R-core Implementation

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Abstract

We argue that a new solution concept by Huang and Sjöström (2002), called the r-core, extends the concept of core to games in partition function forms naturally. Moreover, the r-core can be implemented by a modi…ed Perry and Reny (1994) game in a straightforward manner. We show that: 1) every stationary subgame perfect Nash equilibrium (SSPNE) outcome must be in the r-core; 2) with the additional assumption of superadditivity, every r-core can be implemented as an SSPNE.

1 INTRODUCTION

The concept of core is one of the most important cooperative solutions because of its natural appeal. Competition underlying core is characterized by the unhampered ability with which players can freely sign some binding agreement. Thus an allocation is in the core if no blocking coalition will form to object.

Because of its very intuitive appeal, two extensions seem relatively possible and important. The …rst regards how to extend the core concept to situations where externalities prevail across coalitions. The classical starting point of the core, namely, the characteristic function, simply assumes away this important interaction. A natural and parsimonious starting point is the partition function where externalities are incorporated. Accordingly, solution concepts like the ®-, ¯- or °- cores are de…ned. However, they are often criticized for either being too pessimistic or arbitrary.

The second important extension concerns how to non-cooperatively implement the core. This task seems imminent if the concept is to stand soundly. Kalai, Postlewaite and Roberts (1979) is one of the earliest attempts. However, they show that the set of the strong Nash equilibria coincides with the core. Since the idea of strong equilibria assumes the formation of pro…table blocking coalitions, whether it truly provides a non-cooperative foundation of the core is doubtable. Other contributions include Chatterjee et. al.' (1990) which considers an extension of Rubinstein's bargaining game. They show only some points in the core are implementable. Moldovanu and Winter (1991), on the other hand, implements the core via a class of games. In our view, the implementation of the core is not well settled until Perry and Reny (1994). They provide a non-cooperative game where the moves and counter-moves closely mimic the spirit of competition behind the core, that is, players have unhampered ability to sign binding agreement. They show that for a superadditive game: 1) any Stationary Subgame Perfect Nash Equilibrium (SSPNE) is in the core; 2) with the additional assumption of totally balancedness, an allocation can be supported as an SSPNE outcome if and only if it is in the core.

Since we argue that the …rst extension hasn't been quite settled while the second extension has been well done by Perry and Reny (1994), a qualifying …rst extension must satisfy two properties. First, it must extend the core concept to partition function forms. Second, since Perry and Reny (1994) successfully implements the core in superadditive games in characteristic function form, a modi…ed Perry and Reny (1994) game must also implements the extension at least in superadditive games in partition function form.

This is exactly the purpose of this paper. We shall argue that a new solution concept by Huang and Sjöström (2002), called the r-core,[1] satis…es these two properties straightforwardly. First, the r-core extends the core to partition function forms naturally. Suppose a set of players, denoted N, is contemplating a possible "core" allocation. In order to know whether this allocation is immune to coalitional blocking, every coalition S µ N has to know its value. However

[1] Where r stands for recursive.

the value of S is not de…ned in partition function forms. To solve this, the r-core postulates the following. Since S is calculating its value (which is what it can achieve independently on its own), it must predict what will happen to NnS once S has formed. Because the norm of the society is the core, members in S simply believe that players in NnS will play according to the core. This is typically not the end of the story. To predict what the core of NnS is, every coalition T µ NnS has to know its value. The r-core assumes that players in Nn(S [T) play according to the core and so on. This naturally yields a recursive de…nition of the core where the core solution concept is used to de…ne coalitional values at every recursive step. Second, the r-core, as just roughly de…ned, can be implemented by a modi…ed Perry and Reny (1994) game in a straightforward manner. We will show that: 1) every SSPNE outcome must be in the r-core; 2) with the additional assumption of superadditivity, every r-core can be implemented as an SSPNE. Moreover, the construction of the proof is a straightforward extension of that in Perry and Reny (1994). In our view, this lends extra credit to the r-core being a natural extension of the core because if we believe that Perry and Reny well captures the non-cooperative spirit of the core and its slight modi…cation implements a new solution concept quite straightforwardly, then this new solution concept must have preserved some nice spirit of the core to make its implementation so easily.

(maybe we should mention EBA here)

The rest of the paper is organized as follows. In the next section, we provide an overview of the r-core. For more details, please refer to Huang and Sjöström (2002). The third section lays out the modi…ed Perry and Reny (1994) game informally. The fourth section formally de…ned the game. We prove two main theorems in the …fth section. In the six section we conclude by brie‡y discussing relaxing the assumption of superadditivity.

2 OVERVIEW OF THE R-CORE

Consider a situation in which the underlying opportunities can be summarized by a transferable utility game in partition function form, $(P; N)$, where $N = \{1, 2, \ldots, n\}$ is the set of players and P is the partition function. Precisely, for every partition on the player set N, P_N, for any coalition S in P_N, $P(S \mid P_N) \in R_{++}$ is the value that S can achieve when players partition themselves according to P_N.[2] Restricting our attention to the partition function form is the first natural step to allow for externalities across players since the value of S, unlike in the characteristic function form, crucially depends on what coalitional structure other players $N \backslash S$ form into.

A payoff vector $x \equiv (x_i)_{i \in N}$ is called feasible under the partition P_N if for every coalition S in P_N, $\sum_{i \in S} x_i = P(S \mid P_N)$. Let $C(S \mid S; P_{N \backslash S})$ denote the r-core for coalition S given others $N \backslash S$ partition into $P_{N \backslash S}$. As in the core, in order to determine the r-core, we first need to determine the value of each sub coalition $T \subseteq S$. Let $V(T \mid S; P_{N \backslash S})$ denote the value of each sub coalition $T \subseteq S$ when we are considering the r-core for coalition S given $P_{N \backslash S}$. We can now recursively define the r-core.

For any $i \in N$, and any coalition structure $P_{N \backslash \{i\}}$ for the players in $N \backslash \{i\}$, the value of $\{i\}$ is clearly what it can achieve when other players partition into $P_{N \backslash \{i\}}$. Thus

$$V(\{i\} \mid \{i\}; P_{N \backslash \{i\}}) \equiv P(\{i\} \mid \{i\}; P_{N \backslash \{i\}}). \tag{1}$$

The r-core for the coalition $\{i\}$ given others $N \backslash \{i\}$ partition into $P_{N \backslash \{i\}}$ is the set of feasible payoff vectors such that coalition $\{i\}$ is getting at least its value. Formally,

$$C(\{i\} \mid \{i\}; P_{N \backslash \{i\}}) \equiv \{x \in R^n : x \text{ is feasible under the partition } (\{i\}; P_{N \backslash \{i\}}) \text{ and } x_i \geq V(\{i\} \mid \{i\}; P_{N \backslash \{i\}})\}. \tag{2}$$

[2] Alternatively, we can assume $P(S \mid P_N) \in R_+$ or even allow $P(S \mid P_N)$ to take negative value. These will not change the result but make the exposition unnecessarily messy.

Now make the induction hypothesis that we have defined $C(S \mid S; P_{N \setminus S})$ for all S such that $1 \leq |S| \leq k - 1$ where $|S|$ stands for the cardinality of S, and all partitions $P_{N \setminus S}$ of $N \setminus S$.

Consider S such that $|S| = k$. The value of S is clearly what it can achieve when other players partition into $P_{N \setminus S}$. Thus

$$V(S \mid S; P_{N \setminus S}) \equiv P(S \mid S; P_{N \setminus S}). \tag{3}$$

For any non-empty proper sub coalition $T \subset S$, since players in $N \setminus S$ have partitioned into $P_{N \setminus S}$, to determine its value, we need to know what coalition structure players in $S \setminus T$ will form. This is predicted by the r-core for coalition $S \setminus T$ given other players partition into $(T; P_{N \setminus S})$, $C(S \setminus T \mid S \setminus T; T; P_{N \setminus S})$. Thus

$$V(T \mid S; P_{N \setminus S}) \equiv \min\{x(T) : x \in C(S \setminus T \mid S \setminus T; T; P_{N \setminus S})\}. \tag{4}$$

We take the minimum because there might be multiple coalitional structures that players in $S \setminus T$ could form since as long as every sub coalition in $S \setminus T$ is getting at least its value, the concept of the core makes no further selection. Taking the minimum gives us the least value that coalition T can get when $S \setminus T$ is playing according to the r-core.

