Fault-free Hamiltonian Cycles in Alternating Group Graphs with Conditional Edge Faults

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Abstract. The alternating group graph, which belongs to the class of Cayley graphs, is one of the most versatile interconnection networks for parallel and distributed computing. In this paper, adopting the conditional fault model in which each vertex is assumed to be incident with two or more fault-free edges, we show that an *n*-dimensional alternating group graph can tolerate up to $4n-13$ edge faults, where $n \geq 4$, while retaining a fault-free Hamiltonian cycle exists. The result is optimal with respect to the number of edge faults tolerated. Previously, for the same problem, at most $2n-6$ edge faults can be tolerated if the random fault model is adopted.

Keywords: conditional fault model, embedding, fault tolerance, Hamiltonian, random fault model, alternating group graph, Cayley graph.

1 Introduction

Network topology is a crucial factor for the interconnection networks since it determines the performance of the networks. Many interconnection network topologies have been proposed in the literature for the purpose of connecting hundreds or thousands of processing elements. It can be represented by a graph where nodes represent processors and edges represent links between processors. The alternating group graphs [11], like the well-known star graphs [1] and hypercubes [14], belongs to the class of Cayley graphs [2], [12]. Furthermore, it has been shown in [7] that a class of generalized star graphs called the arrangement graphs also contains alternating group graphs as members. Indeed, a proof given in [7] showed that the *n*-alternating group graph AG_n is isomorphic to the $(n, n-2)$-arrangement graph $A_{n,n-2}$.

Arrangement graphs have been shown to be vertex and edge symmetric, strongly hierarchical, maximally fault tolerant, and strongly resilient [8], and thus of alternating group graphs. Besides, alternating group graphs have sublogarithmic degree and diameter [11]. They are all desirable when we are building an interconnection topology for a parallel and distributed system. An efficient communication algorithm for shortest-path routing is available for alternating group graphs [11].

The study of graph embeddings arises naturally in a number of computational problems: finding storage schemes for logical data structures, layout of circuits in VLSI, portability of algorithms across various parallel architectures, just to mention a few [12]. Among of them, the ring is one of the most fundamental networks for parallel and distributed computation, and it is suitable to develop simple and efficient algorithms. Numerous algorithms that were designed on rings for solving various algebraic problems and graph problems can be found in [3], [13]. A ring can be also used as a control/data flow structure for distributed computation in a network. These applications motivate the embedding of cycles in networks. It was shown that the arbitrary cycles can be embedded in an alternating group graph [11]. Besides, the alternating group graph also can embed some other fundamental networks, such as grids [11], trees [11], and arbitrary paths [6].

Since node faults and/or link faults may occur to networks, it is significant to consider faulty networks. Many fundamental problems such as diameter, routing, broadcasting, gossiping, embedding, *etc*., have been studied on various faulty networks. Among them, two fault models were adopted; one is the random fault model, and the other is the conditional fault model. The *random fault model* assumed that the faults might occur everywhere without any restriction, whereas the *conditional fault model* assumed that the distribution of faults must satisfy some properties, e.g., two or more fault-free links incident to each node. Apparently, it is more difficult to solve problems under the conditional fault model than the random fault model. Previously related work about embedding in faulty networks under the conditional fault model can be found in [4], [5], [9], [15], [16].

In this paper, under the conditional fault model and with the assumption of at least two fault-free links incident to each node, we show that an n-dimensional alternating group graph can tolerate up to $4n-13$ link faults, where $n \geq 4$, while retaining a fault-free Hamiltonian cycle exists. The result is optimal with respect to the number of link faults tolerated. For the same problem, at most $2n-6$ link faults can be tolerated if the random fault model is adopted [10]. With our results, all parallel algorithms developed on rings can be executed as well on an n-dimensional alternating group graph with up to $4n-13$ link faults.

In the next section, the structure of the alternating group graph is reviewed. Necessary definitions, notations, and some properties of the alternating group graph are also introduced in order to prove the main result. Then the main result and its proof are shown in Section 3. Finally, this paper concludes with some remarks in Section 4.

