,$$

where $K(r)$ is the value of $E_r$ in the zeroth iris-hole. The problem is thus "reduced" to the determination of the three functions $F(r)$, $G(r)$, and $K(r)$.

## APPROXIMATE SOLUTION FOR SUFFICIENTLY SMALL VALUES OF $l$

Here we make the assumption that only the lowest mode in the region $a\leq r\leq b$ is excited. This amounts to the assumption that

$$G=F$$

and we have to show that $F$ vanishes for the excited mode or when

$$G=-F.$$

As we will see, this seems to be true for small values of $l$.*

Treating the two cases together we find the integral equation for $F(r)$

$$F(r)=\frac{1\mp e^{-ihl}}{1\pm e^{-ihl}}\sum\frac{e^{ihl}-\cosh\sqrt{(\ )}\,l}{\cosh\sqrt{(\ )}\,l\mp 1}\int_o^a tF(t)\,J_1(\gamma_\nu t)dt\,J_1(\gamma_\nu r)$$

* See note added in Proof.

and the relation between F(r) and K(r)

$$F(r)=\sum\frac{e^{ihl}-\cosh\sqrt{(\quad)}\,l}{\sqrt{(\quad)}\sinh\sqrt{(\quad)}\,l}\int_o^a tK(t)J_1(\gamma_\nu\, t)\;dtJ_1(\gamma_\nu\, r).$$

Here the upper sign is true for ***F= G*** and the lower for ***F= -G.***

At first we consider the case ***F= -G*** and introduce the notation $u(r,$ a) for the step-function

$$u(r,\ a)=\begin{cases}0: r>a\\ 1: r<a\end{cases}$$

$$F(r)=\frac{1+e^{-ihl}}{1-e^{-ihl}}\Big[-F(r)\ \ u(r,a)+e^{ihl}(1+e^{-ihl})\sum\frac{J_1(\gamma_\nu\, r)}{\cosh\sqrt{(\quad)}\,l+1}\times$$

$$\int_o^a tF(t)J_1(\gamma_\nu\, t)\ dt\Big]$$

Thus we get for $r<a$

$$F(r)=\frac{(1+e^{-ihl})^2e^{ihl}}{2}\int_o^a tF(t)\sum\frac{J_1(\gamma_\nu\, t)\ J_1(\gamma_\nu\, r)}{\cosh\sqrt{(\quad)}\,l+1}\ dt.$$

For sufficiently small values of $l$ we can replace the $\cosh\sqrt{(\quad)}\,l$ by unity and obtain

$$F(r)\ (1-\cos^2\frac{hl}{2})=0,\quad h\neq\frac{2\pi m}{l},$$

so that

$$F(r) = 0.$$

For $r>a$ ***we*** get then

$$F(r)=\frac{1+e^{-ihl}}{1-e^{-ihl}}\int_o^a tF(t)\ \ \sum\frac{J_1(\gamma_\nu\, t)\ J_1(\gamma_\nu\, r)}{\cosh\sqrt{(\quad)}\,l+1}dt=0.$$

Thus we see that we can expect that the even and odd modes separate only for so small values of $l$, which make our computation valid.

Now in the case ***G= F we*** can write

$$F(r)=\frac{1-e^{-ihl}}{1+e^{-ihl}}\Big[-F(r)\ \ u(r,\,a)\ +\ (e^{ihl}-1)\int_o^a tF(t)\ \sum\frac{J_1(\gamma_\nu\, t)\ J_1(\gamma_\nu\, r)}{\cosh\sqrt{(\quad)}\,l-1}\ dt$$

or for $r<a$

$$F(r)=\frac{e^{ihl}(1-e^{-ihl})^2}{2}\int_o^a tF(t)\ \sum\frac{J_1(\gamma_\nu\, t)J_1(\gamma_\nu\, r)}{\cosh\sqrt{(\quad)}\ l-1}\ dt.$$