We can now define the r-core for S given others form $P_{N \setminus S}$ since according to the concept of the core, every coalition has to get at least its value and their values have been defined by (3) and (4). Hence

$$\begin{aligned} C(S \mid S; P_{N \setminus S}) \equiv \{x \in R^n :\ & \text{there exists some partition } P_S \text{ of } S \text{ such that} \\ & 1)\ x \text{ is feasible under the coalition structure } (P_S, P_{N \setminus S}); \\ & 2)\ x(S) \geq V(S \mid S; P_{N \setminus S}); \\ & 3)\ x(T) \geq V(T \mid S; P_{N \setminus S}) \text{ for any non-empty proper sub coalition } T \subset S\}. \end{aligned} \tag{5}$$

We have now defined $C(S \mid S; P_{N \setminus S})$ for S such that $|S| = k$ given any partition $P_{N \setminus S}$. Continuing in this fashion, we can define $C(S \mid S; P_{N \setminus S})$ for larger and larger S. The final step occurs when $S = N$. Although the method in the final step is the same as in the preceding steps, it may be helpful to describe this last step explicitly. First, $V(N \mid N)$[3] is what it can achieve. Therefore

$$V(N \mid N) \equiv P(N \mid N).$$

For any non-empty proper sub coalition $T \subset N$, to determine its value, we need to know what coalition structure players in $N \backslash T$ will form. This is predicted by the r-core for coalition $N \backslash T$ given others have formed T, $C(N \backslash T \mid N \backslash T; T)$. Thus

$$V(T \mid N) \equiv \min\{x(T) : x \in C(N \backslash T \mid N \backslash T; T)\}.$$

Then, $C(N \mid N)$ is the set of payoffs where every coalition is getting at least its value. Therefore,

$$\begin{aligned} C(N \mid N) \equiv \{x \in R^n :\ & \text{there exists some partition } P_N \text{ of } N \text{ such that:} \\ & 1)\ x \text{ is feasible under the coalition structure } P_N; \\ & 2)\ x(N) \geq V(N \mid N); \\ & 3)\ x(T) \geq V(T \mid N) \text{ for any non-empty proper sub coalition } T \subset N\}. \end{aligned}$$

We sometimes call $C(N \mid N)$ the r-core because it is what we ultimately care about. The r-core for smaller coalitions are derived in order to recursively derive the r-core for N. Besides, in the process of defining the r-core for S given $P_{N \backslash S}$ in (5), we incidentally get some coalition structures where S could possibly form. The set of these coalition structures of S in $C(S \mid S; P_{N \backslash S})$ will be denoted by $P(S \mid S; P_{N \backslash S})$.

Notice that for the r-core of N to exist, it must exist for every S given any $P_{N \backslash S}$. If at any step of the recursive definition, the r-core for some S given some $P_{N \backslash S}$ does not exist, then the r-core for any larger coalition is not defined because presumably we need to use $C(S \mid S; P_{N \backslash S})$ to predict what S will do. This makes the r-core more difficult to exist than the α- and β- cores.

[3] $V(N \mid N)$ should be written $V(N \mid N; P_{N \backslash N})$ to conform to (3). But $P_{N \backslash N}$ is the trivial partitioning of the empty set since $N \backslash N = \emptyset$; so we will suppress $P_{N \backslash N}$ here and in the following expressions.

However, when it exists, it is a subset of the α- and β- cores.[4] More importantly, rightly because of its recursive nature, it has some perfection built in already. Recall that Theorem 2 in Perry and Reny (1994) requires the game to be totally balanced. The property of totally balanced is necessary because for an SSPNE to exist, on the off-equilibrium path when some coalitions have left, an equilibrium among the remaining players has to exist. However, this equilibrium naturally corresponds to the core among the remaining players. Thus the game has to be totally balanced. On the other hand, in a parallel Theorem of the r-core, we do not need to impose some extra property a la totally balanced. This is because by the recursive nature of the r-core, when it exists, the r-core for any S given any $P_{N\setminus S}$ exists. Thus on the off-equilibrium path when players $N\setminus S$ have left according to $P_{N\setminus S}$, the remaining players of S can simply follow the r-core for S given $P_{N\setminus S}$. This shall become clear when we construct equilibria to support the r-core.

3 THE EXTENSIVE FORM GAME

In this section we will lay out the extensive form game informally. This section follows Section 2 of Perry and Reny (1994) closely. We choose to follow their notations as closely as possible so that the results could be directly comparable.

The game starts at $t = 0$ and time is continuous. At any point of time, a player can either: 1) make a proposal; 2) accept the current proposal; 3) stay quiet or 4) leave.[5]

A proposal $((w_i)_{i\in S}; S)$ by any player who hasn't left consists of a division rule and a coalition S. A division rule is simply a way to distribute S's value where $w_i \in R_+$ represents i's share of the value of S. Thus, $\sum_{i\in S} w_i = 1$ and $w_i \geq 0$ for all $i \in S$. Notice that in Perry and Reny, a player proposes a payoff allocation instead of a division rule. Proposing a payoff allocation does not make

[4] Refer to Huang and Sjöström (2002) for details.

[5] Except that at the very beginning of the game when $t = 0$, players can only choose either 1) or 3).

sense here because unless all other players $N \setminus S$ have left and consumed, players in S might not have any idea about what coalitional structure that others will form. By externalities across coalitions, the coalitional structure determines the value of S, which in turn determines the possible payoff allocations. Proposing a division rule avoids this difficulty because it specifies a way to distribute the value of S for all possible coalitional structures that might form. When a proposal is made, it is effective as long as no new proposal is made. Upon a new proposal is made, the previous proposal is no longer effective. To avoid the simultaneous proposals of distinct proposals, when this happens, no new proposal is effective. So there is at most one effective proposal at any point of time.

When an effective proposal $((w_i)_{i \in S}; S)$ is accepted by all members in S, it becomes binding. In this case, S is a binding coalition. If any player in S chooses to leave, then all players in S leave at the same time. At any point in time, there might be multiple binding proposals among players who haven't left. If a new proposal contains any player in any binding coalition, then it must contain all players in that binding coalition. This reflects the idea that annulment of a binding coalition has to be approved by every member in it. To avoid the problem where a player is involved in two different binding proposals, at the time when a new binding proposal, containing some binding coalitions, becomes binding, the old corresponding binding proposals are annulled. Hence, at any point in time, all binding coalitions are disjoint. Players consume according to their binding proposals only when all have left. Notice that when all players leave, the coalitional structure is uniquely defined. Players simply divide the value of the coalition to which they belong according to the binding proposals they have signed. If some player never leaves the game, all players get $-\infty$.[6]

[6] One can relax this rather strong assumption. For instance, if some binding coalitions have left while others remain in the game forever, it might be argued that although the remaining players might get the worst possible payoffs, any leaving coalition should at least get its value in the coalitional structure where all the leaving coalitions have formed and the remaining players form into the worst possible coalitional structure for this coalition in consideration. Allowing this does not change the result.

Finally, we assume that for every time t and every history up to t, there exists an $\varepsilon > 0$ and two open intervals $(t - \varepsilon; t)$ and $(t; t + \varepsilon)$ in which every strategy of every player asks him to be quiet. Perry and Reny (1994) explain the role that this assumption plays in their Example 1.

4 THE MODEL

We now introduce the notations, the formal game rules and the equilibrium concept. As in the previous section, we choose to follow the notations in Perry and Reny (1994) as closely as possible so that readers familiar with that paper can understand this paper more easily. We also try to make the paper as self-contained as possible so that readers, who haven't read Perry and Reny (1994) before, will not have problems in understanding this paper. We follow Perry and Reny (1994) to divide this section to four sub sections to deal with histories, payoffs, strategies and the equilibrium concept respectively.

4.1 HISTORIES

A proposal, as stated in the previous section, is a division rule and a coalition, $((w_i)_{i \in S}; S)$. Thus, the set of feasible proposals, denoted by P, is

$$P \equiv \{((w_i)_{i \in S}; S) : S \subseteq N; \sum_{i \in S} w_i = 1 \text{ and } w_i \geq 0 \text{ for all } i \in S\}.$$

Denote a the choice to accept the current effective proposal, q the choice to be quiet and l the choice to leave. A history for player i up to time $t > 0$, is a function h_i such that

$$h_i : [0; t) \to P \cup \{a; q; l\}.$$

Since players can only leave once and for all, $h_i^{-1}(l)$ is either empty or singleton. We follow Perry and Reny (1994) to assume that $h_i^{-1}(P \cup \{a\})$ is a finite set. At $t = 0$, since nothing has happened, $h_i(0) = \emptyset$. For convenience, denote a history up to time t by the n-tuple of functions $h \equiv (h_1; h_2; \ldots; h_n)$. Let $H(t)$ denote the set of all histories up to time t and $H \equiv \cup_{t=0}^{\infty} H(t)$ the set of all histories.

