行政院國家科學委員會

獎勵人文與社會科學領域博士候選人撰寫博士論文成果報告

投資型金融商品與退休金制度附加最低保證之評價

核 定 編 號 ：NSC 95-2420-H-002-032-DR
獎 勵 期 間 ：95 年 08 月 01 日至 96 年 07 月 31 日
執 行 單 位 ：國立臺灣大學財務金融學系暨研究所
指 導 教 授 ：楊曉文

博 士 生 ：唐俊華

公 開 資 訊 ：本計畫可公開查詢

中 華 民 國 96 年 08 月 07 日

臺(96)博證字第D00146號
學號：D91723006

國立臺灣大學博士學位證書

唐俊華，中華民國陸拾伍年玖月拾柒日生，於本校財務金融學系博士班研究期滿經博士學位考試合格，依學位授予法之規定授予商學博士學位。此證

校　長　李嗣涔

系主任　陳聖賢

中華民國九十六年六月

校對者：呂學榮

國立臺灣大學

國立臺灣大學管理學院財務金融學系

博士論文

Department of Finance
College of Management
National Taiwan University

Doctoral Dissertation

附收益率保證確定提撥制退休金計畫之研究

The Study of Rate of Return Guarantees under Defined Contribution Pension Plans

唐俊華

Tang, Chun-Hua

指導教授：曾郁仁博士、楊曉文博士

Advisors: Tzeng, Larry Y., Ph.D.
Yang, Sharon S., Ph.D.

中華民國 96 年 6 月

June, 2007

口試委員會審定書

國立臺灣大學博士學位論文
口試委員會審定書

附收益率保證確定提撥制退休金計畫之研究
The Study of Rate of Return Guarantees
under Defined Contribution Pension Plans

本論文係 唐俊華（學號：D91723006）在國立臺灣大學財務金融研究所完成之博士學位論文，於民國九十六年五月十八日承下列考試委員審查通過及口試及格，特此證明

口試委員：

曾有仁　　蔡政憲

廖咸興　　謝明華

楊曉文

系主任、所長：

謝　辭

從踏入台大的第一天算起，經歷大學部、碩士班、博士班，至今已有十四年，這段時間受到許多人的幫助。感謝我的家人，由於父親母親與姊姊的栽培及支持，讓我在學習過程中，能夠專心於學業，並且順利完成博士學位。感謝台大財金所提供多元課程與豐富資源，讓我學習到財務金融、財務工程以及退休金領域的知識，並且結合於博士論文中。感謝博士論文指導教授－曾郁仁老師與楊曉文老師，謝謝曾老師的協助，更謝謝楊老師在研究過程中給予諸多指點，讓我在學術的道路上不斷成長。感謝廖咸興老師、蔡政憲老師與謝明華老師的建議，讓我的博士論文更加完善。感謝曾經教過我的所有老師，尤其是葉小蓁老師、陳業寧老師、巫和懋老師，幫助我在修課過程中建立紮實的研究基礎。感謝林煜宗老師、林筠老師、岳夢蘭老師，謝謝您們給予我在生活上或研究上的幫助。感謝多位學長姊以及博士班的同班同學們，謝謝你們讓我在修課與資格考過程中，有請益與切磋的對象。感謝國科會的獎勵，讓我在博士班的第五年能夠專心致力於撰寫論文。更要感謝上天的保祐，讓我的人生旅程能順利進展。

在接下來的人生中，期望自己能夠將所學貢獻給社會，並且盡一己之力幫助需要幫助的人，存好心、說好話、做好事。與大家共勉之。

中文摘要

本論文包含兩篇文章，皆是以附收益率保證確定提撥制退休金計畫為研究主軸。第一篇文章推導附收益率保證之評價公式，此收益率保證乃是連結至δ年期定存利率，並且兼具互換選擇權、路徑相依選擇權、遠期生效選擇權之性質。本文同時考慮兩種保證型態：到期日保證與多期保證，運用平賭訂價理論以及Margrabe (1978)的方法，在一因子 HJM 利率模型下推導這兩種保證的價格公式解，並且分析重要參數對價格的影響。第二篇文章延續第一篇的保證型態，分析投資策略與行為是否會影響保證成本與退休金收益。本文假設利率動態遵循 CIR 模型，並設定買進持有策略與多種更換投資組合策略，以模擬方式得到保證成本與退休金收益的數值結果。在本文的保證型態與參數設定下，計畫參與者可以買進具有高報酬率與高波動率的投資組合並持有至退休日，以獲得高預期退休金收益；此外，在考慮更換投資組合的行為後，保證成本可能會上升，這反應出：附收益率保證退休金計畫發行者在評估保證成本時，必須將參與者可能的投資行為納入考慮，以免低估成本。

關鍵詞：確定提撥制、退休金計畫、收益率保證、平賭訂價理論、HJM 模型、CIR 模型、行為財務學、錯置效果。

Abstract

This doctoral dissertation contains two essays to study the minimum rate of return guarantees embedded in defined contribution pension plans. The first essay derives the explicit formulae to value these guarantees. Different from previous studies, we analyze the guarantees linked to the δ-year spot rates. Two types of guarantees are considered: maturity guarantee and multi-period guarantee. These guarantees have mixed features of exchange options, path-dependent options and forward start options. We adopt the one-factor HJM model to describe the evolution of interest rates. The closed-form formulae are derived by the martingale pricing theory and Margrabe's (1978) approach. We also present numerical results and analyze how the values change with the parameter estimates.

The second essay investigates how investment strategies and behaviors affect guarantee costs and retirement benefits in defined contribution plans that provide interest rate guarantees. Several investment strategies are considered, including both a buy-and-hold strategy without portfolio modifications and those that allow frequent modifications of the portfolio during the accumulation phase. According to the numerical results offered by simulation techniques, a participant chasing the highest expected retirement benefits should adopt the buy-and-hold strategy and distribute all

contributions in the portfolio with high expected rate of return and high volatility in our guarantee designs. After incorporating frequent modification behaviors, averages and risk measures of guarantee costs may be higher than when the participant always holds the portfolio with high expected rate of return and volatility. Therefore, when the pension plan provides rate of return guarantees, the plan provider cannot ignore the impact of the participant's frequent modification behaviors.

Keywords: Defined Contribution, Pension Plan, Rate of Return Guarantee, Martingale Pricing Theory, HJM Model, CIR model, Behavioral Finance, Disposition Effect.

目　錄

口試委員會審定書 i
謝　辭 ii
中文摘要 iii
Abstract iv

The First Essay: Valuation of Interest Rate Guarantees Embedded in Defined Contribution Pension Plans 1
1. Introduction 2
2. Structures of Guarantees and Financial Model Settings 6
3. Valuation of Interest Rate Guarantees 11
    3.1 Type-I Guarantee (Maturity Guarantee) 11
    3.2 Type-II Guarantee (Multi-Period Guarantee) 19
4. Numerical Results and Sensitivity Analysis 23
5. Extension Research under the Setting of Exponentially Decaying Volatility 30
6. Conclusions 39
References 41
Appendix 44

The Second Essay: Impacts of Investment Strategies and Participants' Behaviors on Guarantee Costs and Retirement Benefits in Defined Contribution Pension Plans 51
1. Introduction 52
2. Structures of Guarantees, Financial Models, and Strategies 58
    2.1 Structures of Guarantees 58
    2.2 Interest Rate and Portfolio Price Dynamics 59
    2.3 Investment Strategies and Participants' Behaviors 61
    2.4 Actual Rate of Return and Income Replacement Ratio 65
3. Numerical Results 66
    3.1 Results for S1 (Buy-and-Hold Strategy) 67
    3.2 Results for Strategies S2-S7 (with Modification) 81
4. Conclusions 92
References 94
Appendix 100

Valuation of Interest Rate Guarantees Embedded in Defined Contribution Pension Plans

Abstract

This paper derives the explicit formulae to value minimum rate of return guarantees embedded in defined contribution pension plans. Different from previous studies, we analyze the guarantees linked to the δ-year spot rates. Two types of guarantees are considered: maturity guarantee and multi-period guarantee. These guarantees have mixed features of exchange options, path-dependent options and forward start options. We adopt the one-factor HJM model to describe the evolution of interest rates. The closed-form formulae are derived by the martingale pricing theory and Margrabe's (1978) approach. We also present numerical results and analyze how the values change with the parameter estimates.

1. Introduction

Many countries have shifted the pension mechanism from the defined benefit (DB) plan to the defined contribution (DC) plan. A DB plan usually promises deterministic benefits to the participant according to some predetermined formulae. Plan providers bear the risks of poor investment performance. Once the participant stops working in one organization and skips to another before retirement, the accumulated pension fund is stopped and cannot be ported with him or her. In contrast, a DC plan offers each participant one portable individual account. The participant regularly contributes parts of wages to this account. Retirement benefits are affected by the investment performance. Disappointing performance will result in uncomfortable retirement life. To lessen this negative impact of a DC plan, minimum rate of return guarantees may be embedded in the plan and the costs are covered by guarantee providers, such as governments or financial organizations.

Among the guarantees commonly seen around the world, some are granted in a deterministic manner, whereas others are stochastic and linked to rates of return on a reference portfolio or interest rates. Lindset (2004) names the deterministic type as the absolute guarantee and the stochastic type as the relative guarantee. In addition, Miltersen and Persson (1999) and Lindset (2004) both mention that guarantees can be

categorized into two forms, maturity guarantee and multi-period guarantee, according to when the guarantee is obliged. The maturity guarantee is binding only at expiration of the pension plan, and the multi-period guarantee specifies a minimum rate of return binding in each period.

The valuation of maturity guarantees or multi-period guarantees associated with pension plans or life insurance contracts has been widely studied. The studies mostly focused on absolute guarantees, including Brennan and Schwartz (1976), Boyle and Schwartz (1977), Ekern and Persson (1996), Boyle and Hardy (1997), Persson and Aase (1997), Grosen and Jørgensen (1997, 2000), Miltersen and Persson (1999), Hansen and Miltersen (2002), and Schrager and Pelsser (2004). Recently, more and more papers have started to analyze relative guarantees, and the guaranteed rate is usually measured by the rate of return on the reference portfolio. The valuation methodology for the relative guarantee is much more complicated than that for the absolute guarantee because it has to incorporate the rate of return process of the reference portfolio. Ekern and Persson (1996) deal with the relative guarantees embedded in unit-linked life insurance contracts. Pennacchi (1999) obtains the values of relative guarantees provided in DC plans by explicit formulae or by using Monte Carlo simulation. Lindset (2004) applies the model of Heath-Jarrow-Morton

(henceforth, HJM) (1992) to deriving closed-form pricing formulae for different pension contracts embedded with minimum guaranteed rates of return. In all of these three papers, guaranteed rates are linked to the performance of the reference portfolio whose price follows the geometric Brownian motion.

Extending prior studies, we consider the relative guarantees linked to the δ-year spot rates, which means that the guarantees are interest rate guarantees. This design is attractive to both guarantee providers and pension participants. Compared with the rate of return on a reference portfolio traded in the equity market, δ-year spot rates are less easily manipulated by individuals and organizations. In addition, the pension plan has to guarantee the rate of return which can be obtained in the market. If not, participants would rather put money in the certificates of deposit issued by banks than join the pension plan.