When $l$ is sufficiently small, we replace $\cosh\sqrt{(\quad)}\,l-1$ by $\frac{l^2}{2}\ (\gamma_\nu{}^2-k^2)$, and with the abbreviation

$$f(h) = -4\sin^2\frac{hl}{2}$$

*we* can write

$$F(r) = \frac{f(h)}{l^2} \int_{o}^{a} tF(t)\, G(r, t)\, dt.$$

Here the infinite sum $G(r, t)$ can be written in closed form

$$G(r, t) = \sum \frac{J_1(\gamma_\nu t)\, J_1(\gamma_\nu r)}{\gamma_\nu^{\,2} - k^2} = \frac{\pi}{2} \begin{cases} J_1(kr)\left[\dfrac{N_0(kb)}{J_0(kb)}\, J_1(kt) - N_1(kt)\right]: & r < t, \\ \left[\dfrac{N_0(kb)}{J_0(kb)}\, J_1(kr) - N_1(kr)\right] J_1(kt): & r > t. \end{cases}$$

This integral equation has the solution

$$F(r) = AJ_1\left(\sqrt{k^2 + \frac{f(h)}{l^2}}\; r\right)$$

to the eigenvalue f(h) which is to be determined from the transcendental equation

$$\frac{\sqrt{k^2 + \frac{f(h)}{l^2}}\; J_0\left(\sqrt{k^2 + \frac{f(h)}{l^2}}\; a\right)}{J_1\left(\sqrt{k^2 + \frac{f(h)}{l^2}}\; a\right)} = k\frac{N_0(kb)J_0(ka) - N_0(ka)J_0(kb)}{N_0(kb)J_1(ka) - N_1(ka)J_0(kb)}.$$

With this result for $F(r)$ when $r < a$ we find now easily the value of $F(r)$ for $r > a$.

$$F(r) = A\frac{2\, J_1\left(\sqrt{k^2 + \frac{f(h)}{l^2}}\; a\right)}{1 + e^{-ihl}} \cdot \frac{J_1(kr)N_0(kb) - N_1(kr)J_0(kb)}{J_1(ka)N_0(kb) - N_1(ka)J_0(kb)}.$$

In order that our result represents a solution of the boundary-value problem, K(r), the value of $E_r$ on the plane $z = 0$, must vanish for $r > a$.*

This is really the case. For we have

$$\int_{o}^{a} tK(t)J_1(\gamma_\nu\, t)\, dt = \frac{1 - e^{-ihl}}{1 + e^{-ihl}} \frac{\sqrt{(\;)}\,\sinh \sqrt{(\;)}\; l}{\cosh \sqrt{(\;)}\; \mathrm{L} - 1} \int_{o}^{a} tF(t)J_1(\gamma_\nu\, t)\, dt$$

Thus we get for sufficiently small values of $l$

$$K(r) = \frac{1 - e^{-ihl}}{1 + e^{-ihl}} \frac{2}{l} \int_{o}^{a} tF(t)\delta(r, t)\, dt.$$

Here

$$\delta(r, t) = \sum J_1(\gamma_\nu\, r) J_1(\gamma_\nu\, t)$$

is the radial a-function satisfying

$$\int_{o}^{a} tF(t)\delta(r, t)\, dt = F(r) \begin{cases} 0: & r > a \\ 1: & r < a \end{cases}$$

Thus we see that all the boundary-conditions of the problem are satisfied

$$K(r) = \frac{1 - e^{-ihl}}{1 + e^{-ihl}} \frac{2}{l} \begin{cases} F(r): & r < a \\ 0\;\;\;: & r > a. \end{cases}$$

* Note added in proof.

In a more careful treatment of the problem one has to take account of the excited modes from the beginning. We have carried out this computation with the result that the excited modes do not contribute in the case of small $l$ because of the boundary condition that $E_r$ must vanish on the screens.

## DISCUSSION

Our result for the propagation-constant $h$ is the same as that of Chu and Hansen[1] apart from the replacement of $h^2$ by $+\frac{4}{l^2}\sin^2\frac{hl}{2}$. This result has been obtained under the assumption that the even and odd modes separate and can be justified only for small values of 1. Thus it is clear that the separate treatment of the fundamental mode, $F=G$ in our notation and the exctied mode, $F=-G$, is possible only for small values of 1. Therefore any further generalization of the result of Chu and Hansen[1] must take into account, that the even and odd modes cannot be treated separately, so that the solution obtained by Walkinshaw[2] seems to be of dubious value.

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1) F. I. Chu and W. W. Hansen, J. Appl. Phys. 18, 996, (1947)
2) W. Walkinshaw, Proc. Phys. Soc. (London) 61, 246, (1948)