We need to introduce more notations.

Let $p(h)$ denote the current effective proposal according to h. To make it well-defined, if according to h, either no proposal has been made, multiple distinct proposals are simultaneously made, the most recently effective proposal has become binding or some member in a binding coalition which is also involved in the most recently effective proposal has exercised to leave, then $p(h) \equiv \emptyset$.[7]

Let $\tau(h)$ for $h \in H(t)$ denote the amount of time that has passed up to time t since $p(h)$ was proposed. Whenever $p(h) = \emptyset$, $\tau(h)$ measures the time that has passed since the previous effective proposal becomes binding. When there is never any effective proposal, $\tau(h)$ measures the time that has passed since time 0.

Let $N(h) \subseteq N$ denote the set of players who have not left and $A(h) \subseteq N(h)$ the set of players who have accepted $p(h)$. Whenever $p(h) = \emptyset$, $A(h) \equiv \emptyset$.

Player i is said to have accepted the current effective proposal $p(h)$ for $h \in H(t)$ if $p(h)$ is made at time $\bar{t} < t$ and $h_i(t') = a$ for some $t' \in (\bar{t}, t)$. If everyone involved in $p(h)$ has accepted it, then $p(h)$ is binding. The coalition associated with $p(h)$ is called a binding coalition.

Let $\Pi(h)$ denote the set of binding proposals among the players in $N(h)$. Since there exists externalities across coalitions, we also need to keep track of those binding coalitions that have left. Let $C(h)$ denote coalitional structure that has formed by the players in $N \setminus N(h)$ according to h. Notice that Perry and Reny (1994) do not need to keep track of $C(h)$ since in characteristic function form the remaining players' values are not affected by the coalitional structure those leaving players form. On the other hand in partition function form, the coalitional structure those leaving players form affects the values of the remain-

[7] When some member in a binding coalition which is also involved in the most recently effective proposal has exercised to leave, we need to reset $p(h)$ to an empty set to avoid the following from happening. Suppose players 1, 2 and 3 remain and the current proposal pertains to them. Suppose player 1 and 2 have accepted the current proposal and 3 hasn't. Suppose coalition {3} is binding. If 3 exercises to leave, then by the rules in Perry and Reny, if $p(h)$ is not reset to an empty set, players 1 and 2 cannot do anything further and they have to stay in the game forever.

ing players a great deal. Finally, let $h|t$ denote the restriction of $h \in H(s)$ to $[0,t)$ where $s \geq t$.

4.2 PAYOFFS

If at some point of time, all players have left so that $N(h) = \emptyset$, that must imply a partition P_N on the player set N has formed. In this case, every binding coalition distributes its value according to its binding division rule. For instance, if $S \in P_N$ and its binding division rule is $(w_i)_{i \in S}$, then player $i \in S$ simply gets $w_i P(S \mid P_N)$. If $N(h) \neq \emptyset$ where $h \in H(\infty)$, implying someone never leaves the game, then everyone gets $-\infty$. The assumption on payoffs reflects the idea that if a proposal $((w_i)_{i \in S}, S)$ becomes binding and any player $i \in S$ leaves, then in contrast to Perry and Reny (1994) he cannot consume immediately since a coalition's value crucially depends on the final coalitional structure P_N that all players form. All he can guarantee by leaving is that coalition S will be one member of the final coalitional structure P_N.

4.3 STRATEGIES

A strategy for any player is a function which maps every possible history to an action. Hence, a strategy for any player $i \in N$, denoted by f_i is:

$$f_i : H \to P \cup \{a, q, l\}.$$

Denote the n-tuple of strategies by $f \equiv (f_1, f_2, ..., f_n)$. There are several restrictions on strategies.

(S0) For $h \in H(0)$, $f_i(h) \in P \cup \{q\}$, since at the very beginning of the game, players can only make an proposal or be quiet.

(S1) If $i \in N(h)$ has accepted the current effective proposal $p(h)$, then $f_i(h) = q$. That is, before the current effective proposal becomes binding, an accepting player can only be quiet. If $i \in N \setminus N(h)$, then $f_i(h) = q$. Thus, a leaving player can only be quiet.

(S2) If $p(h) = ((w_i)_{i \in S}; S)$ and $f_i(h) = ((w_i)_{i \in S'}; S')$, then either $S \cap S' = \emptyset$ or $S \subseteq S'$. Moreover, $S' \subseteq N(h)$. In words, if a new proposal contains some players in the current effective proposal, it has to include all of them. Quite naturally, any new proposal can only contain players who haven't left.

(S3) If a player i is not a member of any binding coalition which hasn't left, then $f_i(h) \neq l$. That is, a player can only leave when it belongs to a binding coalition which hasn't left.

(S4) For all i and $\bar{t} > 0$ and for all $h \in H(\bar{t})$ and $t \in [0; \bar{t})$, there exists an $\epsilon > 0$ such that $f_i(h|\tau) = q$ for all $\tau \geq 0$ and $\tau \in (t - \epsilon; t + \epsilon) \setminus \{t\}$. This assumption serves to make sure that players always have enough time to respond.[8]

Lastly, denote player i's payoff induced by the strategy tuple f after h by $u_i(f|h)$. Let F_i denote the set of strategies for i which satisfies (S0) to (S4).

4.4 THE EQUILIBRIUM CONCEPT

The equilibrium concept in this paper is the stationary subgame perfect Nash equilibrium (SSPNE). Thus, a strategy profile $\hat{f} \equiv (\hat{f}_1; \hat{f}_2; \ldots; \hat{f}_n)$ is an SSPNE if:

(E1) Perfection: For all $i \in N$, $h \in H$ and $f_i \in F_i$,

$$u_i(\hat{f}|h) \geq u_i((f_1; \hat{f}_2; \ldots; \hat{f}_n)|h):$$

(E2) Stationarity: For $h; h' \in H$, if

$$(p(h); \tau(h); N(h); A(h); \phi(h); C(h)) = (p(h'); \tau(h'); N(h'); A(h'); \phi(h'); C(h'));$$

then $\hat{f}(h) = \hat{f}(h')$. Notice that we have one more state variable $C(h)$ because of externalities across coalitions.

[8] Interested readers can refer to Example 1 and the last paragraph on page 806 in Perry and Reny (1994) for explanations.

5 THE THEOREMS

We will prove two parallel theorems to Theorems 1 and 2 in Perry and Reny (1994). The first theorem will show that every SSPNE outcome of the extensive form game we set up is in the r-core of $(P; N)$. The second theorem needs some qualification. We will show that for superadditive game, every r-core of $(P; N)$ can be supported as an SSPNE outcome. The restriction of superadditivity makes the exposition a lot easier and yet it is more than necessary. We will discuss some way to relax it.

Notice however that the proof of the first theorem in this paper is more complicated than that of Theorem 1 in Perry and Reny (1994). This is because for Perry and Reny (1994), the value of each coalition is given by the characteristic function. In the r-core, the value of each coalition has to be derived recursively from the partition function instead. Rightly because of so, we first need to show that if there exists an SSPNE of the extensive form game, then the value of each coalition is well defined. Only until then can we talk about the existence of a r-core which corresponds to this SSPNE outcome.

Proposition 1 Suppose an SSPNE $\hat{\sigma}$ exists.

(1) For any $S \subseteq N$ and any partition $P_{N \setminus S}$ on $N \setminus S$, $C(S \mid S; P_{N \setminus S}) \neq \emptyset$.

(2) Let x denote the equilibrium outcome induced by $\hat{\sigma}$ where players in $N \setminus S$ have left according to $P_{N \setminus S}$ (i.e., in the subgame where the histories are $(p(h); \tau(h); N(h); A(h); \pi(h); C(h)) = (\emptyset; 0; S; \emptyset; \emptyset; P_{N \setminus S})$, there must exist a $y \in C(S \mid S; P_{N \setminus S})$ such that

$$x_i = y_i \text{ for all } i \in S.$$

(3) The coalitional structure induced by $\hat{\sigma}$ must be $(P_S; P_{N \setminus S})$ where $P_S \in P(S \mid S; P_{N \setminus S})$.

Proof. We will proceed by induction.

