AN EXAMPLE OF A SURFACE OF GENERAL TYPE WITH $p_g = q = 2$ AND $K_X^2 = 5$

JUNGKAI A. CHEN AND CHRISTOPHER D. HACON

We give an example of a minimal complex surface of general type with $p_g = q = 2$ and $K_X^2 = 5$.

1. Introduction

Recently, there has been considerable interest in understanding the geometry of irregular complex projective surfaces with $\chi(X, \omega_X) = 1+p_g(X)-q(X) = 1$, and in particular of surfaces with $p_g = q = 2$. Let X be a smooth minimal complex surface of general type. If $\chi(X, \omega_X) = 1$, then one has the bound $1 \leq K_X^2 \leq 9$. If, in addition the surface is irregular, i.e. $q(X) = h^0(X, \Omega_X^1) > 0$, then one also has $K_X^2 \geq 2p_g(X)$ and so $p_g(X) \leq 4$. In [**Be**], it is shown that the case $p_g = q = 4$ corresponds to the product of two curves of genus 2. In [**HP**] and [**Pi**], surfaces with $p_g = q = 3$ are completely classified. When $K_X^2 = 2p_g(X) = 6$ they are symmetric products of curves of genus 3 and when $K_X^2 = 8$ they admit an irrational pencil. The case $p_g = q = 2$ seems considerably more delicate. At any rate Catanese suggests that, analogously to the $p_g = q = 3$ case, a surface of general type with $p_g = q = 2$ and with no fibration over an elliptic curve, is a degree 2 covering of a principally polarized abelian surface (A, Θ) branched along a divisor in the linear series $|2\Theta|$ (cf. [**Zu**]).

In [**Zu**], Zucconi has classified surfaces of general type with $p_g = q = 2$ which admit an irrational pencil. In [**Ma**], Manetti shows that a minimal surface of general type with K_X ample and $K_X^2 = 4$, is a degree 2 covering of a principally polarized abelian surface (A, Θ) branched along a divisor $D \in |2\Theta|$. Ciliberto and Mendes Lopes [**CM**], conjecture that this should be the case for any minimal surface of general type with $p_g = q = 2$ and $K_X^2 = 4$.

The purpose of this note is to give a counter-example to Catanese's conjecture above. The example we construct is birational to a triple cover of an abelian surface. Its canonical divisor K_X is ample, $p_g = q = 2$ and $K_X^2 = 5$. The construction is motivated in §3, where we obtain restrictions on the structure of the sheaf $\mathrm{alb}_{X,*}(\omega_X)$.

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2. Construction and Verification

We will need some results from the theory of Mukai transforms. Let $\hat{A}$ be the dual abelian variety of A and $\mathcal{P}$ be the normalized Poincaré line bundle on $A \times \hat{A}$. Following [**Muk**], define the functor $\hat{\mathcal{S}}$ of $\mathcal{O}_A$-modules into the category of $\mathcal{O}_{\hat{A}}$-modules by

$$\hat{\mathcal{S}}(M) = \pi_{\hat{A},*}(\mathcal{P}\otimes\pi_A^* M).$$

The derived functor $R\hat{\mathcal{S}}$ of $\hat{\mathcal{S}}$ then induces an equivalence of categories between the two derived categories $D(A)$ and $D(\hat{A})$. More precisely, by [**Muk**]: *There are isomorphisms of functors:*

$$R\mathcal{S} \circ R\hat{\mathcal{S}} \cong (-1_A)^*[-g]$$

and

$$R\hat{\mathcal{S}} \circ R\mathcal{S} \cong (-1_{\hat{A}})^*[-g],$$

where $[-g]$ *denotes "shift the complex* g *places to the right".* The Weak Index Theorem (W.I.T.) holds for a coherent sheaf $\mathcal{F}$ on A if there exists an integer $i(\mathcal{F})$ such that for all $j \neq i(\mathcal{F})$, one has $R^j\hat{\mathcal{S}}(\mathcal{F}) = 0$. The coherent sheaf $R^{i(\mathcal{F})}\hat{\mathcal{S}}(\mathcal{F})$ is denoted simply by $\hat{\mathcal{F}}$.