2 Preliminaries

It is convenient to represent a network with a graph G, where each vertex (edge) of G uniquely represents a node (link) of the network. For the graph definition and notation, we follow [17]. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. Given a vertex u in G, we define $N(u) = \{v \mid (u, v) \in E(G)\}$ to be the *neighborhood* of u, which is the set of vertices that are adjacent to u in G. The *degree* of u, denoted by $\deg(u)$, is the size of $N(u)$, i.e., $\deg(u) = |N(u)|$. We use $\delta(G)$ to denote

$\min\{\deg(u) \mid u \in V(G)\}$. Let V' be a vertex subset of G. We define $N(V') = \bigcup_{u \in V'} N(u) - V'$ to be the *neighborhood* of V'. A path $P_{x_0 x_t} = \langle x_0, x_1, \cdots, x_t \rangle$, is a sequence of vertices such that every two consecutive vertices are adjacent. In addition, $P_{x_0 x_t}$ is a cycle if $x_0 = x_t$. A path $\langle x_0, x_1, \cdots, x_t \rangle$ may contain other subpath, denoted as $\langle x_0, x_1, \cdots, x_i, P_{x_i x_j}, x_j, \cdots, x_t \rangle$, where $P_{x_i x_j} = \langle x_i, x_{i+1}, \cdots, x_{j-1}, x_j \rangle$.

A path (or cycle) in G is called a *Hamiltonian path* (or *Hamiltonian cycle*) if it contains every vertex of G exactly once. A graph is called *Hamiltonian* if it has a Hamiltonian cycle. A graph is called *Hamiltonian connected* if every two vertices of G are connected by a Hamiltonian path. All Hamiltonian connected graphs except K_1 and K_2 are Hamiltonian.

Let $p = a_1 a_2 \cdots a_n$ is a permutation on $\{1, 2, \cdots, n\}$. A pair of symbols a_i and a_j in p are said to be an *inversion* if $a_i < a_j$ whenever $i > j$. A permutation is an *even permutation* if it has an even number of inversions. The *alternating group* A_n is the set consisting of all even permutations on $\{1, 2, \cdots, n\}$, where $|A_n| = n!/2$. The following is a formal definition of alternating group graphs, in terms of graph theory.

Definition 1 An *n-dimensional alternating group graph*, denoted by AG_n, has the vertex set $V(AG_n) = \{a_1 a_2 \cdots a_n \mid a_1 a_2 \cdots a_n$ is an even permutation of $1, 2, \cdots, n\}$ and the edge set $E(AG_n) = \{(a_1 a_2 \cdots a_n,\ a_2 a_i \cdots a_{i-1} a_1 a_{i+1} \cdots a_n)$ or $(a_1 a_2 \cdots a_n,\ a_i a_1 \cdots a_{i-1} a_2 a_{i+1} \cdots a_n) \mid a_1 a_2 \cdots a_n \in V(AG_n)$ and $3 \le i \le n\}$.

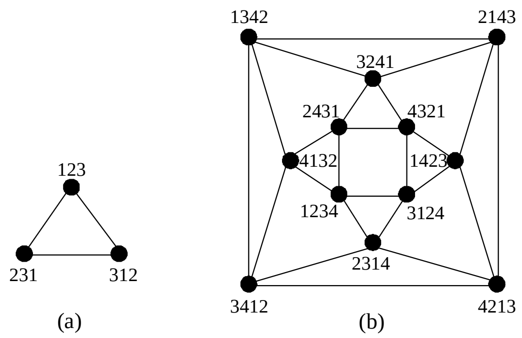

Fig. 1. Examples of alternating group graphs. (a) AG_3. (b) AG_4.

From definition, it is easy to see that the vertex set of AG_n is the alternating group A_n. It has $n!/2$ vertices, each of degree $2(n - 2)$, and has $(n - 2)n!/2$ edges. The alternating group graphs AG_3 and AG_4 are shown in Fig. 1. Let $u = a_1 a_2 \cdots a_n$ be any vertex of the alternating group graph AG_n. The edges $(a_1 a_2 \cdots a_n,\ a_2 a_i \cdots a_{i-1} a_1 a_{i+1} \cdots a_n)$ and $(a_1 a_2 \cdots a_n,\ a_i a_1 \cdots a_{i-1} a_2 a_{i+1} \cdots a_n)$, denoted by $e^{(i)}(u)$, are referred to as *i-dimensional edges* of u, where $3 \le i \le n$. We use $E^{(i)}(AG_n)$ to denote the set of all i-dimensional edges in AG_n.

Alternating group graphs are vertex symmetric, edge symmetric, and strongly hierarchical [11]. For $1 \le k \le n$, let $AG_n(k)$ denote the subgraph of AG_n induced by those vertices u with $a_n = k$. Clearly, each $AG_n(k)$ is isomorphic to AG_{n-1} for $1 \le k \le n$. Due to the strongly hierarchical structure, the alternating group graph can also be defined recursively: AG_n is constructed from n disjoint copies of $(n-1)$-dimensional alternating group graph AG_{n-1}'s. We use $\tilde{E}^{(n)}_{p,q}(AG_n)$ to represent the set of those n-dimensional edges in AG_n that connect $AG_n(p)$ and $AG_n(q)$, where $1 \le p \ne q \le n$.