$|S| = 1$: For any $S = \{i\}$ and any $P_{N \setminus \{i\}}$ on $N \setminus \{i\}$, by (2), $C(\{i\} \mid \{i\}; P_{N \setminus \{i\}}) \neq \emptyset$ since by definition, it consists of every payoff vector where i gets $P(\{i\} \mid \{i\}; P_{N \setminus \{i\}})$ and every coalition $S' \in P_{N \setminus \{i\}}$ gets $P(S' \mid \{i\}; P_{N \setminus \{i\}})$.

In the subgame where all players in $N\setminus\{i\}$ have left and formed $P_{N\setminus\{i\}}$, apparently $\{i\}$ must first propose $((1);\{i\})$ and then leave according to $\hat{\pi}_i$. For otherwise, either i is staying in the game forever and getting -1 or he is getting strictly less than $P(\{i\} \mid \{i\}; P_{N\setminus\{i\}})$. Neither can be an equilibrium strategy. Parts (2) and (3) follow immediately.

Suppose by induction that for any $|S| \le k-1 < n$ and any $P_{N\setminus S}$ on $N\setminus S$, parts (1), (2) and (3) hold.

$|S| = k$: For any such S and any partition $P_{N\setminus S}$ on $N\setminus S$, to show part (1), we first need to make sure that $V(S \mid S; P_{N\setminus S})$ and $V(T \mid S; P_{N\setminus S})$ for any non-empty proper sub coalition $T \subset S$ are well defined. By (3), $V(S \mid S; P_{N\setminus S})$ is certainly well defined since it is simply $P(S \mid S; P_{N\setminus S})$. By (4), $V(T \mid S; P_{N\setminus S})$ is also well defined because $C(S\setminus T \mid S\setminus T; T; P_{N\setminus S}) \neq \emptyset$ by the induction hypothesis since $|S\setminus T| \le k-1$.

Suppose by contradiction that $C(S \mid S; P_{N\setminus S}) = \emptyset$. Let t be the first time that all players in $N\setminus S$ have left according to $P_{N\setminus S}$ and let $h \in H(t')$ where $t' > t$ and $(p(h); \tau(h); N(h); A(h); \Pi(h); C(h)) = (\emptyset; 0; S; \emptyset; \emptyset; P_{N\setminus S})$. Consider the continuing equilibrium outcome x induced by $\hat{\pi}$. Since $C(S \mid S; P_{N\setminus S}) = \emptyset$, there must exist a coalition $S' \subseteq S$ such that $x(S') < V(S' \mid S; P_{N\setminus S})$. Let

$$y_i = x_i + \frac{V(S' \mid S; P_{N\setminus S}) - x(S')}{|S'|} \text{ for all } i \in S';$$

and define

$$w_i = \frac{y_i}{V(S' \mid S; P_{N\setminus S})} \text{ for all } i \in S':$$

Without loss of generalities, let $S' = \{1; 2; \ldots; s'\}$ where $|S'| = s'$. Consider at any time $t'' > t'$, any history $h \in H(t'')$ such that i) $p(h) = ((w_i)_{i\in S'}; S')$, ii) $N(h) = S$, iii) $A(h) = \{1; 2; \ldots; s' - 1\}$, iv) $\Pi(h) = \emptyset$, v) $C(h) = P_{N\setminus S}$ and vi) $\hat{\pi}_i(h) = q$ for all $i \in S$.

Claim 2 According to $\hat{\pi}$, player s' will accept $((w_i)_{i\in S'}; S')$ and thus $((w_i)_{i\in S'}; S')$ will become binding.

Proof of Claim. Notice that player s' can accept the current effective proposal $((w_i)_{i\in S'}; S')$ and leave before anything happens. If this is the case, according to the induction hypothesis, the coalitional structure will be $(P_{S\setminus S'}; S'; P_{N\setminus S})$ where $P_{S\setminus S'} \in P(S\setminus S' \mid S\setminus S'; S'; P_{N\setminus S})$. By (4)

$$V(S' \mid S; P_{N\setminus S}) \quad P(S' \mid S\setminus S'; S'; P_{N\setminus S});$$

because when S' is calculating its value $V(S' \mid S; P_{N\setminus S})$, it expects the worst possible coalitional structure and $(S\setminus S'; S'; P_{N\setminus S})$ is not necessarily the worst possible one. Therefore player s' gets

$$w_{s'} P(S' \mid S\setminus S'; S'; P_{N\setminus S}) \geq y_{s'} > x_{s'}.$$

Suppose to the contrary that either no new proposal is made and player s' never accepts, or some new proposal $((w_i)_{i\in S''}; S'')$ is made before player s' accepts $((w_i)_{i\in S'}; S')$ according to $\hat{\sigma}$. In the first possibility, player s' gets -1 which is impossible. In the second possibility, in the continuing equilibrium, s' must get more than $w_{s'} P(S' \mid S\setminus S'; S'; P_{N\setminus S}) > x_{s'}$. By stationarity this means whenever the history h' yields the states

$$(p(h'); \tau(h'); N(h'); A(h'); \Pi(h'); C(h')) = (((w_i)_{i\in S'}; S'); 0; S; \emptyset; \emptyset; P_{N\setminus S});$$

player s' gets strictly more than $x_{s'}$. But then player s' could have proposed $((w_i)_{i\in S'}; S')$ at time close enough to t'. This is in contradiction to x being a continuing equilibrium outcome. ■

Next consider at any time $t'' > t'$, any history $h \in H(t'')$ such that i) $p(h) = ((w_i)_{i\in S'}; S')$, ii) $N(h) = S$, iii) $A(h) = \{1; 2; \ldots; s' - 2\}$, iv) $\Pi(h) = \emptyset$, v) $C(h) = P_{N\setminus S}$ and vi) $\hat{\sigma}_i(h) = q$ for all $i \in S$.

Claim 3 According to $\hat{\sigma}$, players $s'-1$ and s' will accept $((w_i)_{i\in S'}; S')$ and $((w_i)_{i\in S'}; S')$ will become binding.

Proof of Claim. Notice that player $s'-1$ can accept the proposal to render the states of the previous claim. Thus he can guarantee himself the payoff of

$$w_{s'-1} P(S' \mid S\setminus S'; S'; P_{N\setminus S}) \geq y_{s'-1} > x_{s'-1}.$$

As in the previous claim, suppose to the contrary either no new proposal is made and player $s' - 1$ never accepts or a new proposal is made before both players $s' - 1$ and s' accept. In the first possibility, player $s' - 1$ gets -1, which is impossible. In the second possibility, by an analogous argument, when the new proposal is made, player $s' - 1$ must get strictly more than $x_{s'-1}$. But he could have made that proposal at time close enough to t'. ■

Proceeding step by step, we can finally establish that at any time $t'' > t'$, any history $h \in H(t'')$ such that i) $p(h) = ((w_i)_{i \in S'}; S')$, ii) $N(h) = S$, iii) $A(h) = \{1\}$, iv) $\sigma(h) = \emptyset$, v) $C(h) = P_{N \setminus S}$ and vi) $\hat{\sigma}_i(h) = q$ for all $i \in S$, according to $\hat{\sigma}$, players 2 to s' will accept $((w_i)_{i \in S'}; S')$ and $((w_i)_{i \in S'}; S')$ will become binding.

This contradicts that x is an equilibrium outcome induced by $\hat{\sigma}$ because player 1 can always propose $((w_i)_{i \in S'}; S')$ and subsequently accept it at time close enough to t'. In this deviation, he gets at least $y_1 > x_1$. Thus $C(S \mid S; P_{N \setminus S}) \neq \emptyset$ and part (1) is proved.

For part (2), if x is the equilibrium outcome induced by $\hat{\sigma}$ where players in $N \setminus S$ have left according to $P_{N \setminus S}$ (i.e., in the subgame where the histories are $(p(h); \tau(h); N(h); A(h); \sigma(h); C(h)) = (\emptyset; 0; S; \emptyset; \emptyset; P_{N \setminus S})$, and there does not exist a $y \in C(S \mid S; P_{N \setminus S})$ such that

$$x_i = y_i \text{ for all } i \in S,$$

then there must exist a coalition $S' \subseteq S$ such that $x(S') < V(S' \mid S; P_{N \setminus S})$. We can now run over the same argument all over again to get a contradiction. Part (3) follows from Part (2) immediately. ■

An immediate application of Proposition 1 is summarized in the following theorem to make it directly comparable to Theorem 1 in Perry and Reny (1994).