Consider now (A, M), a simple polarized abelian surface of type $(1, 2)$. Assume that M is symmetric, i.e. $(-1)^*M \cong M$. The linear series $|M|$ has 4 isolated base points $\{o, p, q, r\}$. We may assume that o is the identity of the abelian surface and p, q, r are 2-torsion with $r = p + q$ (see, e.g. [**Ba**]). Each divisor $D \in |M|$ is either a nonsingular curve of genus 3 or a singular curve with a simple node distinct from the base points. $M^\vee$ satisfies the W.I.T. of index 2. Let

$$\mathcal{F} = \widehat{M^\vee} := R^2\hat{\mathcal{S}}(M^\vee)$$

be the Fourier-Mukai transform of $M^\vee$. The vector bundle $\mathcal{F}$ has rank 2. Let $\mathcal{E} = \mathcal{F}^\vee$. One can check that,

$$\dim \operatorname{Hom}(S^3\mathcal{E}, \Lambda^2\mathcal{E}) = h^0(\hat{A}, (S^3\mathcal{E})^\vee \otimes \wedge^2 \mathcal{E}) = 2.$$

According to Miranda's triple covering construction [**Mi**], there is a 2-dimensional family of triple coverings $\hat{f} : \hat{X} \to \hat{A}$ with $\hat{f}_*\mathcal{O}_{\hat{X}} = \mathcal{O}_{\hat{A}} \oplus \mathcal{E}$.

The idea is as follows: in order to construct a triple covering $\hat{f} : \hat{X} \longrightarrow \hat{A}$ over $\hat{A}$ with *Tschirnhausen module* $\mathcal{E}$ (cf. [**Mi**]), we first construct a triple covering $f : X \longrightarrow A$ with Tschirnhausen module $\phi^*_{M^\vee}\mathcal{E}$. In Claim 1, we identify those that descend to a triple covering $f : \hat{X} \longrightarrow \hat{A}$. In Claim 2, we then verify that for a general such covering, the singularities of X are rational. It follows that the singularities of $\hat{X}$ are also rational. Finally, one can compute the invariants of $\hat{X}$ via the invariants of X.

Let $\phi_{M^\vee} : A \longrightarrow \hat{A}$ be the isogeny defined by $M^\vee$. We have the following commutative diagram:

$$\begin{array}{ccc} X = A \times_{\hat{A}} \hat{X} & \xrightarrow{\phi} & \hat{X} \\ f\big\downarrow & & \hat{f}\big\downarrow \\ A & \xrightarrow{\phi_{M^\vee}} & \hat{A} \end{array}$$

Where $\phi : X \to \hat{X}$ is a $4:1$ etale covering and $f : X \to A$ is a triple covering determined by a section of

$$\phi^*_{M^\vee}\mathrm{Hom}(S^3\mathcal{E}, \wedge^2\mathcal{E}) \subset \mathrm{Hom}(S^3\phi^*_{M^\vee}\mathcal{E}, \wedge^2\phi^*_{M^\vee}\mathcal{E}).$$

By [**Muk**], $\phi^*_{M^\vee}\mathcal{E} \cong M^\vee \oplus M^\vee$. Thus

$$\mathrm{Hom}(S^3\phi^*_{M^\vee}\mathcal{E}, \wedge^2\phi^*_{M^\vee}\mathcal{E}) \cong H^0(A, M)^{\oplus 4}.$$

In order to determine the corresponding 2-dimensional subspace, we consider the Heisenberg group action on $H^0(A, M)$. The Heisenberg group can be identified as

$$\mathcal{G}(\delta) := \{(\alpha, t, l) | \alpha \in k^*, t \in \mathbb{Z}_2, l \in \hat{\mathbb{Z}_2}\}$$

with group law $(\alpha, t, l)(\alpha, t', l') = (\alpha\alpha' l'(t), t+t', l+l')$. Moreover, $H^0(A, M)$ corresponds to $\mathrm{Hom}(\mathbb{Z}_2, k)$. The action of $\mathcal{G}(\delta)$ on $\mathrm{Hom}(\mathbb{Z}_2, k)$ is given as $(\alpha, t, l)f(x) = \alpha l(x) f(t + x)$. Let X, Y be the sections in $H^0(A, M)$ corresponding to the characteristic functions of $0, 1$ in $\mathrm{Hom}(\mathbb{Z}_2, k)$ respectively.