Throughout this paper, the paired terms network and graph, node and vertex, and link and edge are used interchangeably. Since AG_n is isomorphic to the $(n, n-2)$-arrangement graph $A_{n,n-2}$, the following lemma of AG_n is deduced from a result of arrangement graphs.

Lemma 1 [10] For any $F' \subseteq V(AG_n) \cup E(AG_n)$, $AG_n - F'$ is Hamiltonian if $|F'| \le 2n-6$, and Hamiltonian connected if $|F'| \le 2n-7$, where $n \ge 4$.

We also present some properties of AG_n in the following. They are necessary in order to show our main result in the next section. Besides, we use F ($\subseteq E(AG_n)$) to denote the set of edge faults in AG_n.

Lemma 2 $|\tilde{E}^{(n)}_{i,j}(AG_n)| = (n-2)!$, where $n \ge 4$.

Proof. Consider $p = p_1 p_2 \cdots p_n \in V(AG_n(i))$, where $p_n = i$. Suppose that p connect to $AG_n(j)$, hence we have $p_1 = j$ or $p_2 = j$. If $p_1 = j$, then there are $(n-2)!/2$ choices for p_2, p_3, $\cdots$, p_{n-1}. The discussion is similar if $p_2 = j$. So $|\tilde{E}^{(n)}_{i,j}(AG_n)| = (n-2)!$. □

Lemma 3 Suppose that $I = \{k_1, k_2, \cdots, k_m\} \subseteq \{1, 2, \cdots, n\}$, where $n \ge 5$ and $m \ge 2$. Let $AG_n(I)$ denote the subgraph of AG_n induced by $\bigcup_{k \in I} V(AG_n(k))$. If $AG_n(k) - F$ is Hamiltonian connected for every $k \in I$ and $|\tilde{E}^{(n)}_{k_j,k_{j+1}}(AG_n) - F| \ge 3$ for all $1 \le j < m$, then there is a Hamiltonian path P_{st} in $AG_n(I) - F$, where $s \in V(AG_n(k_1))$ and $t \in V(AG_n(k_m))$.

Proof. Note that if $v = c_1 c_2 \cdots c_n \in V(AG_n)$, then $v \in N(V(AG_n(c_1))) \cap N(V(AG_n(c_2)))$, thus, the two edges of $e^{(n)}(v)$ incident to different $AG_n(c)$s. Let $u_1 = s$. Since $|\tilde{E}^{(n)}_{k_j,k_{j+1}}(AG_n) - F| > 2$ for all $1 \le j < m$, we can find an edge $(v_1, u_2) \in E^{(n)}(AG_n) - F$ such that $v_1 \ne u_1$ and $u_2 \in V(AG_n(k_2))$. Similarly, we can find edges (v_2, u_3), (v_3, u_4), $\cdots$, $(v_{m-2}, u_{m-1}) \in E^{(n)}(AG_n) - F$, where u_i and v_i are two distinct vertices in $AG_n(k_i)$ for all $i \in \{2, 3, \cdots, m-1\}$. Since $|\tilde{E}^{(n)}_{k_{m-1},k_m}(AG_n) - F| \ge 3$, we can find an edge $(v_{m-1}, u_m) \in E^{(n)}(AG_n) - F$ such that $v_{m-1} \ne u_{m-1}$, $u_m \ne t$, and $u_m \in V(AG_n(k_m))$. Let $v_m = t$. In addition, since $AG_n(k_i) - F$ is Hamiltonian connected for all $k_i \in I$, there is a Hamiltonian path $P_{u_i v_i}$ in $AG_n(k_i) - F$ for all $i \in \{1, 2, \cdots, m\}$. An Hamiltonian path P_{st} in $AG_n(I) - F$ is constructed as follows (see Fig. 2):

$\langle s,\ P_{sv_1},\ v_1,\ u_2,\ P_{u_2v_2},\ v_2,\ \cdots,\ u_{m-1},\ P_{u_{m-1}v_{m-1}},\ v_{m-1},\ u_m,\ P_{u_m t},\ t\rangle$. □

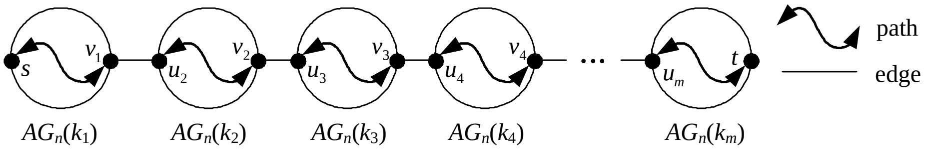

Fig. 2. A Hamiltonian path P_{st} in $AG_n(I) - F$.