Theorem 4 Every SSPNE outcome is in the r-core of $(P; N)$. More formally, suppose an SSPNE $\hat{\sigma}$ exists and induces an equilibrium outcome x. Then $x \in C(N \mid N)$. Moreover, the coalitional structure induced by $\hat{\sigma}$ belongs to the r-core structure $P(N \mid N)$.

A parallel theorem to Theorem 2 in Perry and Reny (1994) can be made easiest with the assumption of superadditivity. We first state the assumption, prove the theorem and finally come back to discuss the assumption after the proof.

A game in partition function form (P, N) is superadditive if for any two non-empty disjoint coalitions S and T and any coalitional structure $P_{N\setminus(S\cup T)}$ on the remaining players,

$$P(S \mid S, T, P_{N\setminus(S\cup T)}) + P(T \mid S, T, P_{N\setminus(S\cup T)}) < P(S \cup T \mid S \cup T, P_{N\setminus(S\cup T)}). \tag{6}$$

For example, the symmetric Bertrand competition with differentiated commodities studied in Deneckere and Davidson (1985) satisfy (6). Supperadditivity greatly simplifies the derivation of the r-core as summarized in the following lemma.

Lemma 5 *If a game (P, N) is superadditive, then for any S and $P_{N\setminus S}$, $P(S \mid S, P_{N\setminus S}) = \{S\}$ provided $C(S \mid S, P_{N\setminus S}) \neq \emptyset$.*

Proof. If $x \in C(S \mid S, P_{N\setminus S}) \neq \emptyset$, then $x(S) \geq V(S \mid S, P_{N\setminus S})$. Since $V(S \mid S, P_{N\setminus S}) = P(S \mid S, P_{N\setminus S})$, by superadditivity, if S breaks up, the sum of payoffs across S must be strictly lower than $P(S \mid S, P_{N\setminus S})$. Thus players in S must stay together in the r-core. ■

With the help of Lemma 5, a theorem parallel to Theorem 2 of Perry and Reny (1994) can be constructed. That is, strategies that support r-core outcomes can be provided.

For any S and $P_{N\setminus S}$, pick a r-core payoff vector from $C(S \mid S, P_{N\setminus S})$. Denote this payoff vector by $x(S \mid S, P_{N\setminus S})$. Hence

$$x(S \mid S, P_{N\setminus S}) \in C(S \mid S, P_{N\setminus S}).$$

This is possible because by the recursive definition of the r-core, if the r-core exists, then it exists for any reduced society S given $P_{N\setminus S}$. We now follow Perry and Reny (1994) to construct two continuing equilibrium payoff vectors: one for the case where the current proposal is rejected, the other for the situation where the current proposal is accepted. That is, for any history h where

$$\Gamma(h) = \{((w_i^1)_{i \in S^1}; S^1); ((w_i^2)_{i \in S^2}; S^2); \ldots; ((w_i^m)_{i \in S^m}; S^m)\};$$

define

$$z_i(h) = \begin{cases} w_i^k \sum_{j \in S^k} x_j(N(h) \mid N(h); C(h)) & \text{if } i \in S^k \text{ where } k \in \{1; 2; \ldots; m\}; \\ x_i(N(h) \mid N(h); C(h)) & \text{if } i \in N(h) \backslash (S^1 \cup S^2 \cup \ldots \cup S^m). \end{cases}$$

Without loss of generality, we can assume that there exists an integer r such that the current proposal $p(h) = ((w_i)_{i \in S}; S)$ contains all the coalitions S^k where $k \le r$. On the other hand, S is disjoint from all the coalitions S^k where $k \geq r + 1$. Thus when the current proposal gets binding, the resulting new set of binding proposals becomes

$$\hat{\Gamma}(h) = \{((w_i)_{i \in S}; S); ((w_i^{r+1})_{i \in S^{r+1}}; S^{r+1}); \ldots; ((w_i^m)_{i \in S^m}; S^m)\}.$$

Define

$$\hat{z}_i(h) = \begin{cases} w_i \sum_{j \in S} x_j(N(h) \mid N(h); C(h)) & \text{if } i \in S; \\ w_i^k \sum_{j \in S^k} x_j(N(h) \mid N(h); C(h)) & \text{if } i \in S^k \text{ where } k \in \{r + 1; \ldots; m\} \\ x_i(N(h) \mid N(h); C(h)) & \text{if } i \in N(h) \backslash (S \cup S^{r+1} \cup \ldots \cup S^m). \end{cases}$$

Intuitively, $z_i(h)$ ($\hat{z}_i(h)$) will be player i's payoff in the continuing equilibrium if the current proposal gets rejected (accepted).

We first provide the equilibrium strategies and then show that they indeed constitute an SSPNE. The proof is essentially a re-write of that of Theorem 2 in Perry and Reny (1994). This is due to the assumption of superadditivity. This similarity hints that the r-core is a very natural extension of the core, at least for superadditive games.

For every $t \geq 0$, $h \in H(t)$ and $i \in N(h)$,

(i) if $\tau(h)$ is not a positive integer, then $f_i(h) = q$;

(ii) if $\tau(h)$ is a positive integer:

(a) if $p(h) = ((w_i)_{i\in S}; S)$, and $\hat{z}_j(h) \geq z_j(h)$ for all $j \in S\setminus A(h)$, then

$$f_i(h) = \begin{cases} a; & \text{if } i \in S\setminus A(h); \\ q; & \text{otherwise;} \end{cases}$$

(b) otherwise

$$f_i(h) = \begin{cases} l; & \text{if } i \in N(h)\setminus A(h) \text{ and } \phi(h) = \{(\cdot; N(h))\}; \\ \left(\left(\frac{z_i(h)}{\sum_{l\in N(h)} x_l(N(h)|N(h);C(h))}\right)_{j\in N(h)}; N(h)\right), & \text{if } i \in N(h)\setminus A(h) \text{ and} \\ & \phi(h) \neq \{(\cdot; N(h))\}; \\ q; & \text{if } i \in A(h); \end{cases}$$

where $\phi(h) = \{(\cdot; N(h))\}$ means the binding coalition is $N(h)$[9] and $\phi(h) \neq \{(\cdot; N(h))\}$ means the binding coalition is not $N(h)$.

These strategies depend only on the state variable. On the equilibrium path, all players propose $\left(\left(\frac{x_i(N|N)}{\sum_{j\in N} x_j(N|N)}\right)_{i\in N}; N\right)$ at time 1, accept at time 2, and leave at time 3. The equilibrium outcome is $x(N \mid N)$ where every $i \in N$ gets $x_i(N \mid N)$. For any history h, in the continuing subgame, the equilibrium outcome is where every $i \in N(h)$ gets $x_i(N(h) \mid N(h); C(h))$.

We first prove two lemmas analogous to Lemmas 1 and 2 in Perry and Reny (1994).

Lemma 6 For any $h \in H$, if $p(h) = ((w_i)_{i\in S}; S)$ and $\phi(h) = \{((w_i^1)_{i\in S^1}; S^1); ((w_i^2)_{i\in S^2}; S^2); \ldots; ((w_i^m)_{i\in S^m}; S^m)\}$, then

(a) $z_i(h) \geq w_i^k \sum_{j\in S^k} x_j(N(h)\setminus S^k \mid N(h)\setminus S^k; S^k; C(h))$ if $i \in S^k$ where $k \in \{1; 2; \ldots; m\}$.

[9] So that there exists a division rule $(w_i)_{i\in N(h)}$ such that $\phi(h) = \{((w_i)_{i\in N(h)}; N(h))\}$.

(b) $\sum_{i \in S^k} z_i(h) = \sum_{i \in S^k} x_i(N(h) \mid N(h); C(h))$ for all $k \in \{1, 2, \ldots, m\}$.

(c) $\sum_{i \in S} z_i(h) = \sum_{i \in S} b_i(h) = \sum_{i \in S} x_i(N(h) \mid N(h); C(h))$.

Proof.