Our objective is to determine the fair values of both maturity and multi-period interest rate guarantees. The interest rate guarantees under analysis have several features. First, the payoff of the pension plan embedded with either of these guarantees is similar to that of the exchange option. Second, the guaranteed rates are dependent on the path of the spot rate during the accumulation phase, which is the working period up to

retirement. This path-dependent feature is also seen in many exotic options, such as the Asian option.[1] Finally, some of the embedded guarantees are calculated from the specific time in the future. This resembles the forward start option.[2] In order to value the interest rate guarantees, the valuation framework has to incorporate two stochastic processes to characterize the price of the underlying investment portfolio and the stochastic interest rate. The price of the underlying investment portfolio is assumed to follow a geometric Brownian motion. The evolution of interest rates is described by the one-factor HJM model. We adopt the martingale pricing theory and Margrabe's (1978) approach to derive the values of both guarantees.

The remaining paper is organized as follows. Section 2 presents the structures of guarantees and financial model settings. The closed-form formulae for both types of guarantees are derived in section 3. We provide numerical results for the values of these guarantees and carry out the sensitivity analysis in section 4. Section 5 extends the proceeding study by setting different diffusion factor in the HJM model. Section 6 concludes this paper and proposes further research directions.

[1] According to Hull (2000), the payoff of the Asian option depends on the arithmetic or geometric average price of the underlying asset during some period within the lifetime of the option.

[2] According to Hull (2000), the forward start option is the option that will start in the future. Bouaziz *et al.* (1994) and Tsao *et al.* (2003) discuss the pricing of forward-starting Asian options, where the price of the underlying asset follows lognormal distribution and the strike price is computed as the arithmetic average of the underlying asset prices over one period in the future. No dynamic of interest rates is considered in Bouaziz *et al.* (1994) and Tsao *et al.* (2003).

2. Structures of Guarantees and Financial Model Settings

First we present the basic structure of the pension plan analyzed in this study. During the accumulation phase, the participant contributes at the first day of each year. The actual rate of return and the guaranteed rate are calculated at several record dates, each of which is the last calendar day of each year. The timeline of the pension plan is shown in Figure 1. One participant starts to work and selects one investment portfolio at t=0. The investment portfolio is held until retirement. The yearly wages in the first year are Y_0. The annual growth rate of wages is k. At the n-th year, the participant's yearly wages are $Y_0(1+k)^{n-1}$. The participant does not retire until time T. During these T years, the percentage of wages contributed by a participant to the individual account per year is fixed at c. We neglect administrative costs and the case that the participant selects different investment portfolios during the accumulation phase.

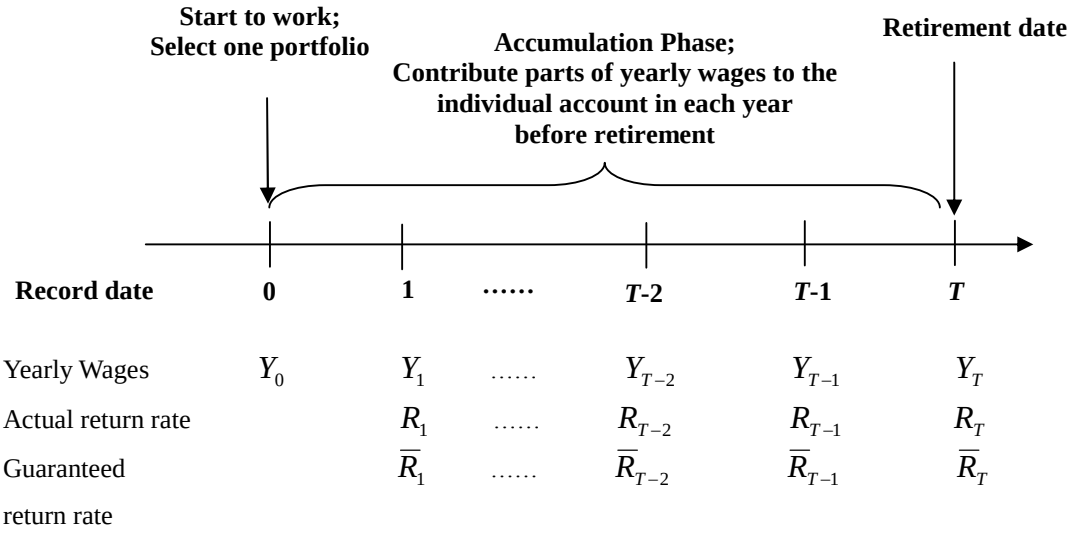


Figure 1. Timeline of the Pension Plan

The amount contributed to the individual account in the first year is $Y_0 \cdot c$. Assume that R_t is the actual rate of return in the *t-th* year. At the first record date (the last calendar day of the first year), the value of the first contribution becomes $Y_0 \cdot c \cdot \exp(R_1)$; at the second record date, it becomes $Y_0 \cdot c \cdot \exp(R_1) \cdot \exp(R_2)$; at time *T*, this amount changes to $Y_0 \cdot c \cdot \prod_{t=1}^{T} \exp(R_t)$ according to the actual investment performance. Similarly, the amount contributed in the *n*-th year changes to $Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \prod_{t=n}^{T} \exp(R_t)$ at time *T*. Summing up actual payoffs of the contributions of each year, we have $\sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \prod_{t=n}^{T} \exp(R_t)$, which are the benefits the participant receives at time *T* without any guarantee and is named as "*actual benefits*."

In each year over the life of the plan, the guarantee provider offers a minimum continuously compounded annual rate of return, which is equal to the δ-year spot rate at each record date. Let $\overline{R}(t,t+\delta)$ be the minimum guaranteed rate of return at time *t*. Two types of guarantees are considered. If the guarantee is binding only at expiration of the pension plan, we called it a "*type-I guarantee*," which is a maturity guarantee. If the minimum guaranteed rate of return is binding for each period, we refer it as a "*type-II guarantee*," which is a multi-period guarantee.

Under the type-I guarantee, we express the retirement benefits a participant claims when retiring at time *T* as

$$\Gamma_T(I) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot Max\left[\prod_{t=n}^{T}\exp(\overline{R}(t,t+\delta)), \prod_{t=n}^{T}\exp(R_t)\right]$$

$$= \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \left\{Max\left[\prod_{t=n}^{T}\exp(\overline{R}(t,t+\delta)) - \prod_{t=n}^{T}\exp(R_t), 0\right] + \prod_{t=n}^{T}\exp(R_t)\right\}, \quad (1)$$

where $\sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot Max\left[\prod_{t=n}^{T}\exp(\overline{R}(t,t+\delta)) - \prod_{t=n}^{T}\exp(R_t), 0\right]$ is the amount the guarantee provider has to compensate for the type-I guarantee.

Under the type-II guarantee, we express the retirement benefits received at time *T* as

$$\Gamma_T(II) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \prod_{t=n}^{T} Max\left[\exp(\overline{R}(t,t+\delta)), \exp(R_t)\right], \quad (2)$$

and $\sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \left\{\prod_{t=n}^{T} Max\left[\exp(\overline{R}(t,t+\delta)), \exp(R_t)\right] - \prod_{t=n}^{T}\exp(R_t)\right\}$ is the amount the guarantee provider has to pay for the type-II guarantee.

Equations (1) and (2) reflect that the payoffs of the defined contribution pension plans embedded with the type-I or type-II guarantees have some interesting features. First, both payoffs have similar structure to the payoff of the exchange option. Second, these payoffs depend on the paths of the actual rate of return on the underlying investment portfolio, R_t, and the guaranteed rate of return, $\overline{R}(t,t+\delta)$. Third, some of the embedded guarantees are not calculated from the inception of the pension plan but from the specific time in the future.[3]

Now we illustrate how to calculate R_t and $\overline{R}(t,t+\delta)$. Under the risk-neutral measure *Q*, the price of the investment portfolio per share, S_t, follows the stochastic process:

$$dS_t = r_t S_t\, dt + \sigma_S S_t\, dZ_t, \tag{3}$$

where $\sigma_S \in \Re^+$ is the constant diffusion factor of the portfolio price, r_t is the risk-free short rate, and $\{Z_t : t \geq 0\}$ is a standard one-dimensional *Q*-measure Brownian motion.

[3] The payoff structures in this paper are different from those in Lindset (2004). We define the guaranteed rate as the δ-year spot rate at time *t*. Instead, Lindset (2004) defines the guaranteed rate in any period *t* as the rate of return on the reference portfolio between time *t*-1 to *t*. Furthermore, we plan to calculate the single premium paid at *t*=0 for the guarantee so we not only take the first contribution into account but also allow that *n*>1. Lindset (2004) only considers $1 contributed initially and implies that the premium is paid in each period.

The actual rate of return on this portfolio R_t is defined as:

$$R_t \equiv \ln(S_t / S_{t-1}), \quad t = 1,2,...,T . \tag{4}$$

When the price of the underlying investment portfolio follows the geometric Brownian motion, R_t is normally distributed.

We apply the HJM framework to deriving short rates and spot rates. According to Heath *et al.* (1992), the instantaneous forward rate process under the risk-neutral measure *Q* is

$$df(t,u) = \left(\sigma_f(t,u) \int_t^u \sigma_f(t,x)dx \right) dt + \sigma_f(t,u) dW_t, \tag{5}$$

where $u \geq t$ and $\{W_t : t \geq 0\}$ is a standard one-dimensional *Q*-measure Brownian motion correlated with Z_t and satisfies $dW_t dZ_t = \rho dt$ for all $\rho \in [-1,+1]$. Therefore, after considering another standard one-dimensional *Q*-measure Brownian motion $W_t^{'}$ independent of W_t, we can rewrite $dZ_t = \rho dW_t + \sqrt{1-\rho^2}\, dW_t^{'}$. The diffusion factor, $\sigma_f(t,u)$, plays an important role in the HJM model. We first assume that $\sigma_f(t,u)$ is a constant, i.e., $\sigma_f(t,u) = \sigma_f > 0$, then we can rewrite equation (5) as

$$df(t,u) = (\sigma_f^2(u-t))dt + \sigma_f dW_t . \tag{6}$$

The HJM model with constant volatility factor can be viewed as the continuous-time limit of the Ho-Lee (1986) model. By equation (6), the instantaneous forward rate is

$$f(t,u) = f(0,u) + \sigma_f^2 t\left(u - \frac{t}{2}\right) + \sigma_f W_t;$$

the short rate r_t is

$$r_t = f(0,t) + \frac{\sigma_f^2}{2}t^2 + \sigma_f W_t; \tag{7}$$

the time-*t* price of a zero-coupon bond maturing at time $t+\delta$ is

$$P(t,t+\delta) = \frac{P(0,t+\delta)}{P(0,t)} \cdot \exp\left[-\frac{\sigma_f^2}{2}t\delta(t+\delta) - \sigma_f \delta W_t\right],$$

where $P(0,t) = \exp\left[-\int_0^t f(0,u)du\right]$; and the time-*t* spot rate for maturity $t+\delta$ is

$$\begin{aligned}\overline{R}(t,t+\delta) &= -\frac{\ln P(t,t+\delta)}{(t+\delta)-t} = \frac{-\ln P(0,t+\delta) + \ln P(0,t) + \frac{\sigma_f^2}{2}t\delta(t+\delta) + \sigma_f \delta W_t}{\delta} \\ &= g_1(t) + \sigma_f W_t, \end{aligned} \tag{8}$$

where $g_1(t) \equiv \frac{-\ln P(0,t+\delta) + \ln P(0,t)}{\delta} + \frac{\sigma_f^2}{2}t(t+\delta)$. From these expressions, we know that the guaranteed rate of return $\overline{R}(t,t+\delta)$ is normally distributed.