Claim 1: The 2-dimensional subspace is determined as

$$\phi^*_{M^\vee}\mathrm{Hom}(S^3\mathcal{E}, \wedge^2\mathcal{E}) \cong \{(sX, tY, -tX, -sY) | s, t \in k\} \subset H^0(A, M)^{\oplus 4}.$$

Grant the claim for the time being. Following Miranda (cf. [**Mi**]) we can then construct a triple covering $f : X \to A$ by using the data $a = sX, b = tY, c = -tX, d = -sY$. Over an affine open subset U of A, the triple covering can be described in $U \times \mathbb{A}^2$ as by the 2×2 minors of

$$\begin{pmatrix} Z + a & W - 2d & c \\ b & Z - 2a & W + d \end{pmatrix}$$

where Z, W are coordinates for $\mathbb{A}^2$.

Following [**Mi**] §4, we have $A = s^2X^2 + stY^2$, $B = (t^2 - s^2)XY$ and $C = s^2Y^2 + stX^2$. The branch locus is defined by $D = B^2 - 4AC \in H^0(M)^{\otimes 4}$ and one can see that it corresponds to a divisor $D_1 + D_2 + D_3 + D_4$ with $D_i \in |M|$. For general choice of s, t, the D_i are all distinct and nonsingular. It is easy to check (cf. [**Mi**] §5) that for general choices of s, t, the only possible singularities of X lie over the 4 base points $\{p, q, r, o\}$. We remark that f is totally ramified only over these 4 base points.

Let $x \in X$ be a point lying over one of $\{p, q, r, o\}$.

Claim 2: For general s, t, the singularity of X at x, is locally isomorphic to a cone over a twisted cubic.

Therefore, X has only rational singularities and so does $\hat{X}$. A resolution $\hat{\mu} : \hat{X}' \to \hat{X}$ can be obtained by blowing up along the singularity. The corresponding resolution $\mu : X' \to X$ is the blow up of X along the 4 points lying over $\{p, q, r, o\}$. Let $\{E_i\}_{i=1,..,4}$ be the exceptional divisors and $\{R_i\}_{i=1,...,4}$ be the proper transform of the D_i, then

$$K_{X'} = \sum_{i=1,...,4} R_i + \sum_{i=1,...,4} E_i.$$

Note that $R_i \cdot R_j = 0$, $R_i \cdot E_j = 1$ for all i, j and $E_i^2 = -3$, $E_i \cdot E_j = 0$ for $i \neq j$. Thus we have $K^2_{X'} = 20$.

$$p_g(X') = h^0(X', \omega_{X'}) = h^2(X', \mathcal{O}_{X'}) = h^2(X, \mathcal{O}_X) =$$

$$h^2(A, \mathcal{O}_A) + 2h^2(A, M^\vee) = 5.$$

Similarly, $q(X') = 2$ and $\chi(X', \omega_{X'}) = 4$. One can also check that $K_{X'}$ is ample.

Since $X' \to \hat{X}'$ is an etale cover of degree 4, one has $\chi(\hat{X}', \omega_{\hat{X}'}) = 1$, $(K_{\hat{X}'})^2 = 5$ and $K_{\hat{X}'}$ is ample. $\hat{X}$ has only rational singularity. It is easy to see that $q(\hat{X}') = 2$ and hence $p_g(\hat{X}') = 2$. So $\hat{X}'$ is a surface of general type with $p_g = q = 2$ and $K^2 = 5$.