(a) By definition $z_i(h) = w_i^k \sum_{j \in S^k} x_j(N(h) \mid N(h); C(h))$. Since $x(N(h) \mid N(h); C(h))$ belongs to the r-core for $N(h)$ given $C(h)$,

$$\begin{aligned} \sum_{j \in S^k} x_j(N(h) \mid N(h); C(h)) &\geq V(S^k \mid N(h); C(h)) \\ &= P(S^k \mid S^k; N(h) \setminus S^k; C(h)) \end{aligned}$$

where the equality follows because by Lemma 5, when S^k breaks off, the remaining players in $N(h) \setminus S^k$ stay together. Since $x(N(h) \setminus S^k \mid N(h) \setminus S^k; S^k; C(h))$ belongs to the r-core for $N(h) \setminus S^k$ given S^k and $C(h)$ and by Lemma 5 players in $N(h) \setminus S^k$ stay together,

$$\sum_{j \in S^k} x_j(N(h) \setminus S^k \mid N(h) \setminus S^k; S^k; C(h)) = P(S^k \mid S^k; N(h) \setminus S^k; C(h)).$$

Hence

$$z_i(h) \geq w_i^k \sum_{j \in S^k} x_j(N(h) \setminus S^k \mid N(h) \setminus S^k; S^k; C(h)).$$

(b) By definition

$$\begin{aligned} \sum_{i \in S^k} z_i(h) &= \sum_{i \in S^k} w_i^k \Big(\sum_{j \in S^k} x_j(N(h) \mid N(h); C(h))\Big) \\ &= \sum_{i \in S^k} x_i(N(h) \mid N(h); C(h)); \end{aligned}$$

since $\sum_{i \in S^k} w_i^k = 1$.

(c) By definition

$$\begin{aligned}\sum_{i\in S} z_i(h) &= \sum_{k\in\{1,\ldots,r\}}\Big(\sum_{i\in S^k} z_i(h)\Big) + \sum_{i\in S\setminus(S^1\cup\ldots\cup S^r)} x_i(N(h)\mid N(h);C(h))\\ &= \sum_{i\in S} x_i(N(h)\mid N(h);C(h)),\end{aligned}$$

where the second equality follows because of (b). By definition

$$\begin{aligned}\sum_{i\in S} \hat{z}_i(h) &= \sum_{i\in S} w_i\Big(\sum_{j\in S} x_j(N(h)\mid N(h);C(h))\Big)\\ &= \sum_{i\in S} x_i(N(h)\mid N(h);C(h)).\end{aligned}$$

■

Lemma 7 For any $t \geq 0$, $h \in H(t)$ and $A(h) = \emptyset$, the outcome generated by the equilibrium strategies $(f_1;\ldots;f_n)$ after h is where player i gets $z_i(h)$ for all $i \in N(h)$.

Proof. Let t_1 be the smallest time at least as large as t such that $t_1 - \tau(h)$ is a positive integer. According to the equilibrium strategies, all players are quiet between $[t;t_1)$.[10] Denote the history generated by $h_1 \in H(t_1)$. There are four possible cases depending on what players will do at t_1 according to the equilibrium strategies.

(1) If $f_i(h_1) = I$, then it must be the case that $\pi(h_1) = \{(\cdot;N(h_1))\}$. This implies there exists a division rule $(w_i)_{i\in N(h_1)}$ such that $\pi(h_1) = \{((w_i)_{i\in N(h_1)};N(h_1))\}$. Thus player i gets $w_i \sum_{j\in N(h_1)} x_j(N(h_1)\mid N(h_1);C(h_1))$ for all $i \in N(h)$ in the continuing equilibrium outcome. However, since everyone is quiet between $[t;t_1)$, $\pi(h) = \pi(h_1)$ and $N(h) = N(h_1)$. By definition

$$\begin{aligned}z_i(h) &= w_i \sum_{j\in N(h)} x_j(N(h)\mid N(h);C(h))\\ &= w_i \sum_{j\in N(h_1)} x_j(N(h_1)\mid N(h_1);C(h_1)),\end{aligned}$$

where the second equality follows because $N(h) = N(h_1)$. Hence player i gets $z_i(h)$.

[10] Note that if $t_1 = t$, then $[t;t_1) = \emptyset$.

(2) If $f_i(h_1) = a$ for some $i \in N(h_1) \setminus A(h_1)$ and $p(h_1) = ((w_i)_{i \in N(h_1)}; N(h_1))$, since everyone is quiet between $[t; t_1)$ and thus $A(h_1) = A(h) = \emptyset$, then it implies $\hat{b}_i(h_1) \geq z_i(h_1)$ for all $i \in N(h_1)$. So the current proposal becomes binding at t_1. By Lemma 6 (c) $\sum_{i \in N(h_1)} z_i(h_1) = \sum_{i \in N(h_1)} \hat{b}_i(h_1)$, hence $\hat{b}_i(h_1) = z_i(h_1)$ for all $i \in N(h_1)$. Moreover, $N(h_1) = N(h)$ and $z_i(h_1) = z_i(h)$ for all $i \in N(h)$ because everyone is quiet between $[t; t_1)$.

By the equilibrium strategies, everyone is quiet between $(t_1; t_1 + 1)$. Let $t_2 = t_1 + 1$ and denote the history generated by $h_2 \in H(t_2)$. Since everyone is quiet between $(t_1; t_2)$, $N(h_1) = N(h_2)$ and $C(h_1) = C(h_2)$. At t_2, since $\Phi(h_2) = \{(\cdot; N(h_2))\}$, so all players leave. Thus player $i \in N(h_2)$ gets

$$
\begin{aligned}
w_i \sum_{j \in N(h_2)} x_j(N(h_2) - N(h_2); C(h_2)) &= w_i \sum_{j \in N(h_1)} x_j(N(h_1) - N(h_1); C(h_1)) \\
&= \hat{b}_i(h_1) = z_i(h_1) = z_i(h).
\end{aligned}
$$

(3) If $f_i(h_1) = \left(\left(\frac{z_j(h_1)}{\sum_{l \in N(h_1)} x_l(N(h_1) - N(h_1); C(h_1))}\right)_{j \in N(h_1)}; N(h_1)\right)$, then everyone is quiet between $(t_1; t_1 + 1)$. Let $t_2 = t_1 + 1$ and denote the history generated by $h_2 \in H(t_2)$. By definition for all $i \in N(h_1)$,

$$
\begin{aligned}
\hat{b}_i(h_2) &= \frac{z_i(h_1)}{\sum_{j \in N(h_1)} x_j(N(h_1) - N(h_1); C(h_1))} \sum_{l \in N(h_1)} x_l(N(h_2) - N(h_2); C(h_2)) \\
&= z_i(h_1) = z_i(h_2).
\end{aligned}
$$

The second equality follows because no one leaves between $[t_1; t_2)$, so $N(h_1) = N(h_2)$ and $C(h_1) = C(h_2)$. The third equality follows because no new proposal binds between $[t_1; t_2)$, so $z_i(h_1) = z_i(h_2)$. Thus all players in $N(h_1)$ accept the proposal at time t_2. According to the equilibrium strategies, everyone is quiet between $(t_2; t_2 + 1)$. Let $t_3 = t_2 + 1$ and denote the history generated by $h_3 \in H(t_3)$. Since no one leaves between $[t_2; t_3)$, so $N(h_2) = N(h_3)$. Hence all players in $N(h_3)$ leave at time t_3. Thus player $i \in N(h_3)$ gets

$$\frac{z_i(h_1)}{\sum_{j \in N(h_1)} x_j(N(h_1) \mid N(h_1); C(h_1))} \sum_{l \in N(h_3)} x_l(N(h_3) \mid N(h_3); C(h_3))$$
$$= z_i(h_1) = z_i(h).$$

The first equality follows because no one leaves between $[t_1; t_3)$. The second equality follows because nothing happens between $[t; t_1)$.

