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Classical Expression of Conservation Laws of Generalized Momenta in Quantum Mechanics

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In a previous paper. an equation is written down for $S\equiv e^{-iW/\hbar}\phi$ instead of the energy eigenfunction ϕ, with W being the classical Jacobi W function solved from the classical Hamilton-Jacobi equation. Coulomb interaction is illustrated as an application of this equation and the independence of S on certain generalized coordinate q is interpreted as the equality (conservation law) of its canonical conjugate momentum p both in the classical system described by W and in the quantum system described by ϕ. The explicit dependence on q of ϕ can be, thus, easily obtained from W, a classical function.

I. INTRODUCTION

INSTEAD of wave functions which is immeasurable in Schrödinger theory, Fano⁽¹⁾ proposed a density matrix formulism for quantum mechanics. When the density matrix is abstractly expressed apart from the coodinate representation, i. e. $|\phi\rangle\langle\phi|$, it is an operator. Recently, Su⁽²·³⁾ expressed this density matrix operator in terms of the Hamiltonian or energy dynamical variable by divergence series method. This replacement is analogous to Heisenberg ' s idea of using the dynamical variable to represent a physical system in matrix mechanics. In Ref. 2, an equation was obtained for a function $S(q,t)$ defined by

$$S(q,t)\equiv e^{-iW(q,t)/\hbar}\phi(q,t), \tag{1.1}$$

where q is the generalized coordinates and $W(q,t)$ is the Jacobi function satisfied the Hamilton-Jacobi equation

$$H(\partial W/\partial q, q) = -\partial W/\partial t \tag{1.2}$$

for the Hamiltonian $H=H(p,q)$. The equation stated as

$$H(-i\hbar\frac{\partial}{\partial q}+\frac{\partial W}{\partial q}, q)S(q,t)=i\hbar\frac{\partial S(q,t)}{\partial t}-\frac{\partial W}{\partial t}S(q,t). \tag{1.3}$$

Directly solving (1.3) for Coulomb interaction (bound state and scattering)

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(1) U. Fano. Revs. Modern. Phys. **29** 74 (1957).

(2) Der-Ruenn Su, Internat. J. Theoret, Phys. 4 171 (1971).

(3) Der-Ruenn Su, Internat. J. Theoret, Phys. **4** 233 (1971).

problem, we find a solution not exactly in form of (1.1) with $W(\boldsymbol{r},t)$ and $\phi(\boldsymbol{r},t)$ as H-atom (or scattering) W function and wave function respectively. But it will reduce to such a case if the following equations are valid

$$\begin{aligned}&\partial S(\boldsymbol{r},\quad t)/\partial t=0\\&\partial S(\boldsymbol{r},\ t)/\partial\varphi=0.\end{aligned}\tag{1.4}$$

Further investigations show that (1.4) indicates the classical expressions for the equalities of E and L_z both given in $W(\boldsymbol{r},t)$ in (1.1) representing the classical state of the system and given in $\phi(\boldsymbol{r},t)$, the quantum state of the system. These equalities are equivalent to the conservation laws as interpreted in Sec. IV.

In Sec. III, (1.4) is generalized to the case if q, p are canonical conjugate variables, then the equation

$$\partial S/\partial q=0$$

or

$$p_{op}S=0\tag{1.4a}$$

denotes the equality (conservation law) of P in the problem. Finally, in Sec. IV, two ways of applying (1.4a) are given.

II. COULOMB INTERACTION

For the Kepler's problem or Coulomb interaction Hamiltonian

$$H=(2\mu)^{-1}p^2-K/r,$$

the Jacobi's W function is solved as

$$W(\boldsymbol{r},t)=\int\sqrt{2\mu E+2\mu K/r-L^2/r^2}dr+L\varphi+Et\tag{2.1}$$

and the differential equation (1.3) can be written as

$$\begin{aligned}&-\hbar^2\left(\frac{1}{r^2}\ \frac{\partial}{\partial r}r^2\frac{\partial S}{\partial r}\right)-i\hbar\frac{1}{r^2}\ \frac{\partial}{\partial r}r^2\sqrt{2\mu E+2\mu K/r-L^2/r^2}S-\\&-i\hbar\sqrt{2\mu E+2\mu K/r-L^2/r^2}\frac{\partial S}{\partial r}+\left(2\mu E+\frac{2\mu K}{r}-\frac{L^2}{r^2}\right)S-\\&-\frac{\hbar^2}{r^2}\left[\frac{1}{\sin\theta}\ \frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}-\frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\varphi^2}+\frac{2iL}{\hbar}\ \frac{\partial}{\partial\varphi}-\frac{L^2}{\hbar^2}\right)\right]S-\\&-2m(K/r+i\hbar\frac{\partial}{\partial t}+E)S=0.\end{aligned}\tag{2.2}$$