Proof. (Claim 1) We follow [**Mum**]. Let $H(M^\vee)$ be the kernel of $\phi_{M^\vee} : A \longrightarrow \hat{A}$, i.e. the set of points $x \in A$ such that $T_x^* M^\vee \cong M^\vee$. Then $H(M) = H(M^\vee)$. Let $\mathcal{G}(M)$ be the set of pairs (x, φ) such that $x \in H(M)$ and φ is an isomorphism $\varphi : M \longrightarrow T_x^* M$. Then $\mathcal{G}(M)$ is a group sitting in the following exact sequence:

$$0 \longrightarrow k^* \longrightarrow \mathcal{G}(M) \longrightarrow H(M) \longrightarrow 0.$$

There is an isomorphism of groups $\mathcal{G}(M) \cong \mathcal{G}(\delta)$. Under this identification, the representation of $\mathcal{G}(M)$ on $H^0(A, M)$ corresponds to the unique representation of $\mathcal{G}(\delta)$ on $V = V(\delta) := \mathrm{Hom}(\mathbb{Z}_2, k)$, which is defined by $((\alpha, t, l)f)(x) = \alpha \cdot l(x) \cdot f(t+x)$. With respect to the ordered basis $\{\chi^0, \chi^1\}$ (characteristic functions of $0, 1$ respectively), this representation is induced by

$$(1,1,1) \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (1,1,0) \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (1,0,1) \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The corresponding $\mathcal{G}(\delta)$ representation on $S^3 V^\vee \otimes \Lambda^2 V \otimes V$ can easily be computed. By [**Muk**] Proposition 3.11, we have

$$\phi^*_{M^\vee} \mathcal{F} \cong H^2(A, M) \otimes M \cong (H^0(A, M) \otimes M^\vee)^\vee \cong \phi^*_{M^\vee} \mathcal{E}^\vee.$$

One sees that

$$H^0(A, S^3\phi^*_{M^\vee}\mathcal{E}^\vee \otimes \Lambda^2\phi^*_{M^\vee}\mathcal{E}) \cong S^3H^0(A,M)^\vee \otimes \Lambda^2 H^0(A,M) \otimes H^0(A,M).$$

This vector space is in turn isomorphic to $\bigoplus_{i=1}^4 H^0(A,M)$. We can now compute the corresponding $\mathcal{G}(M)$ representation in terms of the above $\mathcal{G}(\delta)$ representation.

Let $p_i, i = 1, ..., 4$ denote the projection onto the i-th factor. With respect to the ordered basis

$$\{e_1, e_2, ..., e_8\} = \{p_1^*X, p_1^*Y, ..., p_4^*X, p_4^*Y\} =$$

$$\{\hat{X}^3 \otimes X \wedge Y \otimes X, \hat{X}^3 \otimes X \wedge Y \otimes Y, \hat{X}^2\hat{Y} \otimes X \wedge Y \otimes X, \hat{X}^2\hat{Y} \otimes X \wedge Y \otimes Y,$$

$$\hat{X}\hat{Y}^2 \otimes X \wedge Y \otimes X, \hat{X}\hat{Y}^2 \otimes X \wedge Y \otimes Y, \hat{Y}^3 \otimes X \wedge Y \otimes X, \hat{Y}^3 \otimes X \wedge Y \otimes Y\},$$

we have that $(1,1,1) \to R \in M_8(k)$ with $R_{i,j} = 0$ if $i + j \neq 8$ and $R_{i,8-i} = \{-1, 1, 1, -1, -1, 1, 1, -1\}$, respectively $(1,1,0) \to M \in M_8(k)$ with $M_{i,j} = 0$ if $i + j \neq 8$ and $M_{i,8-i} = \{1,1,1,1,1,1,1,1\}$. In particular $R^2 = M^2 = 1$ and $RM = MR$. There is an induced representation of $H(M^\vee) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. It is easy to see that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ invariant elements are just the subspace

$$\{s(e_1 - e_8) + t(e_4 - e_5) | s, t \in k\} = \{(sX, tY, -tX, -sY) | s, t \in k\}.$$

These invariant elements correspond to the subspace $\phi^*_{M^\vee}\mathrm{Hom}(S^3\mathcal{E}, \Lambda^2\mathcal{E})$. □

Proof. (Claim 2) On a neighborhood of one of the base loci o, p, q, r we may assume that X, Y (or any two distinct sections of $H^0(A,M)$) are local coordinates. By [**Ha**] pg. 14 exercise 1.25, the above mentioned 2×2 minors define a twisted cubic if and only if for all $[u : v] \in \mathbb{P}^1$, the linear forms

$$u(Z + sX) - vtY, u(W + 2sY) - v(Z - 2sX), -utX - v(W - sY)$$

are linearly independent. In other words, if and only if the matrix

$$\begin{pmatrix} us & -vt & u & 0 \\ 2vs & 2us & -v & u \\ -ut & vs & 0 & -v \end{pmatrix}$$

has a nonzero 3×3 minor for every value of u, v By inspection one sees that this is the case for general s, t (more precisely for $t \neq 0$ and $t^2 \neq 9s^2$). □