(4) If $f_i(h_1) = a$ for some $i \in S \setminus A(h_1)$ and $p(h_1) = ((w_i)_{i \in S}; S)$ where $S \neq N(h_1)$, since everyone is quiet between $[t; t_1)$ and thus $A(h_1) = A(h) = \emptyset$, then it implies $\hat{z}_i(h_1) \geq z_i(h_1)$ for all $i \in S$. So the current proposal becomes binding at t_1. By Lemma 6 (c) $\sum_{i \in S} z_i(h_1) = \sum_{i \in S} \hat{z}_i(h_1)$, hence $\hat{z}_i(h_1) = z_i(h_1)$ for all $i \in S$. Denote $\Pi(h_1) = \{((w_i^1)_{i \in S^1}; S^1); ((w_i^2)_{i \in S^2}; S^2); \ldots; ((w_i^m)_{i \in S^m}; S^m)\}$ and without loss of generality, assume that S contains all the coalitions S^k where $k \leq r$ and is disjoint from all the coalitions S^k where $k \geq r + 1$. Then for all $i \in N(h_1) \setminus (S \cup S^{r+1} \cup \ldots \cup S^m)$,

$$\hat{z}_i(h_1) = z_i(h_1) = x_i(N(h_1) \mid N(h_1); C(h_1));$$

and for all $i \in S^k$ where $k \geq r + 1$,

$$\hat{z}_i(h_1) = z_i(h_1) = w_i^k \sum_{j \in S^k} x_j(N(h_1) \mid N(h_1); C(h_1)).$$

Thus $\hat{z}_i(h_1) = z_i(h_1)$ for all $i \in N(h_1)$. Note that all players are quiet between $(t_1; t_1 + 1)$. Let $t_2 = t_1 + 1$ and denote the history generated by $h_2 \in H(t_2)$. Since no one leaves between $[t_1; t_2)$, so $N(h_1) = N(h_2)$.
Since $S \neq N(h_2)$, at time t_2, for all $i \in N(h_2)$ $f_i(h_2) = ((\frac{z_i(h_2)}{\sum_{l \in N(h_2)} x_l(N(h_2) \mid N(h_2); C(h_2))})_{j \in N(h_2)}; N(h_2))$.
Everyone is quiet between $(t_2; t_2 + 1)$. Let $t_3 = t_2 + 1$ and denote the history generated by $h_3 \in H(t_3)$. By definition for all $i \in N(h_2)$,

$$\hat{z}_i(h_3) = \frac{z_i(h_2)}{\sum_{j \in N(h_2)} x_j(N(h_2) \mid N(h_2); C(h_2))} \sum_{l \in N(h_2)} x_l(N(h_3) \mid N(h_3); C(h_3))$$
$$= z_i(h_2) = z_i(h_3).$$

The second equality follows because no one leaves between $[t_2; t_3)$, so $N(h_2) = N(h_3)$ and $C(h_2) = C(h_3)$. The third equality follows because no new offers binds between $[t_2; t_3)$. Thus all players in $N(h_2)$ accept the proposal at time t_3. Everyone is quiet between $(t_3; t_3 + 1)$. Let $t_4 = t_3 + 1$ and denote the history generated by $h_4 \in H(t_4)$.

Since no one leaves between $[t_3; t_4)$, so $N(h_3) = N(h_4)$ and $C(h_3) = C(h_4)$. At t_4, since $\sigma(h_4) = \{(\cdot; N(h_4))\}$, so all players leave. Thus player $i \in N(h_4)$ gets

$$\frac{z_i(h_2)}{\sum_{j \in N(h_2)} x_j(N(h_2) \mid N(h_2); C(h_2))} \sum_{l \in N(h_4)} x_l(N(h_4) \mid N(h_4); C(h_4)) = z_i(h_2).$$

However $z_i(h_2) = \hat{z}_i(h_1)$ because at time t_1, $p(h_1)$ binds. Combining with $\hat{z}_i(h_1) = z_i(h_1)$, thus $z_i(h_2) = z_i(h_1) = z_i(h)$. The last equality follows because nothing happens between $[t; t_1)$.[11] Thus player $i \in N(h)$ gets $z_i(h)$ in the continuing equilibrium.

■

We can now prove a theorem parallel to Theorem 2 in Perry and Reny.

Theorem 8 *If $(P; N)$ is superadditive, then any element of its r-core can be supported as an SSPNE outcome by $(f_1; \ldots; f_n)$.*

Proof. The proof is almost the same as that in Perry and Reny (1994) except with some minor modifications.

For any $t \geq 0$, $h \in H(t)$ and $i \in N(h)\setminus A(h)$, if $f_j(h) = l$ for some $j \in N(h)\setminus\{i\}$, then according to the equilibrium strategies, it must be the case that $\sigma(h) = \{(\cdot; N(h))\}$. Hence player i has to leave anyway. So $f_i(h) = l$ is clearly optimal. Thus we only need to show that for any $t \geq 0$, $h \in H(t)$ and $i \in N(h)\setminus A(h)$, if $f_j(h) \neq l$ for all $j \in N(h)\setminus\{i\}$, then using the equilibrium strategy f_i given all others are using their corresponding equilibrium strategies f_{-i} is optimal for player i.

[11] Hence $N(h) = N(h_1)$.

Consider another strategy f_i' for player i. Notice that f_i' and f_{-i} generate a unique continuation path h' subsequent to h. Since i gets at least $z_i(h) > -1$ by following f_i, he does no better if he never leaves according to h'. Thus suppose i leaves at time $t' \geq t$ with the proposal $((w_j')_{j \in S'}; S')$. Notice that since others are following the equilibrium strategies f_{-i}, everyone must leave the game ultimately. The coalitional structure thus formed, denoted by P_N', must satisfy:

$$S \in P_N' \text{ for all } S \in C(h) \text{ and } S' \in P_N'.$$

That is, it must respect the coalitions that have left. Thus player i obtains the payoff of $w_i' P(S' \mid P_N')$. Since the proposal $((w_j')_{j \in S'}; S')$ must be binding before the coalition S' can leave, thus $((w_j')_{j \in S'}; S') \in \Sigma(h'|t')$. By Lemma 6

$$\begin{aligned} z_i(h'|t') &\geq w_i' \sum_{j \in S'} x_j(N(h'|t') \setminus S' \mid N(h'|t') \setminus S'; S'; C(h'|t')) \\ &= w_i' P(S' \mid P_N'). \end{aligned}$$

The equality follows because the coalitional structure formed is unique, thus $P_N' = \{N(h'|t') \setminus S'\} \cup \{S'\} \cup C(h'|t')$. Moreover, the vector $x(N(h'|t') \setminus S' \mid N(h'|t') \setminus S'; S'; C(h'|t'))$ is in the core $C(N(h'|t') \setminus S' \mid N(h'|t') \setminus S'; S'; C(h'|t'))$, hence the sum of payoffs for players in S' is simply $\sum_{j \in S'} x_j(N(h'|t') \setminus S' \mid N(h'|t') \setminus S'; S'; C(h'|t')) = P(S' \mid P_N')$.

We follow Perry and Reny (1994) to consider three exhaustive cases.

Case A: $A(h) = \emptyset$.

Case B: $p(h) = ((w_j)_{j \in T}; T)$ and either $\hat{b}_i(h) \leq z_i(h)$ or $\hat{b}_j(h) < z_j(h)$ for some $j \in T \setminus A(h)$.

Case C: $p(h) = ((w_j)_{j \in T}; T)$, $\hat{b}_i(h) > z_i(h)$ and $\hat{b}_j(h) \geq z_j(h)$ for all $j \in T \setminus A(h)$.

Notice that case B covers the instances where $i \notin T$ since then $\hat{b}_i(h) = z_i(h)$. Case A covers the instances where $p(h) = \emptyset$. As in Perry and Reny (1994), the argument applying to cases A and B have a common component, thus we treat them together until it is necessary to separate them.

Cases A or B: By Lemma 7, in case A, player i will get $z_i(h)$ by using the equilibrium strategy f_i. In case B, by the equilibrium strategies, either player i does not belong to T, he is the only player who hasn't accepted the current proposal $p(h)$ and will accept it or at least a player will reject the proposal by making a new proposal involving all the players $N(h)$, in any case, player i gets $z_i(h)$. Thus suppose to the contrary that by following f'_i, player i made a profitable deviation. Thus $w'_i P(S' \mid P'_N) > z_i(h)$.

Since $z_i(h'|t') \geq w'_i P(S' \mid P'_N) > z_i(h)$, it must be the case that $t' > t$. This is because $h'|t = h$. Hence, let

$$t^* = \inf\{\tau \in [t, t'] \mid z_i(h'|\tau) > z_i(h)\}.$$

It follows that $z_i(h) \geq z_i(h'|t^*)$. To see this, note if $t^* = t$, then since $h'|t = h$, it is certainly true. If $t^* > t$ and suppose to the contrary that $z_i(h) < z_i(h'|t^*)$, then by (S4), there exists an $\varepsilon > 0$ small enough so that $t^* - \varepsilon > t$ and nothing happens between $[t^* - \varepsilon, t^*)$. Hence $z_i(h'|t^* - \frac{\varepsilon}{2}) = z_i(h'|t^*) > z_i(h)$. But then t^* is not the infimum. Hence $z_i(h) \geq z_i(h'|t^*)$. This implies $t^* \neq t'$ because $z_i(h'|t') > z_i(h)$. Thus $t' > t^*$.