3. Valuation of Interest Rate Guarantees

3.1 Type-I Guarantee (Maturity Guarantee)

This section derives the value of the type-I guarantee. Define $V_T(I)$ as the terminal market value of the type-I guarantee, which is

$$V_T(I) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot Max\left[\prod_{t=n}^{T} \exp(\overline{R}(t,t+\delta)) - \prod_{t=n}^{T} \exp(R_t), 0 \right]. \quad (9)$$

Also define $\pi_T^{(n)}(I)$ as the time-T value of the type-I guarantee for one dollar contributed in the n-th year, $n \le T$, i.e.,

$$\pi_T^{(n)}(I) = Max\left[\prod_{t=n}^{T} \exp(\overline{R}(t,t+\delta)) - \prod_{t=n}^{T} \exp(R_t), 0 \right].$$

Let $A_{1T}^{(n)} = \prod_{t=n}^{T} \exp(\overline{R}(t,t+\delta))$ and $A_{2T}^{(n)} = \prod_{t=n}^{T} \exp(R_t)$, then we have $\pi_T^{(n)}(I) = Max[A_{1T}^{(n)} - A_{2T}^{(n)}, 0]$.

Proposition 1: Given that the time-T value of the type-I guarantee for one dollar contributed in the n-th year is $\pi_T^{(n)}(I)$. The time-0 market value of $\pi_T^{(n)}(I)$ is $\pi_0^{(n)}(I) = A_{10}^{(n)} \Phi(d_1^{(n)}) - A_{20}^{(n)} \Phi(d_2^{(n)})$, where

$$A_{10}^{(n)} = \exp\left\{ g_2(n,T) + \frac{\sigma_f^2}{6} \cdot [T + n(n-1)(n-2)] \right\},$$

$$A_{20}^{(n)} = \exp\left[-\int_0^{n-1} f(0,u)du \right],$$

$$g_1(t) = \frac{-\ln P(0,t+\delta) + \ln P(0,t)}{\delta} + \frac{\sigma_f^2}{2} t(t+\delta),$$

$$g_2(n,T) = \left(\sum_{t=n}^{T} g_1(t) \right) - \int_0^T f(0,u)du - \frac{\sigma_f^2 T^3}{6},$$

$$d_1^{(n)} = \frac{\ln(A_{10}^{(n)} / A_{20}^{(n)}) + (\hat{\sigma}^{(n)})^2 T/2}{(\hat{\sigma}^{(n)})\sqrt{T}}, \quad d_2^{(n)} = d_1^{(n)} - (\hat{\sigma}^{(n)})\sqrt{T},$$

$$(\hat{\sigma}^{(n)})^2 T = (T-n+1)\left(\frac{\sigma_f^2}{3} + \sigma_S^2 - \rho\sigma_f\sigma_S\right), \text{ and}$$

$\Phi(\cdot)$ denotes the cumulative probability function for a variable followed by a standardized normal distribution.

After considering the contributions in each year and total accumulations, the value of the type-I guarantee is $V_0(I) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \pi_0^{(n)}(I)$.

Proof: The complete proof is broken into six steps.

- Step 1: Changing numeraire.

We follow the approach in Margrabe (1978). Treating the asset with time-*T* value of $A_{2T}^{(n)}$ as numeraire, the time-0 value of one contingent claim whose payoff at expiration date *T* is equal to $\pi_T^{(n)}(I)$ can be derived by the following equation:

$$\frac{\pi_0^{(n)}(I)}{A_{20}^{(n)}} = \mathrm{E}_0^{\hat{\mathrm{P}}}\left[\frac{\pi_T^{(n)}(I)}{A_{2T}^{(n)}}\right],$$

where $\mathrm{E}_0^{\hat{\mathrm{P}}}[\cdot]$ denotes the expectation function under $\hat{\mathrm{P}}$ measure conditional on the market information up to time 0. This equation comes from the martingale pricing approach and reflects that the process of $\frac{\pi_t^{(n)}(I)}{A_{2t}^{(n)}}$ is a martingale under $\hat{\mathrm{P}}$ measure. Then,

$$\pi_0^{(n)}(I) = A_{20}^{(n)} \cdot \mathrm{E}_0^{\hat{\mathrm{P}}}\left[\frac{Max(A_{1T}^{(n)} - A_{2T}^{(n)}, 0)}{A_{2T}^{(n)}}\right]$$

$$= A_{20}^{(n)} \cdot \mathrm{E}_0^{\hat{\mathrm{P}}}\left[Max\left(\frac{A_{1T}^{(n)}}{A_{2T}^{(n)}} - 1, 0\right)\right].$$

We view $Max\left(\frac{A_{1T}^{(n)}}{A_{2T}^{(n)}} - 1, 0\right)$ as the time-*T* payoff of one option whose strike price is unity and the time-*T* price of the underlying asset is $A_{1T}^{(n)} / A_{2T}^{(n)}$. $A_{1T}^{(n)}$, $A_{2T}^{(n)}$, and

$A_{1T}^{(n)} / A_{2T}^{(n)}$ are all lognormally distributed, so we can apply the formula in Hull (2000) and obtain that

$$\pi_0^{(n)}(I) = A_{20}^{(n)}\left[\frac{A_{10}^{(n)}}{A_{20}^{(n)}}\Phi(d_1^{(n)}) - 1\cdot\Phi(d_2^{(n)})\right]$$

$$= A_{10}^{(n)}\Phi(d_1^{(n)}) - A_{20}^{(n)}\Phi(d_2^{(n)}),$$

where $A_{10}^{(n)}$ is the time-0 price of $A_{1T}^{(n)}$, $A_{20}^{(n)}$ is the time-0 price of $A_{2T}^{(n)}$, $d_1^{(n)} = \frac{\ln(A_{10}^{(n)} / A_{20}^{(n)}) + (\hat{\sigma}^{(n)})^2 T/2}{(\hat{\sigma}^{(n)})\sqrt{T}}$, $d_2^{(n)} = d_1^{(n)} - (\hat{\sigma}^{(n)})\sqrt{T}$, and $(\hat{\sigma}^{(n)})^2 T$ is equal to $Var_0^Q[\ln(A_{1T}^{(n)} / A_{2T}^{(n)})] = Var_0^Q\left(\ln A_{1T}^{(n)}\right) + Var_0^Q\left(\ln A_{2T}^{(n)}\right) - 2Cov_0^Q\left(\ln A_{1T}^{(n)}, \ln A_{2T}^{(n)}\right)$. We calculate $A_{10}^{(n)}$, $A_{20}^{(n)}$, $Var_0^Q\left(\ln A_{1T}^{(n)}\right)$, $Var_0^Q\left(\ln A_{2T}^{(n)}\right)$ and $Cov_0^Q\left(\ln A_{1T}^{(n)}, \ln A_{2T}^{(n)}\right)$ in the following steps.

- Step 2: Find $A_{10}^{(n)}$, the time-0 price of $A_{1T}^{(n)}$.

Since $A_{10}^{(n)}$ is not necessary observed in the market, we have to calculate the value of $A_{10}^{(n)}$. By the martingale pricing approach, $A_{10}^{(n)}$ is equivalent to $\mathrm{E}_0^Q[A_{1T}^{(n)} B_T^{-1}]$, where $E_0^Q[\cdot]$ is the expectation function under the risk-neutral measure *Q* conditional on the market information up to time 0, and B_T is the cash account at time *T*. With risk-free rate r_u, B_T is defined as $B_T = B_0 \exp\left[\int_0^T r_u du\right]$ and B_0 =1. Then,

$$A_{10}^{(n)} = \mathrm{E}_0^Q[A_{1T}^{(n)} B_T^{-1}]$$

$$= \mathrm{E}_0^Q\left[\prod_{t=n}^{T}\exp(\overline{R}(t,t+\delta))\cdot\exp(-\int_0^T r_u du)\right]$$

$$= \mathrm{E}_0^Q\left\{\exp\left[\sum_{t=n}^{T}\left(g_1(t)+\sigma_f W_t\right)-\int_0^T f(0,u)du-\frac{\sigma_f^2 T^3}{6}-\sigma_f\int_0^T W_u du\right]\right\}$$

$$\left(\text{Let } \; g_2(n,T)=\left(\sum_{t=n}^{T} g_1(t)\right)-\int_0^T f(0,u)du-\frac{\sigma_f^2 T^3}{6}\right)$$

$$= \mathrm{E}_0^Q\left\{\exp\left[g_2(n,T)+\sum_{t=n}^{T}\sigma_f W_t-\sigma_f\int_0^T W_u du\right]\right\}$$

$$= \exp\left\{g_2(n,T)+\frac{1}{2}Var_0^Q\left(\sum_{t=n}^{T}\sigma_f W_t-\sigma_f\int_0^T W_u du\right)\right\}$$

$$= \exp\left\{g_2(n,T)+\frac{\sigma_f^2}{6}\cdot\left[T+n(n-1)(n-2)\right]\right\}.$$

The derivation of the last equation can be read in Appendix (A).

- Step 3: Find $A_{20}^{(n)}$, the time-0 price of $A_{2T}^{(n)}$.