3. Computation of $\mathrm{alb}_{X,*}(\omega_X)$

In this section, using the techniques of [**HP**], we find restrictons on the structure of the coherent sheaf $\mathrm{alb}_{X,*}(\omega_X)$. It was this computation that suggested to us the possibility of constructing the example of §2.

Proposition 3.1. *([**CM**] Proposition 2.3) Let X be a minimal surface of general type with $p_g = q = 2$. Then $a := \mathrm{alb}_X : X \longrightarrow \mathrm{Alb}(X) =: A$ is not surjective if and only if $B := a(X)$ is a curve of genus 2 and $a : X \longrightarrow B$ has smooth connected fibers of genus 2 with constant modulus and $K_X^2 = 8$.*

We now therefore consider the situation where $a : X \longrightarrow \mathrm{Alb}(X) =: A$ is surjective. For any coherent sheaf $\mathcal{F}$ on X, define

$$V^i(X, \mathcal{F}) := \{P \in \mathrm{Pic}^0(X) | h^i(X, \mathcal{F} \otimes P) \neq 0\}.$$

Since a is generically finite, $R^i a_* \omega_X = 0$ for all $i > 0$ and so $V^i(X, \omega_X) = V^i(A, a_*\omega_X)$ for all i.

Lemma 3.2. *Let X be a minimal surface with $p_g = q = 2$ and surjective Albanese map. If $\dim V^1(X, \omega_X) \geq 1$, then there exists an elliptic pencil $X \longrightarrow E$ with $g(E) = 1$.*

Proof. By the generic vanishing theorems of Green and Lazarsfeld, $\dim V^1(X, \omega_X) < 2$ and if T is a component of $V^1(X, \omega_X)$ of dimension 1, then T is a translate of an elliptic curve $T_0 \subset \mathrm{Pic}^0(X)$. The pencil $X \longrightarrow E$ is induced by $a : X \longrightarrow \mathrm{Alb}(X)$ composed with the dual map of abelian varieties $\mathrm{Alb}(X) \longrightarrow E := T_0^\vee$. □

An immediate consequence is the following:

Corollary 3.3. *Let X be a minimal surface of general type with $p_g = q = 2$ without irrational pencils. Then $a : X \longrightarrow A$ is surjective with $V^1(A, a_*\omega_X)$ supported on finitely many points.*

A vector bundle U on an abelian variety A is unipotent if it has a filtration

$$0 = U_0 \subset U_1 \subset ... \subset U_{n-1} \subset U_n = U$$

such that $U_i/U_{i-1} \cong \mathcal{O}_A$. A vector bundle is homogeneous if and only if it is isomorphic to $\oplus_{i=1}^n (P_i \otimes U_i)$ with $P_i \in \mathrm{Pic}^0(A)$ and U_i unipotent vector bundles. By [**Muk**], there is an one-to-one corresponce between sheaves supported on finitely many points and homogeneous vector bundles via the Fourier-Mukai transform.

Lemma 3.4. *Let $\mathcal{F}$ be a coherent sheaf on an abelian surface, then $R^i\mathcal{S}R^j\hat{\mathcal{S}}(\mathcal{F}) = 0$ for $(i,j) \in \{(1,2),(2,2),(0,0),(1,0)\}$ and there is an injection (resp. surjection) $d : R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} \longrightarrow R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F}$ (resp. $d' : R^0\mathcal{S}R^2\hat{\mathcal{S}}\mathcal{F} \longrightarrow R^2\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F}$). In particular $R^0\hat{\mathcal{S}}\mathcal{F}$ (resp. $R^2\hat{\mathcal{S}}\mathcal{F}$) satisfies the W.I.T. of index 2 (resp. 0).*

