Boundary spectral method for potential flows

as Gaussian quadrature or trapezoidal rules successfully. The singularities of the integral equations are completely removed by the Gauss flux theorem and the equipotential theorem. Therefore, all the integrals are non-singular and can be executed accurately and efficiently. By spectral approximation, the unknown variable is expressed as a truncated series of basis functions, which are orthogonal usually. Instead of solving the variables at collocation points in the conventional methods, only the coefficients of basis functions are determined in the present approach. It is shown that the new method reduces a lot of number of unknowns, storage of matrix elements, and computer time for solving the algebraic equations.

Keywords: boundary integral equation; spectral method; potential flows

(singularities)

(adaptive integration)

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Abstract

A boundary spectral method is developed for solving three-dimensional exterior potential problems. This method directly applies spectral approximations and numerical integration such

In the last three decades, the boundary element method (BEM) has become very popular and has been applied to solve a wide range of engineering problems. Since the BEM frequently uses local elements, their basis functions are usually low-order polynomials. In the present paper, a spectral approach is proposed to solve boundary integral equations and it is quite different from traditional BEMs.

For potential flows, since Hess and Smith published their famous work on the BEMs (or called panel methods), a large progress has

been reached. [1] Most of the original BEMs, developed to solve the strength of singularities such as source, vortex, or doublet, are called indirect methods. Later, a Green's formula based method was developed in which the primary unknown was the velocity potential. [2] Both methods are still popularly used currently, but only the direct method is adopted in the present paper.

The spectral method has also been developed for decades and very successful in solving partial differential equations when very high accuracy is needed. It is well known that a truncated Fourier series can approximate any function and its rate of convergence of such a series depends on the smoothness of function itself. Until recently, spectral methods have not been widely used in solving boundary integral equations.

The difficulty of combining the spectral method and the boundary integral equation is due to the singular integrals in the boundary integral equation. There are several methods in the literature to calculate singular integrals such as polar transformations, adaptive sub-division scheme and etc. However, those methods are inefficient when compared with regular integration. In order to avoid such singular integrals, in the present paper we use the subtracting and adding-back technique to evaluate singular integrals. By this scheme, the singularities of the integral kernels can be regularized, and the resultant integrand in the modified Green's formula is bounded. [3] The bounded integrands are excellent for numerical quadrature. The numerical formulation can yield very accurate solutions with significantly fewer terms by most numerical quadrature method.

In the literature, some authors already tried to develop the boundary spectral method. Some of them use analytic formulas to solve two-dimensional cases, and call it boundary strip methods or boundary spectral strip methods. [4,5] But in general their method cannot be extended into three-dimensional cases. The author in the previous papers already developed a method to handle three-dimensional smooth bodies for exterior flow field. [6,7] This method can directly be applied in the two-dimensional cases without any difficulty. In this paper, we discuss the detailed variation for the uniform flows past different spheroids, and try to understand the limitation of boundary spectral methods.

For simplicity, in the following three axisymmetrical cases are presented. Since the flow is axisymmetric, we can choose one-dimensional basis functions and the Chebyshev polynomial, $T_n(x)$, is used. The first numerical example is for the scattering problem of a uniform flow past a sphere. The velocity potential for a uniform flow of magnitude U approaching a sphere of radius a, centered at origin, along positive x axis can be expressed as

$$\Phi(R,\theta) = U(R + \frac{a^3}{2R^2})\cos\theta \qquad (1)$$

where θ is the angle between the x-axis and the position vector, and R the distance to the origin. The velocity potential on the sphere surface is $\Phi(a, \theta) = 3Ux/2$ when R = a. In Table 1, a 15-point Gauss-Legendre formula is applied in the x direction and trapezoidal rules is used in the θ direction. The first 6 terms of Chebyshev polynomials are calculated, and the maximum error, which occurs at two ends x=±1, is less than 10^{-4}.

The second example is the problem of a uniform flow past a prolate spheroid. The prolate spheroid is an elongated ellipsoid of revolution with major axis of length a on the x-axis and minor axis of length b. The uniform flow of magnitude U travels along the positive x-axis. The exact solution derived in Milne-Thomson is $\Phi = \alpha Ux$, where α depends on a and b. Again, Chebyshev polynomials, the 15-point Gauss-Legendre formula in the x direction, and trapezoidal rules in the θ

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direction are applied. Table 2 shows the values of coefficient c_1 for different slenderness of spheroids. For elongate spheroid these main coefficients are very accurate. The other terms are negligible as shown in Fig. 1. For oblate spheroid, the accuracy becomes less and less when b/a becomes larger. Part of the reason is the variation of velocity potential becomes larger. In the case b/a=4, the relative error of c_1 is about 1%. In Fig. 2, we show the trend of coefficients for the case b/a=4.