Also by the martingale pricing approach, we have $A_{20}^{(n)} = \mathrm{E}_0^Q[A_{2T}^{(n)} B_T^{-1}]$. Then,

$$A_{20}^{(n)} = \mathrm{E}_0^Q[A_{2T}^{(n)} B_T^{-1}] = \mathrm{E}_0^Q[\prod_{t=n}^{T}\exp(R_t)\cdot B_T^{-1}]$$

$$= \mathrm{E}_0^Q\left[\frac{S_T}{S_{n-1}}\cdot B_T^{-1}\right]$$

$$= \mathrm{E}_0^Q\left[\frac{S_0\cdot\exp\left(\int_0^T r_u du-\frac{1}{2}\sigma_S^2 T+\sigma_S Z_T\right)}{S_0\cdot\exp\left(\int_0^{n-1} r_u du-\frac{1}{2}\sigma_S^2(n-1)+\sigma_S Z_{n-1}\right)}\cdot\exp\left(-\int_0^T r_u du\right)\right]$$

$$= \mathrm{E}_0^Q\left\{\exp\left[-\int_0^{n-1} r_u du-\frac{1}{2}\sigma_S^2(T-n+1)+\sigma_S(Z_T-Z_{n-1})\right]\right\}$$

$$= \mathrm{E}_0^Q\left\{\exp\left[-\int_0^{n-1} f(0,u)du - \frac{\sigma_f^2(n-1)^3}{6} - \sigma_f\int_0^{n-1} W_u du\right.\right.$$

$$\left.\left. -\frac{1}{2}\sigma_S^2(T-n+1) + \sigma_S\int_{n-1}^{T} dZ_u\right]\right\}$$

$$\left(\text{Let } \; h(n,T) = -\int_0^{n-1} f(0,u)du - \frac{\sigma_f^2(n-1)^3}{6} - \frac{1}{2}\sigma_S^2(T-n+1)\right)$$

$$= \mathrm{E}_0^Q\left\{\exp\left[h(n,T) - \sigma_f\int_0^{n-1}(n-1-u)dW_u + \sigma_S\int_{n-1}^{T} dZ_u\right]\right\}$$

$$= \exp\left\{h(n,T) + \frac{1}{2}\cdot Var_0^Q\left[-\sigma_f\int_0^{n-1}(n-1-u)dW_u + \sigma_S\int_{n-1}^{T} dZ_u\right]\right\}$$

$$= \exp\left\{h(n,T) + \frac{1}{2}\cdot Var_0^Q\left[-\sigma_f\int_0^{n-1}(n-1-u)dW_u\right] + \frac{1}{2}\cdot Var_0^Q\left[\sigma_S\int_{n-1}^{T} dZ_u\right]\right\}$$

$$= \exp\left[h(n,T) + \frac{1}{2}\cdot\sigma_f^2\int_0^{n-1}(n-1-u)^2 du + \frac{1}{2}\cdot\sigma_S^2\int_{n-1}^{T} du\right]$$

$$= \exp\left[h(n,T) + \frac{1}{6}\cdot\sigma_f^2(n-1)^3 + \frac{1}{2}\cdot\sigma_S^2(T-n+1)\right]$$

$$= \exp\left[-\int_0^{n-1} f(0,u)du\right].$$

- Step 4: Find $Var_0^Q\left(\ln A_{1T}^{(n)}\right)$.

$$Var_0^Q\left(\ln A_{1T}^{(n)}\right)$$

$$= Var_0^Q\left[\ln\prod_{t=n}^{T}\exp(\bar{R}(t,t+\delta))\right]$$

$$= Var_0^Q\left[\sum_{t=n}^{T}\bar{R}(t,t+\delta)\right]$$

$$= Var_0^Q\left[\sum_{t=n}^{T}\sigma_f W_t\right]$$

$$= \sigma_f^2\cdot\left[n(T-n+1)^2 + \frac{(T-n)(T-n+1)(2T-2n+1)}{6}\right].$$

- Step 5: Find $Var_0^Q\left(\ln A_{2T}^{(n)}\right)$.

$$Var_0^Q\left(\ln A_{2T}^{(n)}\right)$$

$$= Var_0^Q[\ln(S_T / S_{n-1})]$$

$$= Var_0^Q\left[\ln\left(\frac{S_0 \cdot \exp(\int_0^T r_u du - \frac{1}{2}\sigma_S^2 T + \sigma_S Z_T)}{S_0 \cdot \exp(\int_0^{n-1} r_u du - \frac{1}{2}\sigma_S^2 (n-1) + \sigma_S Z_{n-1})}\right)\right]$$

$$= Var_0^Q\left[\sigma_f \int_{n-1}^T W_u du + \sigma_S \int_{n-1}^T dZ_u\right]$$

$$= \sigma_f^2 \cdot Var_0^Q\left[\int_0^T (T-u)dW_u - \int_0^{n-1}(n-1-u)dW_u\right] + \sigma_S^2 \cdot Var_0^Q\left[\int_{n-1}^T dZ_u\right]$$

$$+ 2\sigma_f \cdot \sigma_S \cdot Cov_0^Q\left[\int_0^T (T-u)dW_u - \int_0^{n-1}(n-1-u)dW_u, \rho\int_{n-1}^T dW_u\right]$$

$$= \sigma_f^2 \cdot \left\{Var_0^Q\left[\int_0^T (T-u)dW_u\right] + Var_0^Q\left[\int_0^{n-1}(n-1-u)dW_u\right]\right.$$

$$\left. - 2Cov_0^Q\left[\int_0^T (T-u)dW_u, \int_0^{n-1}(n-1-u)dW_u\right]\right\}$$

$$+ \sigma_S^2 \cdot Var_0^Q\left[\int_{n-1}^T dZ_u\right] + 2\rho\sigma_f\sigma_S \cdot Cov_0^Q\left[\int_{n-1}^T (T-u)dW_u, \int_{n-1}^T dW_u\right]$$

$$= \sigma_f^2 \cdot \left\{\int_0^T (T-u)^2 du + \int_0^{n-1}(n-1-u)^2 du - 2\int_0^{n-1}(T-u)(n-1-u)du\right\}$$

$$+ \sigma_S^2 \cdot \int_{n-1}^T du + 2\rho\sigma_f\sigma_S \cdot \int_{n-1}^T (T-u)du$$

$$= \sigma_f^2 \cdot \left[\frac{T^3}{3} + \frac{(n-1)^3}{3} - \frac{(n-1)^2(3T-n+1)}{3}\right] + \sigma_S^2 \cdot (T-n+1)$$

$$+ \rho\sigma_f\sigma_S (T-n+1)^2 .$$

- Step 6: Find $Cov_0^Q\left(\ln A_{1T}^{(n)}, \ln A_{2T}^{(n)}\right)$.

$$
\begin{aligned}
& Cov_0^Q\left(\ln A_{1T}^{(n)}, \ln A_{2T}^{(n)}\right) \\
&= Cov_0^Q\left[\ln \prod_{t=n}^{T} \exp(\bar{R}(t, t+\delta)), \ln(S_T / S_{n-1})\right] \\
&= Cov_0^Q\left[\sum_{t=n}^{T} \sigma_f W_t,\ \sigma_f \int_{n-1}^{T} W_u du + \sigma_S \int_{n-1}^{T} dZ_u\right] \\
&= Cov_0^Q\left[\sum_{t=n}^{T} \sigma_f W_t,\ \sigma_f \int_{n-1}^{T} W_u du + \rho\sigma_S \int_{n-1}^{T} dW_u\right] \\
&= \sigma_f^2 \cdot Cov_0^Q\left[\sum_{t=n}^{T} W_t, \int_{n-1}^{T} W_u du\right] + \rho\sigma_f\sigma_S \cdot Cov_0^Q\left[\sum_{t=n}^{T} W_t, \int_{n-1}^{T} dW_u\right].
\end{aligned}
$$

In the last equation,

$$
\begin{aligned}
& Cov_0^Q\left[\sum_{t=n}^{T} W_t, \int_{n-1}^{T} W_u du\right] \\
&= Cov_0^Q\left[\sum_{t=n}^{T} \int_0^t dW_u, \int_0^T (T-u) dW_u - \int_0^{n-1} (n-1-u) dW_u\right] \\
&= \sum_{t=n}^{T} \left\{ Cov_0^Q\left[\int_0^t dW_u, \int_0^T (T-u) dW_u\right] - Cov_0^Q\left[\int_0^t dW_u, \int_0^{n-1} (n-1-u) dW_u\right] \right\} \\
&= \sum_{t=n}^{T} \left[\int_0^t (T-u) du - \int_0^{n-1} (n-1-u) du\right] \\
&= \sum_{t=n}^{T} \left[Tt - \frac{t^2}{2} - \frac{(n-1)^2}{2}\right] \\
&= \frac{(T-n+1)[T(T+n) - (n-1)^2]}{2} - \frac{T(T+1)(2T+1) - n(n-1)(2n-1)}{12},
\end{aligned}
$$

and

$$
\begin{aligned}
& Cov_0^Q\left[\sum_{t=n}^{T} W_t, \int_{n-1}^{T} dW_u\right] = Cov_0^Q\left[\sum_{t=n}^{T} \int_0^t dW_u, \int_{n-1}^{T} dW_u\right] \\
&= \sum_{t=n}^{T} Cov_0^Q\left[\int_0^t dW_u, \int_{n-1}^{T} dW_u\right] \\
&= \sum_{t=n}^{T} Cov_0^Q\left[\int_0^t dW_u, \int_{n-1}^{t} dW_u\right]
\end{aligned}
$$

$$= \sum_{t=n}^{T} \int_{n-1}^{t} du = \sum_{t=n}^{T} (t-n+1) = \frac{(T-n+1)(T-n+2)}{2}.$$

Then the expression of $Cov_0^Q\left(\ln A_{1T}^{(n)}, \ln A_{2T}^{(n)}\right)$ is written as

$$Cov_0^Q\left(\ln A_{1T}^{(n)}, \ln A_{2T}^{(n)}\right)$$

$$= \sigma_f^2 \cdot \left\{ \frac{(T-n+1)[T(T+n)-(n-1)^2]}{2} - \frac{T(T+1)(2T+1)-n(n-1)(2n-1)}{12} \right\}$$

$$+ \rho \sigma_f \sigma_S \cdot \frac{(T-n+1)(T-n+2)}{2}. \quad \square$$

3.2 Type-II Guarantee (Multi-Period Guarantee)

This section derives the value of the type-II guarantee. Define $V_T(II)$ as the terminal market value of the type-II guarantee, which is

$$V_T(II) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \left\{ \prod_{t=n}^{T} Max\left[\exp(\bar{R}(t,t+\delta)), \exp(R_t)\right] - \prod_{t=n}^{T} \exp(R_t) \right\}. \quad (10)$$

Also define $\pi_T^{(n)}(II)$ as the time-T value of the type-II guarantee for one dollar contributed in the n-th year, $n \le T$. $\pi_T^{(n)}(II)$ is expressed as:

$$\pi_T^{(n)}(II) = \prod_{t=n}^{T} Max\left[\exp(\bar{R}(t,t+\delta)), \exp(R_t)\right] - \prod_{t=n}^{T} \exp(R_t).$$

Let $A_{3T}^{(n)} = \prod_{t=n}^{T} Max\left[\exp(\bar{R}(t,t+\delta)), \exp(R_t)\right]$ and $A_{2T}^{(n)} = \prod_{t=n}^{T} \exp(R_t)$, then we have $\pi_T^{(n)}(II) = A_{3T}^{(n)} - A_{2T}^{(n)}$.