Proof. As mentioned above, by [**Muk**], there is an isomorphism of functors

$$R\mathcal{S} \circ R\hat{\mathcal{S}} \cong (-1_A)^*[-2].$$

In particular there is a spectral sequence $E_2^{p,q} = R^p\mathcal{S}R^q\hat{\mathcal{S}}\mathcal{F}$ with $E_\infty^{p,q} = 0$ if $p + q \neq 2$. The only possibly non-vanishing differentials d_2 are

$$d : R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} \longrightarrow R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \ \text{ and } \ d' : R^0\mathcal{S}R^2\hat{\mathcal{S}}\mathcal{F} \longrightarrow R^2\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F}.$$

One sees that $E_2^{p,q} = E_\infty^{p,q} = 0$ for $(p,q) \in \{(1,2),(2,2),(0,0),(1,0)\}$. Moreover, $ker(d) = E_3^{0,1} = E_\infty^{0,1} = 0$. So d is an injection. Similarly d' is a surjection. □

Theorem 3.5. *Let X be a minimal surface of general type with $p_g = q = 2$ without any irrational pencil. Then there exist homogeneous vector bundles $\mathcal{H}$, and a negative definite line bundle L on $\hat{A} = \mathrm{Pic}^0(A)$ (i.e. $L^\vee$ is ample) such that $a_*\omega_X$ fits into the following exact sequences*

$$0 \longrightarrow \mathcal{O}_A \longrightarrow a_*\omega_X \longrightarrow \mathcal{F} \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{H} \longrightarrow \hat{L} \longrightarrow (-1_A)^*\mathcal{F} \longrightarrow 0.$$

Proof. Notice that $\omega_A = \mathcal{O}_A$. By assumption X has no irrational pencils, therefore $a : X \longrightarrow A$ is surjective and $\dim V^1(X, \omega_X) = 0$, hence $V^1(X,\omega_X) = \{\mathcal{O}_X, P_1, ..., P_n\}$. Let $\mathcal{F}$ be the coherent sheaf defined by the short exact sequence

$$0 \longrightarrow \mathcal{O}_A \longrightarrow a_*\omega_X \longrightarrow \mathcal{F} \longrightarrow 0.$$

Since $R^i a_*\omega_X = 0$ for $i > 0$, one sees that for $i \geq 0$,

$$H^i(A, a_*\omega_X) \cong H^i(X,\omega_X) \cong H^i(A,\omega_A)$$

and therefore $h^1(\mathcal{F}) = h^2(\mathcal{F}) = 0$. Moreover, for all $\mathcal{O}_X \neq P \in \mathrm{Pic}^0(A)$, one has $h^i(A, \mathcal{F}\otimes P) = h^i(X, \omega_X \otimes P)$ for all i. In particular $V^2(A,\mathcal{F}) = \emptyset$ and $V^1(A,\mathcal{F}) = \{P_1, ..., P_n\}$. We have that $R^2\hat{\mathcal{S}}\mathcal{F} = 0$ and $R^1\hat{\mathcal{S}}\mathcal{F} = \oplus B_i$ where the sheaves B_i are supported at the points P_i (and are Artinian $\mathcal{O}_{\hat{A},P_i}$-modules cf. [**Muk**] Example 2.9). In particular, $R^1\hat{\mathcal{S}}\mathcal{F}$ satisfies the W.I.T. of index 0. Consider now the spectral sequence of the proof of Lemma 3.4. The only non-zero E_2 terms are $E_2^{0,1}$ and $E_2^{2,0}$. Therefore, one has the following exact sequence

$$0 \longrightarrow R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} \longrightarrow R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \longrightarrow (-1_A)^*\mathcal{F} \longrightarrow 0.$$

First note that $R^1\hat{\mathcal{S}}\mathcal{F}$ is supported on finitely many points. It follows that $R^0\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F} = R\mathcal{S}R^1\hat{\mathcal{S}}\mathcal{F}$ is a homogeneous vector bundle, call it $\mathcal{H}$. It suffices to show that $R^0\hat{\mathcal{S}}\mathcal{F}$ is a negative line bundle.