3 Main Result

In this section, we would show that with the assumption of two or more fault-free edges incident to each vertex, an n-dimensional alternating group graph can tolerate up to $4n - 13$ edge faults, while retaining a fault-free Hamiltonian cycle exists, where $n \geq 4$.

Theorem 1. $AG_n - F$ is Hamiltonian if $|F| \leq 4n - 13$ and $\delta(AG_n - F) \geq 2$, where $n \geq 4$.

Proof. We can proceed by induction on n. When $n = 4$, we can use a computer program to check that the result is true [18]. Assume that the result holds for AG_n for some $n \geq 4$. Consider AG_{n+1} with $|F| \leq 4n - 9$ and $\delta(AG_{n+1} - F) \geq 2$. For brevity, assume that $|F| = 4n - 9$. Without loss of generality, assume that $|E^{(n+1)}(AG_{n+1}) \cap F| \geq |E^{(n)}(AG_{n+1}) \cap F| \geq \cdots \geq |E^{(3)}(AG_{n+1}) \cap F|$. If $n \geq 7$, we have $|E^{(n+1)}(AG_{n+1}) \cap F| \geq \lceil (4n-9) / (n-1) \rceil \geq 4$, $|F - E^{(n+1)}(AG_{n+1})| \leq 4n - 13$, and $|E(AG_{n+1}(r)) \cap F| \leq 4n - 13$ for all $1 \leq r \leq n + 1$. In addition, when $4 \leq n \leq 6$, we have $|E^{(n+1)}(AG_{n+1}) \cap F| \geq 3$, $|F - E^{(n+1)}(AG_{n+1})| \leq 4n - 12$, and $|E(AG_{n+1}(r)) \cap F| \leq 4n - 12$ for all $1 \leq r \leq n + 1$. Without loss of generality, assume that $|E(AG_{n+1}(n + 1)) \cap F| \geq |E(AG_{n+1}(n)) \cap F| \geq \cdots \geq |E(AG_{n+1}(1)) \cap F|$. Note that when $n \geq 5$, by Lemma 2, $|\tilde{E}^{(n+1)}_{j,k}(AG_{n+1})| = ((n+1) - 2)! = (n-1)! > 4n - 6 = |F| + 3$, hence we have $|\tilde{E}^{(n+1)}_{j,k}(AG_{n+1}) - F| \geq 3$ for all j, $k \in \{1, 2, \cdots, n+1\}$ and $j \neq k$. If $n = 4$, we have $|\tilde{E}^{(5)}_{j,k}(AG_5)| = (5-2)! = 6 < 7 = 4n - 9$ for all j, $k \in \{1, 2, \cdots, 5\}$ and $j \neq k$. Hence, it is possible that $|\tilde{E}^{(5)}_{j',k'}(AG_5) - F| < 3$ for some j', $k' \in \{1, 2, \cdots, 5\}$ and $j' \neq k'$. We use i_1, i_2, $\cdots$, i_{n+1} to denote the $n + 1$ distinct integers from 1 to $n + 1$ (i.e., $\{i_1, i_2, \cdots, i_{n+1}\} = \{1, 2, \cdots, n + 1\}$). Four cases are considered:

Case 1. $|E(AG_{n+1}(n + 1)) \cap F| \leq 2n - 7$. In this case, $\delta(AG_{n+1}(r) - F) \geq 2$ for all $1 \leq r \leq n + 1$. Let $i_1 = n + 1$ and $I = \{1, 2, \cdots, n\}$. We can find $u_1, v_1 \in V(AG_{n+1}(i_1))$ such that (v_1, u_2), $(u_1, v_{n+1}) \in E^{(n+1)}(AG_{n+1}) - F$, where $u_2 \in V(AG_{n+1}(i_2))$ and $v_{n+1} \in V(AG_{n+1}(i_{n+1}))$. By Lemma 1 and Lemma 3, we can find a Hamiltonian path $P_{u_1v_1}$ in $AG_{n+1}(i_1) - F$ and a Hamiltonian path $P_{u_2v_{n+1}}$ in $AG_{n+1}(I) - F$ (when $n = 4$ and if j', k' exist, let $\{j', k'\} \neq \{i_r, i_{r+1}\}$ for $2 \leq r \leq n$). Hence $\langle u_1, P_{u_1v_1}, v_1, u_2, P_{u_2v_{n+1}}, v_{n+1}, u_1 \rangle$ form a Hamiltonian cycle in $AG_{n+1} - F$.