Proposition 2: Given that the time-*T* value of the type-II guarantee for one dollar contributed in the *n*-th year is $\pi_T^{(n)}(II)$. The time-0 market value of $\pi_T^{(n)}(II)$ is $\pi_0^{(n)}(II) = A_{30}^{(n)} - A_{20}^{(n)}$, where

$$A_{30}^{(n)} = \left\{ \prod_{t=n}^{T} \left[\exp\left(g_3(t) + \frac{\sigma_f^2}{6} \right) \cdot \Phi(d_{3t}) - \Phi(d_{4t}) + 1 \right] \right\} \cdot \exp\left[- \int_0^{n-1} f(0,u)du \right],$$

$$A_{20}^{(n)} = \exp\left[- \int_0^{n-1} f(0,u)du \right],$$

$$d_{3t} = \frac{g_3(t) + \frac{\sigma_f^2}{6} + \frac{\tilde{\sigma}_t^2 \cdot t}{2}}{\tilde{\sigma}_t \cdot \sqrt{t}},$$

$$d_{4t} = d_{3t} - \tilde{\sigma}_t \cdot \sqrt{t},$$

$$g_1(t) = \frac{-\ln P(0,t+\delta) + \ln P(0,t)}{\delta} + \frac{\sigma_f^2}{2} t(t+\delta),$$

$$g_3(t) = -\int_{t-1}^{t} f(0,u)du - \frac{\sigma_f^2}{6}(3t^2 - 3t + 1) + g_1(t),$$

$$\tilde{\sigma}_t^2 \cdot t = \frac{\sigma_f^2}{3} + \sigma_S^2 - \rho\sigma_f\sigma_S, \text{ and}$$

$\Phi(\cdot)$ denotes the cumulative probability function for a variable followed by a standardized normal distribution.

After considering the contributions in each year and total accumulations, the value of the type-II guarantee is $V_0(II) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \pi_0^{(n)}(II)$.

Proof:

The time-0 value of $A_{2T}^{(n)}$, or $A_{20}^{(n)}$, has been derived. We need only to calculate the time-0 value of $A_{3T}^{(n)}$, or $A_{30}^{(n)}$. By the martingale pricing approach,

$$A_{30}^{(n)} = \mathrm{E}_0^Q[A_{3T}^{(n)} B_T^{-1}]$$

$$= \mathrm{E}_0^Q\left\{\prod_{t=n}^{T} Max\left[\exp(\bar{R}(t,t+\delta)), \exp(R_t)\right]\cdot \exp(-\int_0^T r_u du)\right\}$$

$$= \mathrm{E}_0^Q\left\{\prod_{t=n}^{T} Max\left[\exp(\bar{R}(t,t+\delta)), S_t / S_{t-1}\right]\cdot \exp(-\int_0^T r_u du)\right\}$$

$$= \mathrm{E}_0^Q\left\{\prod_{t=n}^{T} Max\left[\exp\left(g_1(t)+\sigma_f W_t\right), \exp\left(\int_{t-1}^{t} r_u du - \frac{1}{2}\sigma_S^2 + \sigma_S \int_{t-1}^{t} dZ_u\right)\right]\right.$$

$$\left.\cdot \prod_{t=n}^{T} \exp(-\int_{t-1}^{t} r_u du)\cdot \exp(-\int_0^{n-1} r_u du)\right\}$$

$$= \mathrm{E}_0^Q\left\{\prod_{t=n}^{T} Max\left[\exp\left(-\int_{t-1}^{t} r_u du + g_1(t)+\sigma_f W_t\right), \exp\left(-\frac{1}{2}\sigma_S^2 + \sigma_S \int_{t-1}^{t} dZ_u\right)\right]\right.$$

$$\left.\cdot \exp(-\int_0^{n-1} r_u du)\right\}.$$

According to Appendix (B), the terms $-\int_{t-1}^{t} r_u du + g_1(t) + \sigma_f W_t$ can be simplified to $g_3(t) + \sigma_f \int_{t-1}^{t} (u - t + 1) dW_u$, where $g_3(t) = -\int_{t-1}^{t} f(0,u) du - \frac{\sigma_f^2}{6}(3t^2 - 3t + 1) + g_1(t)$.

Therefore, $A_{30}^{(n)}$ can be further reduced so that

$$A_{30}^{(n)} = \mathrm{E}_0^Q\left\{\prod_{t=n}^{T} Max\left[\exp\left(g_3(t)+\sigma_f \int_{t-1}^{t} (u-t+1) dW_u\right), \exp\left(-\frac{1}{2}\sigma_S^2 + \sigma_S \int_{t-1}^{t} dZ_u\right)\right]\right.$$

$$\left.\cdot \exp(-\int_0^{n-1} r_u du)\right\}$$

$$= \prod_{t=n}^{T} \mathrm{E}_0^Q\left\{Max\left[\exp\left(g_3(t)+\sigma_f \int_{t-1}^{t} (u-t+1) dW_u\right), \exp\left(-\frac{1}{2}\sigma_S^2 + \sigma_S \int_{t-1}^{t} dZ_u\right)\right]\right\}$$

$$\cdot \mathrm{E}_0^Q\left[\exp(-\int_0^{n-1} r_u du)\right].$$

The last equation comes from the property of independent increment of Brownian motion. It is simple to prove that $\mathrm{E}_0^Q\left[\exp(-\int_0^{n-1} r_u du)\right]$ is equal to

$\exp\left[-\int_0^{n-1} f(0,u)du\right]$. To calculate the first expectation value in the last equation, let $X_{1t} = \exp\left(g_3(t) + \sigma_f \int_{t-1}^{t} (u-t+1)dW_u\right)$ and $X_{2t} = \exp\left(-\frac{1}{2}\sigma_S^2 + \sigma_S \int_{t-1}^{t} dZ_u\right)$, then

$$\mathrm{E}_0^Q\left\{Max\left[\exp\left(g_3(t) + \sigma_f \int_{t-1}^{t} (u-t+1)dW_u\right), \exp\left(-\frac{1}{2}\sigma_S^2 + \sigma_S \int_{t-1}^{t} dZ_u\right)\right]\right\}$$

$$= \mathrm{E}_0^Q[Max(X_{1t}, X_{2t})]$$

$$= \mathrm{E}_0^Q[Max(X_{1t}B_t - X_{2t}B_t, 0)\cdot B_t^{-1}] + \mathrm{E}_0^Q(X_{2t}).$$

Both $X_{1t}B_t$ and $X_{2t}B_t$ are lognormally distributed. Consider one contingent claim whose payoff at time *t* is $Max(X_{1t}B_t - X_{2t}B_t, 0)$, which can be viewed as the payoff of an exchange option exercised at time *t*. Then $\mathrm{E}_0^Q[Max(X_{1t}B_t - X_{2t}B_t, 0)\cdot B_t^{-1}]$ is the time-0 value of this contingent claim under the risk-neutral measure *Q*. Given that there exists no arbitrage opportunity, the time-0 value of $Max(X_{1t}B_t - X_{2t}B_t, 0)$ should be unique no matter what asset is applied to be the numeraire in valuation. Instead of directly computing $\mathrm{E}_0^Q[Max(X_{1t}B_t - X_{2t}B_t, 0)\cdot B_t^{-1}]$, we again follow Margrabe's (1978) pricing method so that the time-0 value of this claim is $\mathrm{E}_0^Q(X_{1t})\Phi(d_{3t}) - \mathrm{E}_0^Q(X_{2t})\Phi(d_{4t})$, where $\mathrm{E}_0^Q(X_{1t}) = \mathrm{E}_0^Q(X_{1t}B_t \cdot B_t^{-1})$ is the time-0 price of $X_{1t}B_t$, $\mathrm{E}_0^Q(X_{2t}) = \mathrm{E}_0^Q(X_{2t}B_t \cdot B_t^{-1})$ is the time-0 price of $X_{2t}B_t$, $d_{3t} = \frac{\ln[\mathrm{E}_0^Q(X_{1t})/\mathrm{E}_0^Q(X_{2t})] + \tilde{\sigma}_t^2 \cdot t/2}{\tilde{\sigma}_t \cdot \sqrt{t}}$, $d_{4t} = d_{3t} - \tilde{\sigma}_t \cdot \sqrt{t}$, and $\tilde{\sigma}_t^2 \cdot t$ is defined as $Var_0^Q\left[\ln\left(\frac{X_{1t}B_t}{X_{2t}B_t}\right)\right] = Var_0^Q\left[\ln\left(\frac{X_{1t}}{X_{2t}}\right)\right]$.

It is easy to show that $\mathrm{E}_0^Q(X_{1t}) = \exp\left(g_3(t) + \frac{\sigma_f^2}{6}\right)$ and $\mathrm{E}_0^Q(X_{2t}) = 1$. From Appendix (C), we know that $\tilde{\sigma}_t^2 \cdot t = \frac{\sigma_f^2}{3} + \sigma_S^2 - \rho\sigma_f\sigma_S > 0$. Consequently, we have this proposition. □

4. Numerical Results and Sensitivity Analysis

This section provides the numerical results for $V_0(I)$, $V_0(II)$, and shows how these values vary with some key parameter estimates, while keeping all the other parameter estimates unchanged. To justify the values derived by Proposition 1 and Proposition 2, we also list the values obtained by Monte Carlo simulation. The procedure of simulation is described as follows.

Step 1: Given the parameters estimates, simulate weekly prices of the investment portfolio by equation (3). We consider the discretization error here. The details about this are reported in Appendix (D).

Step 2: At the last day of each year, calculate the actual rate of return by equation (4) and the guaranteed rate by equation (8). Record the actual rate of return and the guaranteed rate.

Step 3: At time *T*, calculate $V_T(I)$ by equation (9) and $V_T(II)$ by equation (10). Then the present value of $V_T(I)$ for this path is computed as

$\exp\left(-\int_0^T r_u du\right)\cdot V_T(I) \equiv \hat{V}_0(I)$, and the present value of $V_T(II)$ for this path is computed as $\exp\left(-\int_0^T r_u du\right)\cdot V_T(II) \equiv \hat{V}_0(II)$. From this step we know that the price derived by simulation is under the risk-neutral measure *Q*. We consider the discretization error when deriving $\exp\left(-\int_0^T r_u du\right)$. The details about this are described in Appendix (D).

Step 4: Repeat steps 1-3 for 50,000 paths. Calculate the arithmetic average of $\hat{V}_0(I)$ (or $\hat{V}_0(II)$) over the 50,000 paths and then derive the fair value of the type-I (or type-II) guarantee. The results are reported in column (B) of Table 1.

The basic parameter estimates are as follows:[4]

$T = 30$, $Y_0 = 1$, $k = 0.03$, $c = 0.06$,

$S_0 = 1$, $\delta = 2$, $\sigma_S = 0.1$, $\sigma_f = 0.01$, $\rho = -0.2$.

Furthermore, we also assume that the initial term structure is flat and fixed at 3%, so $P(0,t) = e^{-0.03t}$. Under these settings, $V_0(I)$ is 0.32106 and $V_0(II)$ is 1.81088. Numerical results are listed in Table 1 and Figures 2-3. We have some findings.

(i) Table 1 shows that the difference between the value derived by the explicit formula and the value derived by simulation is tiny. This demonstrates that the formulae provide one accurate way to value the interest rate guarantees. The term

[4] From Proposition 1 and Proposition 2, we know that the values of both guarantees are irrelevant to the initial price of the investment portfolio, S_0. Therefore, we do not need to set up S_0 when valuing both guarantees by these two propositions. However, in order to produce the weekly price of the investment portfolio when carrying out simulation, the setting of S_0 is required.

of the pension plan is usually very long. Deriving the value by simulation is very time-consuming.[5] It is an efficient way to price these guarantees by our formulae.