Let $U = \mathrm{Pic}^0(A) - \{\mathcal{O}_A, P_1, ..., P_n\}$, then for all $P \in U$

$$h^0(A, \mathcal{F}\otimes P) = h^0(A, a_*\omega_X \otimes P) = \chi(X, \omega_X) = 1.$$

Thus $R^0\hat{\mathcal{S}}\mathcal{F}|U$ is locally free of rank 1. Let $L = (R^0\hat{\mathcal{S}}\mathcal{F})^{\vee\vee}$. Then L is a reflexive sheaf of rank 1 on a non-singular surface and hence a line bundle. Since $R^0\hat{\mathcal{S}}\mathcal{F} = R^0\hat{\mathcal{S}}a_*\omega_X$ is torsion free, we have an exact sequence of coherent sheaves on $\hat{A}$:

$$0 \longrightarrow R^0\hat{\mathcal{S}}\mathcal{F} \longrightarrow L \longrightarrow Q \longrightarrow 0$$

where Q is supported at most on the points P_i.

We claim that $Q = 0$. Suppose on the contrary that $Q \neq 0$. By Lemma 3.4, $R^i\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} = 0$ for $i = 0, 1$, hence $R^0L \cong R^0Q$. So for general $P \in A = \mathrm{Pic}^0(\hat{A})$ one has $h^0(L \otimes P) = h^0(Q \otimes P) \neq 0$ since $Q \neq 0$ is supported on points. It follows that L is an ample line bundle and therefore satisfying I.T of index 0. In particular, $R^2\mathcal{S}L = 0$. On the other hand, since Q is supported on points, we have that $R^1\mathcal{S}Q = 0$. The exact sequence

$$R^1\mathcal{S}Q \to R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \to R^2\mathcal{S}L,$$

yields $R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} = 0$. It follows that $\mathcal{F} = 0$ since there is a surjection $R^2\mathcal{S}R^0\hat{\mathcal{S}}\mathcal{F} \to (-1)^*\mathcal{F}$. One concludes that $\mathcal{O}_A = a_*\omega_X$, and in particular $X \longrightarrow A$ is birational, which is the required contradiction.

We may therefore assume that $Q = 0$ and hence $L = R^0\hat{\mathcal{S}}\mathcal{F}$ is a line bundle. By Lemma 3.4, L satifies W.I.T of index 2, hence it is a negative definite line bundle. □

Remark. It follows that if the degree of $X \longrightarrow A$ is 2, then $rk(\mathcal{F}) = 1$ The only possibility is that $a_*\omega_X = \mathcal{O}_A \oplus \mathcal{O}_A(-\Theta)$ where Θ is a principal polarization. This is a $2:1$ covering branched along a divisor $D \in |2\Theta|$.

We have given an example with $a_*\omega_X = \mathcal{O}_A \oplus \hat{L}$, where $L^\vee$ is an ample line bundle of type $(1, 2)$. Unluckily we have not been able to rule out the cases in which $\mathcal{H} \neq 0$. For example is it possible to have $a_*\omega_X = \mathcal{O}_A \oplus \mathcal{F}$ with $\mathcal{F}$ as follows?

Example. Let (A, L) be a general polarized abelian surface of type $(1, 3)$ and $x \in A$ a closed point. Then $h^i(A, L \otimes \mathcal{I}_x \otimes P) = 0$ for all $i > 0$ and all $P \in \mathrm{Pic}^0(A)$. Let $\mathcal{E} = \widehat{L \otimes \mathcal{I}_x}$ and $\mathcal{F} = \mathcal{E}^\vee$. Then we have an exact sequence

$$0 \longrightarrow P_x^\vee \longrightarrow \hat{L}^\vee \longrightarrow \mathcal{F} \longrightarrow 0.$$

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Jungkai Alfred Chen
Department of Mathematics
National Taiwan University
1 Sec. 4, Roosevelt Rd.
Taipei, 106, Taiwan
E-mail address: jkchen@math.ntu.edu.tw

Christopher Derek Hacon
Department of Mathematics
University of Utah
155 South 1400 East, JWB 233
Salt Lake City, Utah 84112-0090, USA
E-mail address: hacon@math.utah.edu

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The first author is a member of Mathematics Division, National Center for Theoretical Sciences at Taipei.