The third example is the problem for a sphere in a doublet flow. A doublet of strength $\sim$ locates at $(l, 0, 0)$ along the positive x direction outside a sphere, centered at the origin, with radius a $(a < l)$. The flow field is constructed by the doublet of strength μ at $(l, 0, 0)$ and a doublet with strength $-\sim a^3/l^3$ at its image point $(a^2/l, 0, 0)$. The corresponding velocity potential on the sphere surface is

$$\Phi(a,_{,,}) = \frac{\sim}{4\pi l} \frac{a^2 - l^2}{(a^2 + l^2 - 2lx)^{3/2}} \quad (2)$$

In this case, 25 Gaussian points and 15 terms of Chebyshev polynomials are used for the example $l/a = 2$ and $4\pi a^2/\sim = 1$. The comparison of exact and numerical coefficients of Chebyshev polynomials is greatly consistent. Only first 10 terms are shown in Table 3, while 15 terms are calculated actually. The maximum deviation of coefficients is 0.00004. When $i > 3$, the ratio between two consecutive terms is approximately half. It indicates the coefficient sequence has exponential convergence. The non-dimensional velocity potential versus to x is plotted in Fig. 3. The maximum error at the calculated points is about 0.01%. This error is of the same order as 25-point formulation by the direct method without the spectral series. It confirms that the reduction of the matrix size does not affect the accuracy significantly. Since the size of matrix A is small, the memory space, and computing time for forming and solving matrix A are also saved lots of amount.

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Table 1. The comparison of analytical and series of solutions for a uniform flow past a sphere

i	exact	c_i
0	0	-0.37E-08
1	1.5	1.49998
2	0	-0.90E-09
3	0	-0.15E-03
4	0	0.27E-07
5	0	0.50E-04

Table 2. The comparison of analytical and series of solutions for a uniform flow past a spheroid

i	exact	c_i
0	-0.354515	-0.354512
1	-0.483844	-0.483844
2	-0.292130	-0.292120
3	-0.167360	-0.167340
4	-0.093099	-0.093080
5	-0.050819	-0.050810
6	-0.027377	-0.027356
7	-0.014606	-0.014565
8	-0.007734	-0.007711
9	-0.004071	-0.004056
10	-0.002133	-0.002132

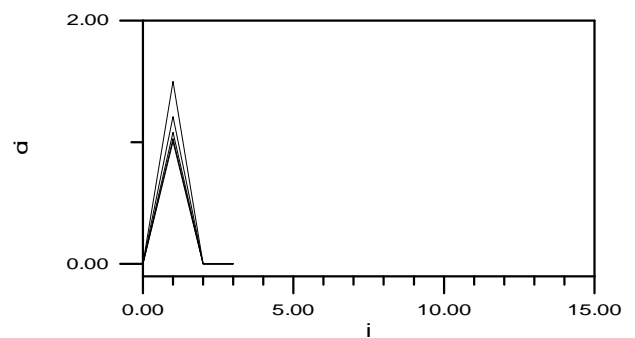| b/a	exact	c_1
1/20	1.00679	1.00657
1/8	1.02925	1.02917
1/4	1.08156	1.08151
1/2	1.21002	1.20999
1/1	1.5	1.50000
2/1	2.11506	2.11640
4/1	3.37429	3.41547

Fig. 1. The generalized Fouier coefficients for the uniform flow past a spheroid for different slenderness.

Table 3. The comparison of analytical and series of solutions for a sphere in a doublet flow.
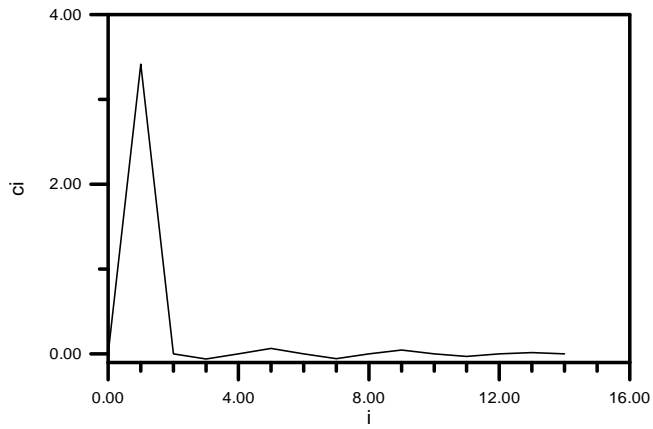
Figure 2. The generalized Fouier coefficients for the uniform flow past a spheroid with slenderness b/a=4
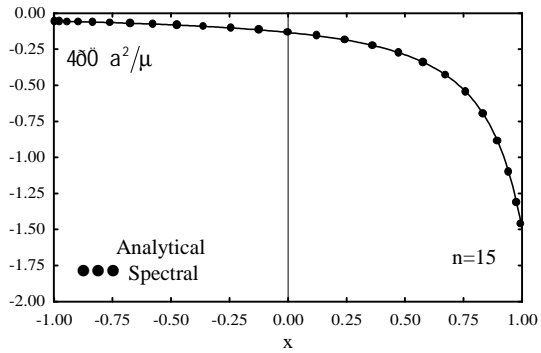

Figure 3. Velocity potential on the surface of sphere in a doublet flow

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