(ii) Figures 2 and 3 reveal that all values decrease as the correlation estimate becomes higher, except for the cases that $\sigma_f = 0$ or $\sigma_S = 0$. This is due to the fact that $(\hat{\sigma}^{(n)})^2$ in Proposition 1 and $\tilde{\sigma}_t^2$ in Proposition 2 are both decreasing functions of ρ when $\sigma_f > 0$ and $\sigma_S > 0$. If $\sigma_f = 0$ or $\sigma_S = 0$, $(\hat{\sigma}^{(n)})^2$ and $\tilde{\sigma}_t^2$ are irrelevant to ρ therefore the values do not change with ρ.

(iii) High forward rate volatility (σ_f) or volatility of the investment portfolio price (σ_S) has two effects. As σ_f or σ_S increases, the probability that the actual rate of return on the investment portfolio significantly falls below the guaranteed rate of return increases as well. However, the first derivative of $(\hat{\sigma}^{(n)})^2$ and $\tilde{\sigma}_t^2$ with respect to σ_f (or σ_S) may be negative, especially in the case that ρ is close to +1 and σ_f (or σ_S) is very small. These effects work together so that higher σ_f (or σ_S) does not necessary cause higher $V_0(I)$ and $V_0(II)$, depending on the estimates of ρ and σ_f (or σ_S), as shown in Figures 2 and 3.

(iv) The type-I guarantee is not more expensive than the type-II guarantee. For the type-I guarantee, the effects of large guaranteed rates in some years can be

[5] For example, it takes almost an hour to derive the values in Table 1 by simulation.

mitigated by smaller guaranteed rates appeared in other years. For the type-II guarantee, such mitigation does not work and being multiplied by subsequent maximum terms enhances the effects of large guaranteed rates. Thus, the value of the type-II guarantee is large and no less than that of the type-I guarantee.

Table 1. Fair Values of the Type-I and Type-II Guarantees: $\sigma_f(t,u) = \sigma_f > 0$

T	$V_0(I)$				
	Formula (A)	Simulation (B)	Absolute Diff. (C)=(A)-(B)	Relative Diff. (D)	S.E. (E)
20	0.16326	0.16319	0.00008	0.00048	0.00072
25	0.23529	0.23371	0.00158	0.00673	0.00101
30	**0.32106**	**0.31992**	**0.00114**	**0.00356**	**0.00136**
35	0.42220	0.41999	0.00220	0.00521	0.00182
40	0.54075	0.54076	-0.00001	-0.00001	0.00243
T	$V_0(II)$				
	Formula (A)	Simulation (B)	Absolute Diff. (C)=(A)-(B)	Relative Diff. (D)	S.E. (E)
20	0.68575	0.68610	-0.00035	-0.00051	0.00085
25	1.15691	1.15586	0.00105	0.00091	0.00143
30	**1.81088**	**1.80776**	**0.00312**	**0.00172**	**0.00242**
35	2.69403	2.69311	0.00092	0.00034	0.00404
40	3.86506	3.86868	-0.00363	-0.00094	0.00667

Notes: Column (A) lists the values derived by the formulae in the propositions. Column (B) lists the values derived by simulation based on 50,000 paths. Column (C) describes the absolute difference between (A) and (B), and column (D) describes the relative difference, which is equal to the value in (C) divided by the value in (A). Column (E) reports the standard error of the simulation estimates.

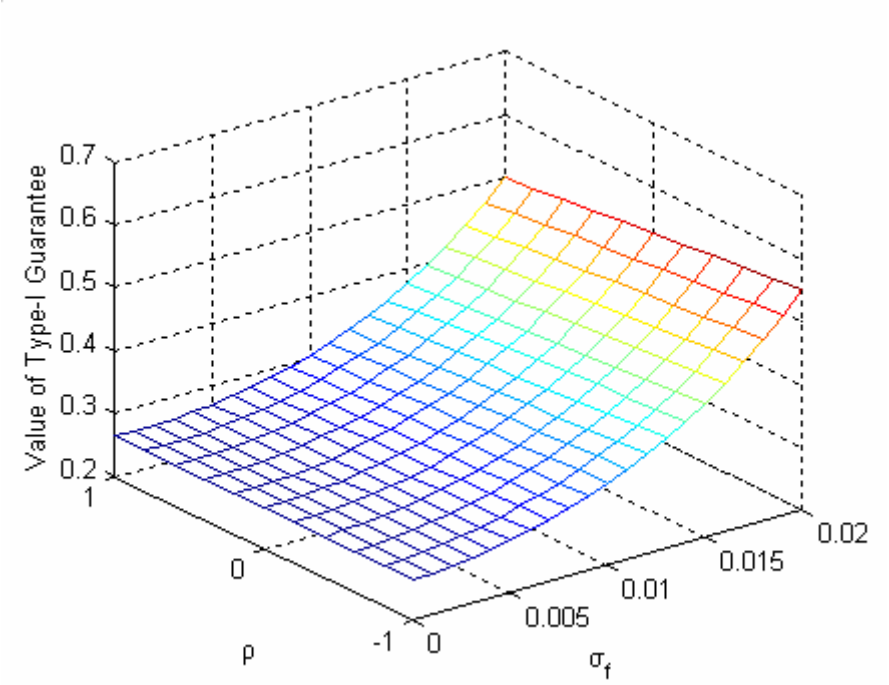


Figure 2-I. Value of Type-I Guarantee, Given Different ρ and σ_f

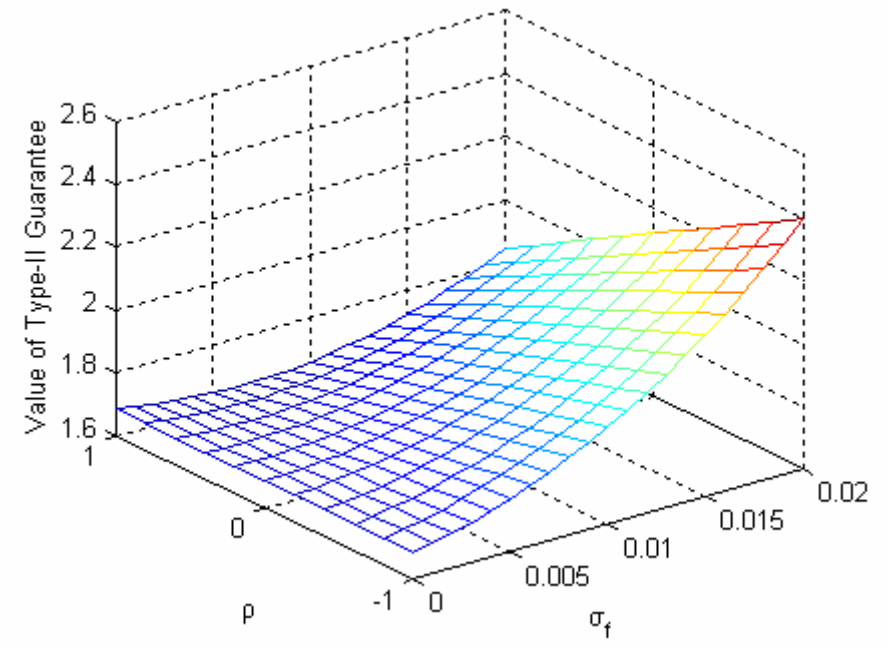


Figure 2-II. Value of Type-II Guarantee, Given Different ρ and σ_f

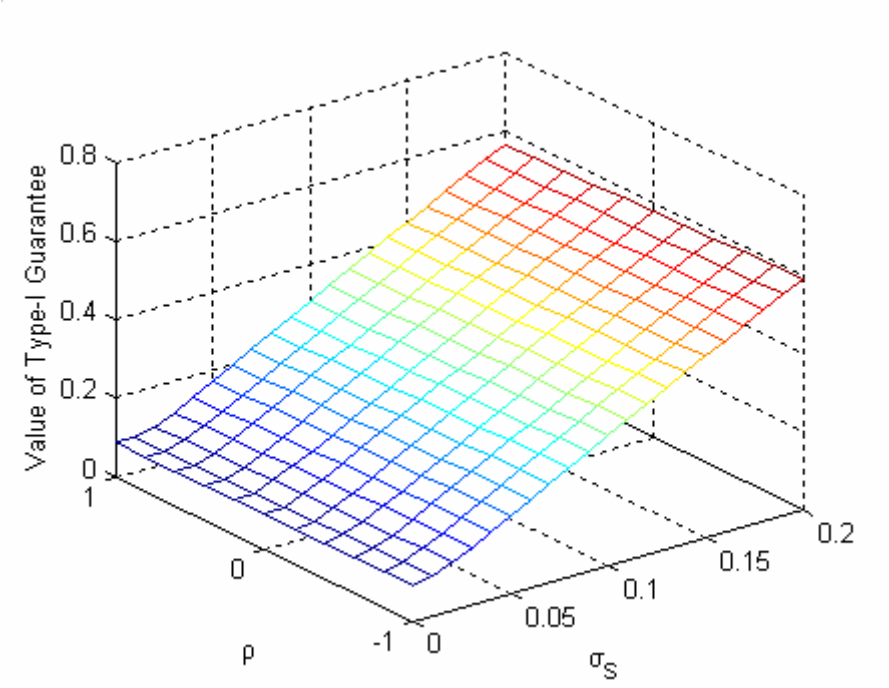


Figure 3-I. Value of Type-I Guarantee, Given Different ρ and σ_S

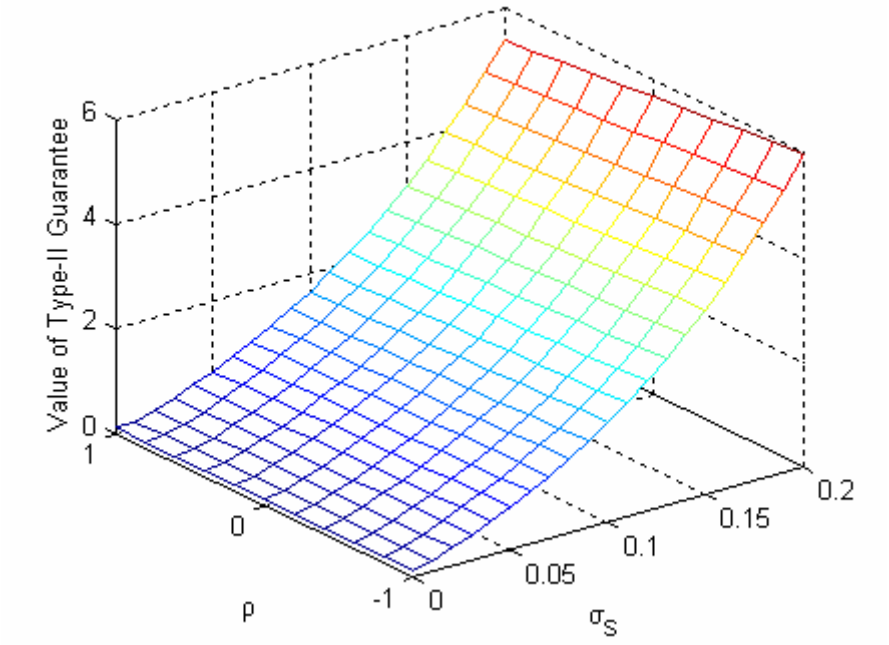


Figure 3-II. Value of Type-II Guarantee, Given Different ρ and σ_S

5. Extension Research under the Setting of Exponentially Decaying Volatility

The preceding two propositions are derived by assuming that the volatility term in the HJM model is constant. Flesaker (1993) uses GMM test to demonstrate that the constant-volatility HJM model is rejected in the daily data for Eurodollar futures options. However, from Amin and Morton (1994) we know that the constant-volatility model is preferable to other one-parameter and two-parameter models since its parameter estimates are more stable. Kuo and Paxson (2006) find that among the simple one-factor HJM models, the EXP-based model produces lower fitting errors than the constant-volatility model, but the latter outperforms the former in detecting mispriced options. Hence, it is better to consider two forms of volatility in our research. This section focuses on pricing the type-I guarantee under the assumption that the volatility term in the HJM model is described by an exponentially decaying structure, i.e., $\sigma_f(t,u) = \sigma \cdot e^{-\lambda(u-t)} > 0$, where $u \geq t$, $\sigma > 0$ and $\lambda > 0$. By definition and simple calculation, we obtain that