Case 2. $2n-6 \le |E(AG_{n+1}(n+1)) \cap F| \le 4n-13$ and $|E(AG_{n+1}(n)) \cap F| \le 2n-7$. Then we have $|E(AG_{n+1}(r)) \cap F| \le 2n-7$ for all $r \in \{1, 2, \cdots, n\}$. Let $i_1 = n+1$. Three cases are further considered:

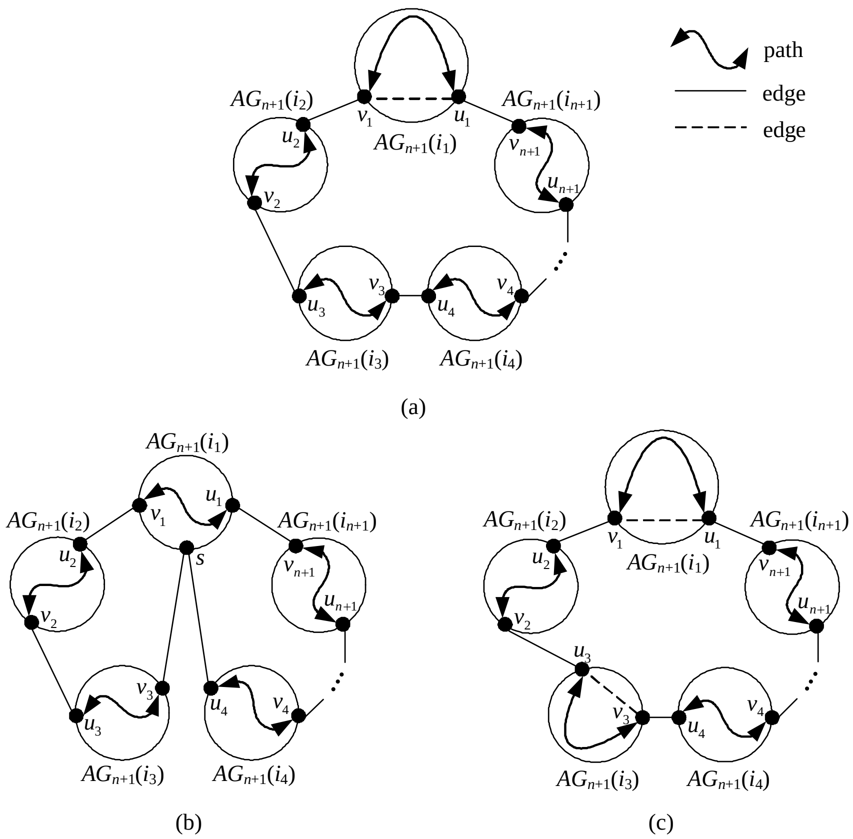

Fig. 3. A Hamiltonian cycle in $AG_{n+1} - F$. (a) $|E(AG_{n+1}(n+1)) \cap F| \ge 2n-6$, $|E(AG_{n+1}(n)) \cap F| \le 2n-7$, and $\delta(AG_{n+1}(i_1) - F) \ge 1$. (b) $|E(AG_{n+1}(n+1)) \cap F| \ge 2n-6$, $|E(AG_{n+1}(n)) \cap F| \le 2n-7$, and $\delta(AG_{n+1}(i_1) - F) = 0$. (c) $|E(AG_{n+1}(n+1)) \cap F| = |E(AG_{n+1}(n)) \cap F| = 2n-6$.

Case 2.1. $\delta(AG_{n+1}(i_1) - F) \ge 2$. The induction hypothesis assures that there exists a Hamiltonian cycle C in $AG_{n+1}(i_1) - F$. We can find $(u_1, v_1) \in C$ such that (v_1, u_2), $(u_1, v_{n+1}) \in E^{(n+1)}(AG_{n+1}) - F$, where $u_2 \in V(AG_{n+1}(i_2))$ and $v_{n+1} \in V(AG_{n+1}(i_{n+1}))$. Let $P_{u_1v_1} = C - \{(u_1, v_1)\}$ and $I = \{1, 2, \cdots, n\}$. By Lemma 3, we can find a Hamiltonian path $P_{u_2v_{n+1}}$ in $AG_{n+1}(I) - F$ (when $n = 4$ and if j', k' exist, let $\{j', k'\} \ne \{i_r, i_{r+1}\}$ for $2 \le r \le n$). Hence $\langle u_1, P_{u_1v_1}, v_1, u_2, P_{u_2v_{n+1}}, v_{n+1}, u_1\rangle$ form a Hamiltonian cycle in $AG_{n+1} - F$ (see Fig. 3(a)).