Because $z_i(h) \geq z_i(h'|t^*)$ and t^* is the infimum, there must exist a sequence of positive numbers $\{\varepsilon_n\}$ where $\lim_{n \to \infty} \varepsilon_n = 0$ and $z_i(h'|t^* + \varepsilon_n) > z_i(h) \geq z_i(h'|t^*)$ for every ε_n. By (S4), there must exist an n^* large enough such that nothing happens between $(t^*, t^* + \varepsilon_{n^*})$. Thus something must happen at time t^* for otherwise it cannot be the case that $z_i(h'|t^* + \varepsilon_{n^*}) > z_i(h'|t^*)$. Thus either the current proposal $p(h'|t^*)$ which contains player i becomes binding at t^* or someone leaves at t^*. The latter cannot happen because according to f'_i, player i leaves at $t' > t^*$. Players other than i cannot leave at t^* either for otherwise since they are playing according to the equilibrium strategies f_{-i}, when one leaves, all must leave, contradicting player i leaves at $t' > t^*$. Therefore, the current proposal $p(h'|t^*)$ becomes binding at t^*. Let $p(h'|t^*) = ((w^*_j)_{j \in S}, S)$. Notice that $i \in S$.

Since the current proposal $p(h'|t^*)$ becomes binding at t^* and nothing happens between $(t^*, t^* + \varepsilon_{n^*})$, this implies $\hat{z}_i(h'|t^*) = z_i(h'|t^* + \varepsilon_{n^*})$. Because

$z_i(h'|t^* + \varepsilon_{n^*}) > z_i(h'|t^*)$, it follows $\hat{z}_i(h'|t^*) > z_i(h'|t^*)$. By part (c) of Lemma 6 $\sum_{j \in S} z_j(h'|t^*) = \sum_{j \in S} \hat{z}_j(h'|t^*)$. Because i is in S, this implies there exists a player k in S such that $\hat{z}_k(h'|t^*) < z_k(h'|t^*)$. We now separate the discussion for cases A and B.

Case A: Since $p(h'|t^*)$ becomes binding at t^* and at time t no one has accepted any proposal because $A(h) = \emptyset$, thus player k must accept $p(h'|t^*)$ at some point of time $t_k \in [t; t^*]$. At that time, since k is playing according to the equilibrium strategy, so $\hat{z}_k(h'|t_k) \geq z_k(h'|t_k)$. Note that from $\pi(h'|t_k) = \pi(h'|t^*)$, $N(h'|t_k) = N(h'|t^*)$ and $C(h'|t_k) = C(h'|t^*)$ since $p(h'|t_k) = p(h'|t^*)$. Hence $\hat{z}_k(h'|t_k) = \hat{z}_k(h'|t^*)$ and $z_k(h'|t_k) = z_k(h'|t^*)$. This implies $\hat{z}_k(h'|t^*) \geq z_k(h'|t^*)$, yielding a contradiction. Hence there is no profitable deviation for case A.

Case B: Note that $p(h) = ((w_j)_{j \in T}; T)$ will not bind. This is because if there exists some $j \neq i$ where $j \in T \backslash A(h)$ such that $\hat{z}_j(h) < z_j(h)$, then he will not accept the proposal and will make another proposal pertaining to $N(h)$ at the next integer time if no one has done so. If there exists no $j \neq i$ where $j \in T \backslash A(h)$ such that $\hat{z}_j(h) < z_j(h)$, then either $\hat{z}_i(h) = z_i(h)$ or $\hat{z}_i(h) < z_i(h)$. When $\hat{z}_i(h) = z_i(h)$, according to the equilibrium strategies, all $j \neq i$ where $j \in T \backslash A(h)$ will accept the proposal at the next integer time. Thus if player i accepts as well, the proposal will bind. However, once it binds, say at time t'', then $p(h'|t'') = \emptyset$. Since we have shown in Case A that no profitable deviation is possible, player i's optimal strategy is to follow the equilibrium strategy f_i from t'' on. This implies i's payoff will be $\hat{z}_i(h) = z_i(h)$ by using f_i'. Since $\hat{z}_i(h) = w_i' P(S' | P_N')$. This is in contradiction to $w_i' P(S' | P_N') > z_i(h)$. When $\hat{z}_i(h) < z_i(h)$, there are two possibilities. Either there exists an $j \neq i$ where $j \in T \backslash A(h)$ or $\{i\} = T \backslash A(h)$. In the first situation when there exists an $j \neq i$ where $j \in T \backslash A(h)$, then according to player j's equilibrium strategy, he will not accept the proposal and will make another proposal pertaining to $N(h)$ if no on has done so. In the second situation where $\{i\} = T \backslash A(h)$, player i will not accept the proposal. For if he did, say at time t''', then $p(h'|t''') = \emptyset$. Again, since we have shown in Case A that no profitable deviation is possible from t''' on, player i gets $\hat{z}_i(h) < z_i(h)$, yielding a contradiction. Therefore in all possible situations, $p(h) = ((w_j)_{j \in T}; T)$ will not bind.

Since $p(h'|t^*)$ becomes binding at t^* and $p(h) = ((w_j)_{j \in T}; T)$ will not bind, $p(h'|t^*)$ must be proposed at time t or later but before t^*. Hence player k must accept $p(h'|t^*)$ at some point of time $t_k \in [t; t^*]$. Now apply exactly the same logic in case A to get a contradiction. Thus there is no profitable deviation for case B.

Case C: We will show that player i has no profitable deviation. If player i plays according to the equilibrium strategy f_i, since all others are also playing the equilibrium strategies, his payoff is $\hat{z}_i(h) > z_i(h)$. If instead player i deviates to another strategy f_i', there are two possibilities.

In the first possibility $p(h)$ becomes binding. This implies player i accepts $p(h)$ at some time t''. Since all other players accept $p(h)$ by the equilibrium strategies at the next integer time, say t''', this implies $p(h)$ becomes binding at $\max\{t''; t'''\}$. Therefore $p(h'|\max\{t''; t'''\}) = \emptyset$. By the argument in case A, it is optimal for player i to follow the equilibrium strategy from time $\max\{t''; t'''\}$ on. Hence player i's payoff from using f_i' is at most $\hat{z}_i(h)$.

In the second possibility $p(h)$ does not become binding. This implies either player i makes another proposal at some time t'' or i leaves before accepting. In the first situation, since others are playing according to the equilibrium strategies, if the next integer time arrives before than or at t'', all others accept $p(h)$ at the next integer time except player i. If the next integer time arrives after t'', this new proposal is made before anyone has accepted it. Both imply $p(h'|t'') = \emptyset$. By the argument in case A, it is optimal for player i to follow the equilibrium strategy from time t'' on. Hence player i's payoff from using f_i' is at most $z_i(h)$. In the second situation where i leaves before accepting, it must be the case that player i is in a binding coalition S^k. After S^k leaves, all players still play according to the equilibrium strategies. Thus $x(N(h) \setminus S^k \mid N(h) \setminus S^k; S^k; C(h))$ is expected in the continuing equilibrium.

Hence player i's payoff is $w_i^k \quad \sum_{j \in S^k} x_j(N(h) \setminus S^k - N(h) \setminus S^k; S^k; C(h)) \quad z_i(h)$ by part (a) of Lemma 6.

Thus there is no profitable deviation for case C. ■

6 Conclusion

There are some difficulties in extending the proof of Theorem 8 to cases where superadditivity does not hold. For example, suppose the r-core structure of N should consist of three sub coalitions S_1, S_2 and S_3. To implement a r-core payoff consistent with this coalitional structure, without loss of generality, suppose S_1 has to form first, S_2 second and S_3 last.

When S_1 forms, the relevant partition function is no longer $(P; N)$. Instead, we should now treat S_1 as a composite player and consider the new partition function where S_1 is treated as a player. This implies some sort of consistency is needed. For instance, a player in S_2 may not want to deviate originally but now since players in S_1 cannot break apart, he may get a new incentive to deviate. Imposing some consistency on the partition functions presumably could rule this kind of deviation out.

Moreover, once S_1 has formed, it can leave first. When this happens, according to 4, we expect the r-core for $N \setminus S_1$ to occur in the continuing equilibrium. In this situation, we naturally want the r-core structure for $N \setminus S_1$ to consist of S_2 and S_3 because we have to make sure that S_1 does not have the incentive to deviate to leave so fast to induce another continuing equilibrium.

How to solve difficulties of these kinds is worth further consideration if not impossible. We leave this for future research.

References

[1] To be filled.