The solution for (2.2) is

$$S(\boldsymbol{r},\ t)=\exp\left(-\frac{i}{\hbar}\int\sqrt{2\mu E+\frac{2\mu K}{r}-\frac{L^2}{r^2}}dr\right)r^{-1}R_l(r)P_l^{m'+L/\hbar}(\theta)e^{irm'\varphi}e^{-i\bar{E}t\hbar}\tag{2.3}$$

and $R_l(r)$ satisfies the differential equation

$$\frac{d^2}{dr^2}R_l(r)-\left[\frac{l(l+1)}{r^2}-\frac{2mK}{\hbar^2 r}-\frac{2m(E+E)}{\hbar^2}\right]R_l(r)=0\tag{2.4}$$

which is H-atom ($E+\bar{E}<0$) or Coulomb scattering ($E+E>0$) differential equation for energy $E+\bar{E}$.(4) In order to make $S(\boldsymbol{r},t)$ clearer, let us rewrite (2.3) more compact as

$$S(\boldsymbol{r},\ t)=\exp-\frac{i}{\hbar}\left(\int\sqrt{2\mu E+\frac{2\mu K}{r}-\frac{L^2}{r^2}}dr+L\varphi-Et\right)\times$$
$$\times r^{-1}R_l(r)P_l^{m'+L/\hbar}(\theta)e^{i(m'+L/\hbar)\varphi}e^{-i(E+\bar{E})t/\hbar}$$
$$\equiv e^{-iW(\boldsymbol{r},t)/\hbar}\phi_{Elm}(\boldsymbol{r},\ t). \tag{2.3a}$$

It is clear that the second equality of (2.3a) is valid only if

$$m'=0$$
$$\bar{E}=0$$

and $\quad L/\hbar=m=$ integers

in which the third one is the quantization of L_z due to the single-valuedness. As pointed out in Ref. 2, the classical trajectory of ϕ (I, t) is the one described by $W(\boldsymbol{r},\ t)$. It is clear, on comparing L_z and E in (2.3a), that the first two equations of (2.5) indicate the equalities of L_z and E in the quantum state $\phi(\boldsymbol{r},t)$ and in its classical correspondence described by $W(\boldsymbol{r}$, t).

III. GENERALIZATION

Return back to (2.3), we see that the first two equations of (2.5) are equivalent to

$$\partial S/\partial\varphi=0,$$
$$\partial S/\partial t=0.$$

Since $\partial S/\partial\varphi=0$ means the equality of L_z in both quantum state and its classical correspondence, similarly $\partial S/\partial t=0$, the equality of E, because L_z and φ, E and t are canonical conjugate variables, we generalize our result to any pair of canonical conjugate variables P and q which do have classical correspondence. Then our result states that the equality of P in ϕ and in its classical correspondence is followed from the condition

$$\partial S/\partial q=0 \quad \text{or} \quad p_{op}S=0. \tag{3.1}$$

The general proof of (3.1) is given as follows: (3.1) implies

$$S\partial W/\partial q-i\hbar e^{-iW/\hbar}\partial\phi/\partial q=0.$$

By the classical Hamilton- Jacobi theory

$$\partial W/\partial q=p,$$

we obtain

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(4) cf. eq. A Messiah, Quantum *Mechanics*, Vol. I (North-Holland Pub. Co. Amsterdam 1961).

$$-i\hbar\partial\phi/\partial q=p\phi, \qquad (3.2)$$

i. e. p is also a good quantum number of ϕ. Q. E. D.

The equality of p is equivalent to the law of conservation of p. This point is going to be clear in the next section.

IV. METHODS OF APPLICATION

We shall mention two ways of applying (3.1). First way, since in classical problems, especially the collision problems, we use the conservation laws to solve the problems, there are no quantum mechanical correspondences so far. But there are problems, such as Compton effect etc., needed to be solved by such a way. (3.1) provides a way of this kind of calculation. This way is stated as follows. If p_I is conserved, i. e. $[p_I, H]=0$, then (3.2), thus (3.1), must be satisfied, since the energy eigenfunction ϕ must be also an eigenfunction of p_I. Then $S(q_j, t)$ is not a function of q_I explicitly, viz.

$$S(q_k, t)=e^{-iW(q_j,t)/\hbar}\phi(q_j, t)=f(q_k, t),\ k\neq I, \qquad (4.1)$$

where $f(q_k,t)$ is some function of q_k and t but not q_I. Thus, from (4.1), the explicit dependence of $\phi(q_j,t)$ on q_I is determined entirely on the ***classical*** function $W(q_j,t)$.

Second way, we apply (3.1) to treate scattering problems. In scattering problems, the incident (initial) state is usually off from interaction region. The corresponding classical trajectory, with $\hbar\to 0, \lambda\to 0$, is in general simple and known. For example, for the plane wave incident, as usual the case, its classical trajectory is straight lines. For the Coulomb (long range) interaction, as given in Sec. II above, in which L_z is conserved, parabolic trajectory is followed. Anyhow, the classical W function for the initial state is obvious. After applying the conservation laws, or (3.1), we can easily obtain the dependence on q_I of ϕ which is the wave function describing the interaction, directly from W function. Both of these two ways can be used to interprete for φ- and t-dependences in Coulomb scattering above in Sec. II.

Conclusively, (3.1) provides us a method to obtain quantum mechanical state from the classical W function.