Case 2.2. $\delta(AG_{n+1}(i_1) - F) = 1$. There is exactly one vertex v_1 with degree one in $AG_{n+1}(i_1) - F$. Since $\delta(AG_{n+1} - F) \geq 2$, we have $(v_1, u_2) \in E^{(n+1)}(AG_{n+1}) - F$, for some $u_2 \in V(AG_{n+1}(i_2))$. Select $(u_1, v_1) \in E(AG_{n+1}(i_1)) \cap F$ such that $(u_1, v_{n+1}) \in E^{(n+1)}(AG_{n+1}) - F$, where $v_{n+1} \in V(AG_{n+1}(i_{n+1}))$. We can always find such (u_1, v_1) since $|\{z \mid (v_1, z) \in F$ and $z \in V(AG_{n+1}(i_1))\}| = 2n - 5$, $|e^{(n+1)}(z)| = 2$ for all $z \in V(AG_{n+1}(i_1))$, $|E^{(n+1)}(AG_{n+1}) \cap F| \leq (4n - 9) - (2n - 5) = 2n - 4$, and $2(2n - 5) > 2n - 4$ when $n \geq 4$. Moreover, since $|E(AG_{n+1}(i_1)) \cap (F - \{(u_1, v_1)\})| \leq 4n - 14$, the induction hypothesis assures that there is a Hamiltonian cycle C in $AG_{n+1}(i_1) - (F - \{(u_1, v_1)\})$. Note that (u_1, v_1) must be contained in C. Then the construction of a Hamiltonian cycle in $AG_{n+1} - F$ is similar to Fig. 3(a) (when $n = 4$ and if j', k' exist, let $\{j', k'\} \neq \{i_r, i_{r+1}\}$ for $2 \leq r \leq n$).

Case 2.3. $\delta(AG_{n+1}(i_1) - F) = 0$. Note that $4n - 13 = 3 < 4 = 2(n - 2)$ when $n = 4$, hence this case will occur only when $n \geq 5$, thus, j' and k' do not exist. There is exactly one vertex s with degree zero in $AG_{n+1}(i_1) - F$. Since $\delta(AG_{n+1} - F) \geq 2$, we have $e^{(n+1)}(s) \cap F = 0$. Let (v_3, s), $(s, u_4) \in e^{(n+1)}(s)$ where $v_3 \in V(AG_{n+1}(i_3))$ and $u_4 \in V(AG_{n+1}(i_4))$. Additionally, there exist (v_1, u_2), $(u_1, v_{n+1}) \in E^{(n+1)}(AG_{n+1}) - F$, where u_1, v_1 are two distinct vertices in $V(AG_{n+1}(i_1)) - \{s\}$, $u_2 \in V(AG_{n+1}(i_2))$, and $v_{n+1} \in V(AG_{n+1}(i_{n+1}))$. In addition, let $F' = \{s\} \cup (E(AG_{n+1}(i_1)) \cap F - \{(s, z) \mid z \in V(AG_{n+1}(i_1))\})$. Note that $|F'| \leq 2n - 8$. By Lemma 1, we can find a Hamiltonian path $P_{u_1 v_1}$ in $AG_{n+1}(i_1) - F'$. Let $I_1 = \{i_2, i_3\}$ and $I_2 = \{1, 2, \cdots, n + 1\} - \{i_1, i_2, i_3\}$. By Lemma 3, we can find a Hamiltonian path $P_{u_2 v_3}$ in $AG_{n+1}(I_1) - F$, and a Hamiltonian path $P_{u_4 v_{n+1}}$ in $AG_{n+1}(I_2) - F$. Hence $\langle u_1, P_{u_1 v_1}, v_1, u_2, P_{u_2 v_3}, v_3, s, u_4, P_{u_4 v_{n+1}}, v_{n+1}, u_1\rangle$ form a Hamiltonian cycle in $AG_{n+1} - F$ (see Fig. 3(b)).