$$df(t,u) = \left(\sigma^2 e^{-\lambda(u-t)} \int_t^u e^{-\lambda(x-t)} dx \right) dt + \sigma e^{-\lambda(u-t)} dW_t$$

$$= \left(\frac{\sigma^2}{\lambda} e^{-\lambda(u-t)} \left(1 - e^{-\lambda(u-t)}\right) \right) dt + \sigma e^{-\lambda(u-t)} dW_t ,$$

$$f(t,u) = f(0,u) + \frac{\sigma^2}{\lambda^2} \left[e^{-\lambda u} (e^{\lambda t} - 1) - \frac{e^{-2\lambda u} (e^{2\lambda t} - 1)}{2} \right] + \sigma \int_0^t e^{-\lambda(u-v)} dW_v ,$$

$$r_t = f(0,t) + \frac{\sigma^2}{2\lambda^2}(1-e^{-\lambda t})^2 + \sigma\int_0^t e^{-\lambda(t-v)}dW_v \,,$$

and

$$\overline{R}(t,t+\delta) = g_4(t) + \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\sigma\int_0^t e^{-\lambda(t-v)}dW_v \,,$$

where $g_4(t) = \dfrac{-\ln P(0,t+\delta)+\ln P(0,t)}{\delta}$

$$+\left(\frac{\sigma^2}{\lambda^3\delta}\right)\left[\left(e^{\lambda t}-1\right)\left(e^{-\lambda t}-e^{-\lambda(t+\delta)}\right)-\frac{\left(e^{2\lambda t}-1\right)\left(e^{-2\lambda t}-e^{-2\lambda(t+\delta)}\right)}{4}\right].$$

We can rewrite $\overline{R}(t,t+\delta)$ as

$$\overline{R}(t,t+\delta) = \frac{-\ln P(0,t+\delta)+\ln P(0,t)}{\delta} + \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\left[r_t - f(0,t)\right]$$

$$+\frac{1}{2}\left(\frac{1-e^{-\lambda\delta}}{\lambda}\right)^2\cdot\frac{\sigma^2(1-e^{-2\lambda t})}{2\lambda\delta}. \qquad (11)$$

Proposition 3: Given that time-T value of the type-I guarantee for one dollar contributed in the n-th year is $\pi_T^{(n)}(I)$. Under the setting that $\sigma_f(t,u) = \sigma\cdot e^{-\lambda(u-t)} > 0$, the time-0 market value of $\pi_T^{(n)}(I)$ is $\pi_0^{(n)}(I) = A_{10}^{(n)}\Phi(d_1^{(n)}) - A_{20}^{(n)}\Phi(d_2^{(n)})$. In this expression,

$$A_{10}^{(n)} = \exp\left\{\sum_{t=n}^{T} g_4(t) - \int_0^T f(0,u)du - \frac{\sigma^2}{2\lambda^2}\left[T - \frac{2}{\lambda}(1-e^{-\lambda T}) + \frac{1}{2\lambda}(1-e^{-2\lambda T})\right]\right\}$$

$$\cdot\exp\left\{\frac{1}{2}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)^2 Var_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right)\right] + \frac{1}{2}Var_0^Q\left[\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]\right\}$$

$$\cdot\exp\left\{(-1)\cdot\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot Cov_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right), \sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]\right\},$$

$$A_{20}^{(n)} = \exp\left[-\int_0^{n-1} f(0,u)du\right],$$

$$d_1^{(n)} = \frac{\ln(A_{10}^{(n)} / A_{20}^{(n)}) + (\hat{\sigma}^{(n)})^2 T/2}{(\hat{\sigma}^{(n)})\sqrt{T}},\ \ d_2^{(n)} = d_1^{(n)} - (\hat{\sigma}^{(n)})\sqrt{T}\ ,$$

$$(\hat{\sigma}^{(n)})^2 T = Var_0^Q\left(\ln A_{1T}^{(n)}\right) + Var_0^Q\left(\ln A_{2T}^{(n)}\right) - 2Cov_0^Q\left(\ln A_{1T}^{(n)}, \ln A_{2T}^{(n)}\right),$$

where

(i) $g_4(t) = \dfrac{-\ln P(0,t+\delta) + \ln P(0,t)}{\delta}$

$$+\left(\frac{\sigma^2}{\lambda^3\delta}\right)\left[\left(e^{\lambda t}-1\right)\left(e^{-\lambda t}-e^{-\lambda(t+\delta)}\right) - \frac{\left(e^{2\lambda t}-1\right)\left(e^{-2\lambda t}-e^{-2\lambda(t+\delta)}\right)}{4}\right]$$

(ii) $Var_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right)\right] = \sigma^2\cdot\sum_{i=n}^{T}\sum_{j=n}^{T}\frac{1}{2\lambda}\cdot e^{-\lambda(i+j)}\left(e^{2\lambda\min(i,j)}-1\right)$

(iii) $Var_0^Q\left[\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right] = \dfrac{\sigma^2}{\lambda^2}\left[T - \frac{2}{\lambda}(1-e^{-\lambda T}) + \frac{1}{2\lambda}(1-e^{-2\lambda T})\right]$

(iv) $Cov_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right), \sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]$

$$= \frac{\sigma^2}{\lambda^2}\left\{(T-n+1) - \left(\frac{e^{-\lambda(T+1)}-e^{-\lambda n}}{e^{-\lambda}-1}\right) - \left(\frac{e^{\lambda}-e^{-\lambda(T-n)}}{2(e^{\lambda}-1)}\right) + \left(\frac{e^{-\lambda(2T+1)}-e^{-\lambda(T+n)}}{2(e^{-\lambda}-1)}\right)\right\}$$

(v) $Var_0^Q\left(\ln A_{1T}^{(n)}\right) = \dfrac{\sigma^2}{2\lambda}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)^2\sum_{i=n}^{T}\sum_{j=n}^{T}\left[e^{-\lambda(i+j)}\left(e^{2\lambda\min(i,j)}-1\right)\right]$

(vi) $Var_0^Q\left(\ln A_{2T}^{(n)}\right) = \dfrac{\sigma^2}{2\lambda^3}\left(e^{-\lambda T}-e^{-\lambda(n-1)}\right)^2\left(e^{2\lambda(n-1)}-1\right)$

$$+\frac{\sigma^2}{\lambda^2}\left[(T-n+1) - \left(\frac{2(1-e^{-\lambda(T-n+1)})}{\lambda}\right) + \left(\frac{1-e^{-2\lambda(T-n+1)}}{2\lambda}\right)\right]$$

$$+\sigma_S^2(T-n+1) + \frac{2\rho\sigma\sigma_S}{\lambda}\cdot\left[(T-n+1) - \left(\frac{1-e^{-\lambda(T-n+1)}}{\lambda}\right)\right]$$

(vii) $Cov_0^Q\left(\ln A_{1T}^{(n)}, \ln A_{2T}^{(n)}\right)$

$$= \sigma\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\sum_{t=n}^{T}\left[\left(\frac{1-e^{-\lambda(t-n+1)}}{\lambda}\right)\left(\frac{\sigma}{\lambda}+\rho\sigma_S\right)\right.$$

$$\left.+\frac{\sigma}{2\lambda^2}\left(e^{-\lambda(T+t)}+e^{-\lambda(t-n+1)}-e^{-\lambda(t+n-1)}-e^{-\lambda(T-t)}\right)\right]$$

(viii) $\Phi(\cdot)$ denotes the cumulative probability function for a variable followed by a standardized normal distribution.

After considering the contributions in each year and total accumulations, the value of the type-I guarantee is $V_0(I) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \pi_0^{(n)}(I)$.

Proof:

We segregate the complete proof into several steps.

- Step 1: The same as the step 1 in the proof of Proposition 1.
- Step 2: Find $A_{10}^{(n)}$, the time-0 price of $A_{1T}^{(n)}$.

$$A_{10}^{(n)} = \mathrm{E}_0^Q[A_{1T}^{(n)} B_T^{-1}]$$

$$= \mathrm{E}_0^Q\left[\prod_{t=n}^{T}\exp(\overline{R}(t,t+\delta))\cdot\exp(-\int_0^T r_u du)\right]$$

$$= \mathrm{E}_0^Q\left\{\exp\left[\sum_{t=n}^{T}\overline{R}(t,t+\delta)\right]\cdot\exp(-\int_0^T r_u du)\right\}$$

$$= \mathrm{E}_0^Q\left\{\exp\left[\sum_{t=n}^{T}\left(g_4(t)+\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right)\right]\right.$$

$$\left.\cdot\exp\left[-\int_0^T\left(f(0,u)+\frac{\sigma^2}{2\lambda^2}(1-e^{-\lambda u})^2+\sigma\int_0^u e^{-\lambda(u-v)}dW_v\right)du\right]\right\}$$

$$= \mathrm{E}_0^Q\left\{\exp\left[g_5(n,T) + \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right) - \sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]\right\}$$

$$= \exp\left[g_5(n,T)\right]$$

$$\cdot\exp\left\{\frac{1}{2}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)^2 Var_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right)\right] + \frac{1}{2}Var_0^Q\left[\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]\right\}$$

$$\cdot\exp\left\{(-1)\cdot\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot Cov_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right), \sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]\right\},$$

where

(i) $g_5(n,T) = \sum_{t=n}^{T} g_4(t) - \int_0^T\left(f(0,u) + \frac{\sigma^2}{2\lambda^2}(1-e^{-\lambda u})^2\right)du$

$$= \sum_{t=n}^{T} g_4(t) - \int_0^T f(0,u)du - \frac{\sigma^2}{2\lambda^2}\left[T - \frac{2}{\lambda}(1-e^{-\lambda T}) + \frac{1}{2\lambda}(1-e^{-2\lambda T})\right]$$

(ii) $Var_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right)\right] = \sigma^2\cdot\sum_{i=n}^{T}\sum_{j=n}^{T}\frac{1}{2\lambda}\cdot e^{-\lambda(i+j)}\left(e^{2\lambda\min(i,j)} - 1\right)$

(iii) $Var_0^Q\left[\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right] = \frac{\sigma^2}{\lambda^2}\left[T - \frac{2}{\lambda}(1-e^{-\lambda T}) + \frac{1}{2\lambda}(1-e^{-2\lambda T})\right]$

(iv) $Cov_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right), \sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]$

$$= \frac{\sigma^2}{\lambda^2}\left\{(T-n+1) - \left(\frac{e^{-\lambda(T+1)} - e^{-\lambda n}}{e^{-\lambda} - 1}\right) - \left(\frac{e^{\lambda} - e^{-\lambda(T-n)}}{2(e^{\lambda} - 1)}\right) + \left(\frac{e^{-\lambda(2T+1)} - e^{-\lambda(T+n)}}{2(e^{-\lambda} - 1)}\right)\right\}$$

We describe how to derive the latter three equations in Appendixes (E)-(G).