Case 3. $|E(AG_{n+1}(n + 1)) \cap F| = 4n - 12$. Note that this case will occur only when $4 \leq n \leq 6$. In addition, j' and k' do not exist and $|E(AG_{n+1}(r)) \cap F| = 0$ for all $r \in \{1, 2, \cdots, n\}$. Let $i_1 = n + 1$. Three cases are further considered:

Case 3.1. $\delta(AG_{n+1}(i_1) - F) \geq 2$. We can find $(u_1, v_1) \in E(AG_{n+1}(i_1)) \cap F$ such that (v_1, u_2), $(u_1, v_{n+1}) \in E^{(n+1)}(AG_{n+1}) - F$, where $u_2 \in V(AG_{n+1}(i_2))$ and $v_{n+1} \in V(AG_{n+1}(i_{n+1}))$. Since $|E(AG_{n+1}(i_1)) \cap (F - \{(u_1, v_1)\})| = 4n - 13$, the induction hypothesis assures that there is a Hamiltonian cycle C in $AG_{n+1}(i_1) - (F - \{(u_1, v_1)\})$. Assume that C contains (u_1, v_1) (otherwise, the discussion is easier). Then the construction of a Hamiltonian cycle in $AG_{n+1} - F$ is similar to Fig. 3(a).

Case 3.2. $\delta(AG_{n+1}(i_1) - F) = 1$. There is exactly one vertex v_1 with degree one in $AG_{n+1}(i_1) - F$. Since $\delta(AG_{n+1} - F) \geq 2$, we have $(v_1, u_2) \in E^{(n+1)}(AG_{n+1}) - F$, for some $u_2 \in V(AG_{n+1}(i_2))$. Select $(u_1, v_1) \in E(AG_{n+1}(i_1)) \cap F$ such that $(u_1, v_{n+1}) \in E^{(n+1)}(AG_{n+1}) - F$, where $v_{n+1} \in V(AG_{n+1}(i_{n+1}))$. We can always find such (u_1, v_1) since $|\{z \mid (v_1, z) \in F$ and $z \in V(AG_{n+1}(i_1))\}| = 2n - 5$, $|e^{(n+1)}(z)| = 2$ for all $z \in V(AG_{n+1}(i_1))$, $|E^{(n+1)}(AG_{n+1}) \cap F| = (4n - 9) - (4n - 12) = 3$, and $2(2n - 5) > 3$ when $n \geq 4$. Moreover, since $|E(AG_{n+1}(i_1)) \cap (F - \{(u_1, v_1)\})| = 4n - 13$, the induction hypothesis assures that there is a Hamiltonian cycle C in $AG_{n+1}(i_1) - (F - \{(u_1, v_1)\})$. Note that (u_1, v_1) must be contained in C. Then the construction of a Hamiltonian cycle in $AG_{n+1} - F$ is similar to Case 3.1.

Case 3.3. $\delta(AG_{n+1}(i_1) - F) = 0$. There is exactly one vertex s with degree zero in $AG_{n+1}(i_1) - F$. Since $\delta(AG_{n+1} - F) \geq 2$, we have $e^{(n+1)}(s) \cap F = 0$. Let (v_3, s), $(s, u_4) \in e^{(n+1)}(s)$ where $v_3 \in V(AG_{n+1}(i_3))$ and $u_4 \in V(AG_{n+1}(i_4))$. Additionally, there exist (v_1, u_2), $(u_1, v_{n+1}) \in E^{(n+1)}(AG_{n+1}) - F$, where $u_1, v_1 \in V(AG_{n+1}(i_1)) - \{s\}$, $u_2 \in V(AG_{n+1}(i_2))$, and $v_{n+1} \in V(AG_{n+1}(i_{n+1}))$. In addition, let $F' = \{s\} \cup (E(AG_{n+1}(i_1)) \cap F - \{(s, z) \mid z \in V(AG_{n+1}(i_1))\})$. Note that $|F'| \leq 2n - 7$. By Lemma 1, we can find a Hamiltonian path $P_{u_1v_1}$ in $AG_{n+1}(i_1) - F'$. Let $I_1 = \{i_2, i_3\}$ and $I_2 = \{1, 2, \cdots, n+1\} - \{i_1, i_2, i_3\}$. By Lemma 3, we can find a Hamiltonian path $P_{u_2v_3}$ in $AG_{n+1}(I_1) - F$ and a Hamiltonian path $P_{u_4v_{n+1}}$ in $AG_{n+1}(I_2) - F$. Hence $\langle u_1, P_{u_1v_1}, v_1, u_2, P_{u_2v_3}, v_3, s, u_4, P_{u_4v_{n+1}}, v_{n+1}, u_1 \rangle$ form a Hamiltonian cycle in $AG_{n+1} - F$ (see Fig. 3(b)).

Case 4. $|E(AG_{n+1}(n+1)) \cap F| = |E(AG_{n+1}(n)) \cap F| = 2n - 6$. Note that this case will occur only when $4 \leq n \leq 6$. In addition, j' and k' do not exist in this case and $|E(AG_{n+1}(r)) \cap F| = 0$ for all $r \in \{1, 2, \cdots, n-1\}$. In this case, we also have $\delta(AG_{n+1}(r) - F) \geq 2$ for all $1 \leq r \leq n+1$. Let $i_1 = n+1$ and $i_3 = n$. The induction hypothesis assures that there are a Hamiltonian cycle C_1 in $AG_{n+1}(i_1) - F$ and a Hamiltonian cycle C_3 in $AG_{n+1}(i_3) - F$. We can find $(u_1, v_1) \in C_1$ and $(u_3, v_3) \in C_3$ such that (v_1, u_2), (v_2, u_3), (v_3, u_4), $(u_1, v_{n+1}) \in E^{(n+1)}(AG_{n+1}) - F$, where $u_2, v_2 \in V(AG_{n+1}(i_2))$, $u_4 \in V(AG_{n+1}(i_4))$, and $v_{n+1} \in V(AG_{n+1}(i_{n+1}))$. Let $I = \{1, 2, \cdots, n+1\} - \{i_1, i_2, i_3\}$, $P_{u_1v_1} = C_1 - \{(u_1, v_1)\}$, and $P_{u_3v_3} = C_3 - \{(u_3, v_3)\}$. By Lemma 1 and Lemma 3, we can find a Hamiltonian path $P_{u_2v_2}$ in $AG_{n+1}(i_2) - F$ and a Hamiltonian path $P_{u_4v_{n+1}}$ in $AG_{n+1}(I) - F$. Hence $\langle u_1, P_{u_1v_1}, v_1, u_2, P_{u_2v_2}, v_2, u_3, P_{u_3v_3}, v_3, u_4, P_{u_4v_{n+1}}, v_{n+1}, u_1 \rangle$ form a Hamiltonian cycle in $AG_{n+1} - F$ (see Fig. 3(c)). □

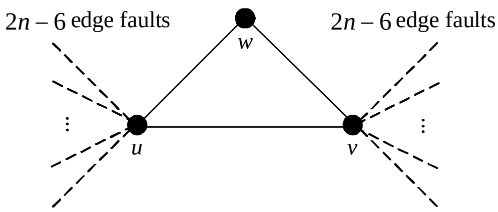

Fig. 4. A distribution of $4n - 12$ edge faults over AG_n.

The result is optimal with respect to the number of edge faults tolerated. Since alternating group graphs can be constructed recursively, it is easy to see that there exists a 3-cycle in AG_n, where $n \geq 4$. Fig. 4 shows a distribution of $4n - 12$ edge faults over AG_n, where $\langle w, u, v, w \rangle$ is a cycle and (u, v), (u, w) (respectively, (v, u), (v, w)) are the only two fault-free edges incident to u (respectively, v). It is easy to see that no fault-free Hamiltonian cycle exists in the faulty AG_n.

4 Concluding Remarks

Since processor faults and/or link faults may occur to multiprocessor systems, it is both practically significant and theoretically interesting to study the fault tolerance of multiprocessor systems. Most of previous works used the random fault model, which assumed that the faults might occur everywhere without any restriction. There was another fault model, i.e., the conditional fault model, which assumed that the fault distribution must satisfy some properties.

In this paper, adopting the conditional fault model and assuming that there were two or more fault-free edges incident to each vertex, we constructed a fault-free Hamiltonian cycle of an n-dimensional alternating group graph with up to $4n - 13$ edge faults. The main construction methods were demonstrated in Fig. 3. This result is optimal with respect to the number of edge faults tolerated. Since an n-dimensional alternating group graph AG_n is isomorphic to an $(n, n - 2)$-arrangement graph $A_{n,n-2}$, our results and methods might be useful for people who want to solve Hamiltonian problems on faulty arrangement graphs under conditional fault model and the same assumption.

Acknowledgments. The authors would like to thank the National Science Council of the Republic of China, Taiwan for financially supporting this research under Contract No. NSC 95-2221-E-239-002-.

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