- Step 3: Find $A_{20}^{(n)}$, the time-0 price of $A_{2T}^{(n)}$.

$$A_{20}^{(n)} = \mathrm{E}_0^Q[A_{2T}^{(n)}B_T^{-1}] = \mathrm{E}_0^Q[\prod_{t=n}^{T}\exp(R_t)\cdot B_T^{-1}]$$

$$= \mathrm{E}_0^Q\left[\frac{S_T}{S_{n-1}}\cdot B_T^{-1}\right]$$

$$= \mathrm{E}_0^Q\left[\frac{S_0\cdot\exp\left(\int_0^T r_u du-\frac{1}{2}\sigma_S^2 T+\sigma_S Z_T\right)}{S_0\cdot\exp\left(\int_0^{n-1} r_u du-\frac{1}{2}\sigma_S^2(n-1)+\sigma_S Z_{n-1}\right)}\cdot\exp\left(-\int_0^T r_u du\right)\right]$$

$$= \mathrm{E}_0^Q\left\{\exp\left[-\int_0^{n-1} r_u du-\frac{1}{2}\sigma_S^2(T-n+1)+\sigma_S(Z_T-Z_{n-1})\right]\right\}$$

$$= \mathrm{E}_0^Q\left\{\exp\left[-\int_0^{n-1}\left[f(0,u)+\frac{\sigma^2}{2\lambda^2}(1-e^{-\lambda u})^2+\sigma\int_0^u e^{-\lambda(u-v)}dW_v\right]du\right.\right.$$

$$\left.\left.-\frac{1}{2}\sigma_S^2(T-n+1)+\sigma_S(Z_T-Z_{n-1})\right]\right\}$$

$$\left(\text{Let } \; h(n,T)=-\int_0^{n-1}\left[f(0,u)+\frac{\sigma^2}{2\lambda^2}(1-e^{-\lambda u})^2\right]du-\frac{\sigma_S^2}{2}(T-n+1)\right)$$

$$= \mathrm{E}_0^Q\left\{\exp\left[h(n,T)-\sigma\int_0^{n-1}\int_0^u e^{-\lambda(u-v)}dW_v du+\sigma_S\int_{n-1}^T dZ_v\right]\right\}$$

$$= \mathrm{E}_0^Q\left\{\exp\left[h(n,T)-\frac{\sigma}{\lambda}\int_0^{n-1}\left(1-e^{-\lambda(n-1-v)}\right)dW_v+\sigma_S\int_{n-1}^T dZ_v\right]\right\}$$

$$= \mathrm{E}_0^Q\left\{\exp\left[h(n,T)-\frac{\sigma}{\lambda}\int_0^{n-1}\left(1-e^{-\lambda(n-1-u)}\right)dW_u+\sigma_S\int_{n-1}^T dZ_u\right]\right\}$$

$$= \exp\left[h(n,T)+\frac{1}{2}Var_0^Q\left[-\frac{\sigma}{\lambda}\int_0^{n-1}\left(1-e^{-\lambda(n-1-u)}\right)dW_u+\sigma_S\int_{n-1}^T dZ_u\right]\right]$$

$$= \exp\left[h(n,T)+\frac{1}{2}Var_0^Q\left[\frac{\sigma}{\lambda}\int_0^{n-1}\left(1-e^{-\lambda(n-1-u)}\right)dW_u\right]+\frac{1}{2}Var_0^Q\left(\sigma_S\int_{n-1}^T dZ_u\right)\right]$$

$$= \exp\left[h(n,T)+\frac{1}{2}\left[\frac{\sigma^2}{\lambda^2}\int_0^{n-1}\left(1-e^{-\lambda(n-1-u)}\right)^2 du\right]+\frac{1}{2}\sigma_S^2\int_{n-1}^T du\right]$$

$$= \exp\left[-\int_0^{n-1} f(0,u)du\right].$$

The last equation holds because $\int_0^{n-1}(1-e^{-\lambda(n-1-u)})^2 du=\int_0^{n-1}(1-e^{-\lambda u})^2 du$.

- Step 4: Find $Var_0^Q\left(\ln A_{1T}^{(n)}\right)$.

$$Var_0^Q\left(\ln A_{1T}^{(n)}\right)=Var_0^Q\left[\sum_{t=n}^{T}\overline{R}(t,t+\delta)\right]$$

$$= Var_0^Q\left[\sum_{t=n}^{T}\left(\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right)\right]$$

$$= \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)^2 Var_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right)\right]$$

$$= \frac{\sigma^2}{2\lambda}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)^2\sum_{i=n}^{T}\sum_{j=n}^{T}\left[e^{-\lambda(i+j)}\left(e^{2\lambda\min(i,j)}-1\right)\right].$$

The last equality holds because of the proof in Appendix (E).

- Step 5: Find $Var_0^Q\left(\ln A_{2T}^{(n)}\right)$.

$$Var_0^Q\left(\ln A_{2T}^{(n)}\right)$$

$$= Var_0^Q\left[\int_{n-1}^{T} r_u du + \sigma_S(Z_T - Z_{n-1})\right]$$

$$= Var_0^Q\left[\sigma\int_{n-1}^{T}\int_0^u e^{-\lambda(u-v)}dW_v du + \sigma_S\int_{n-1}^{T} dZ_v\right]$$

$$= Var_0^Q\left[\sigma\int_0^{n-1}\int_{n-1}^{T} e^{-\lambda(u-v)}du dW_v + \sigma\int_{n-1}^{T}\int_v^T e^{-\lambda(u-v)}du dW_v + \sigma_S\int_{n-1}^{T} dZ_v\right]$$

$$= Var_0^Q\left[-\frac{\sigma}{\lambda}\left(e^{-\lambda T}-e^{-\lambda(n-1)}\right)\int_0^{n-1} e^{\lambda v}dW_v + \frac{\sigma}{\lambda}\int_{n-1}^{T}\left(1-e^{-\lambda(T-v)}\right)dW_v + \sigma_S\int_{n-1}^{T} dZ_v\right]$$

$$= Var_0^Q\left[-\frac{\sigma}{\lambda}\left(e^{-\lambda T}-e^{-\lambda(n-1)}\right)\int_0^{n-1} e^{\lambda v}dW_v\right]$$

$$+ Var_0^Q\left[\frac{\sigma}{\lambda}\int_{n-1}^{T}\left(1-e^{-\lambda(T-v)}\right)dW_v\right]$$

$$+ Var_0^Q\left[\sigma_S\int_{n-1}^{T} dZ_v\right] + 2\cdot Cov_0^Q\left[\frac{\sigma}{\lambda}\int_{n-1}^{T}\left(1-e^{-\lambda(T-v)}\right)dW_v, \sigma_S\int_{n-1}^{T} dZ_v\right]$$

$$= \frac{\sigma^2}{2\lambda^3}\left(e^{-\lambda T}-e^{-\lambda(n-1)}\right)^2\left(e^{2\lambda(n-1)}-1\right)$$

$$+ \frac{\sigma^2}{\lambda^2}\left[(T-n+1) - \left(\frac{2(1-e^{-\lambda(T-n+1)})}{\lambda}\right) + \left(\frac{1-e^{-2\lambda(T-n+1)}}{2\lambda}\right)\right]$$

$$+ \sigma_S^2(T-n+1) + \frac{2\rho\sigma\sigma_S}{\lambda}\cdot\left[(T-n+1) - \left(\frac{1-e^{-\lambda(T-n+1)}}{\lambda}\right)\right].$$

- Step 6: Find $Cov_0^Q\left(\ln A_{1T}^{(n)}, \ln A_{2T}^{(n)}\right)$.

$$Cov_0^Q\left(\ln A_{1T}^{(n)}, \ln A_{2T}^{(n)}\right)$$

$$= Cov_0^Q\left[\sum_{t=n}^{T}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\sigma\int_0^t e^{-\lambda(t-v)}dW_v, \int_{n-1}^{T} r_u\,du + \sigma_S(Z_T - Z_{n-1})\right]$$

$$= Cov_0^Q\left[\sum_{t=n}^{T}\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\sigma\int_0^t e^{-\lambda(t-v)}dW_v,\right.$$

$$\left. -\frac{\sigma}{\lambda}\left(e^{-\lambda T} - e^{-\lambda(n-1)}\right)\int_0^{n-1} e^{\lambda v}dW_v - \frac{\sigma}{\lambda}\int_{n-1}^{T}\left(e^{-\lambda(T-v)} - 1\right)dW_v + \sigma_S\int_{n-1}^{T} dZ_v\right]$$

$$= \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\frac{\sigma^2}{\lambda}\cdot\left(-e^{-\lambda T} + e^{-\lambda(n-1)}\right)\cdot\sum_{t=n}^{T} Cov_0^Q\left[\int_0^t e^{-\lambda(t-v)}dW_v, \int_0^{n-1} e^{\lambda v}dW_v\right]$$

$$+\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\frac{\sigma^2}{\lambda}\cdot\sum_{t=n}^{T} Cov_0^Q\left[\int_0^t e^{-\lambda(t-v)}dW_v, -\int_{n-1}^{T}\left(e^{-\lambda(T-v)} - 1\right)dW_v\right]$$

$$+\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\sigma\sigma_S\cdot\sum_{t=n}^{T} Cov_0^Q\left[\int_0^t e^{-\lambda(t-v)}dW_v, \int_{n-1}^{T} dZ_v\right]$$

$$= \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\frac{\sigma^2}{\lambda}\cdot\left(-e^{-\lambda T} + e^{-\lambda(n-1)}\right)\cdot\sum_{t=n}^{T}\int_0^{n-1} e^{-\lambda(t-2v)}dv$$

$$+\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\frac{\sigma^2}{\lambda}\cdot\sum_{t=n}^{T}\int_{n-1}^{t} e^{-\lambda(t-v)}\left(1-e^{-\lambda(T-v)}\right)dv$$

$$+\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\sigma\sigma_S\cdot\sum_{t=n}^{T}\int_{n-1}^{t} e^{-\lambda(t-v)}\rho\,dv$$

$$= \left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\cdot\frac{\sigma^2}{\lambda}\cdot\left(-e^{-\lambda T} + e^{-\lambda(n-1)}\right)\cdot\sum_{t=n}^{T}\left(\frac{e^{-\lambda(t-2n+2)} - e^{-\lambda t}}{2\lambda}\right)$$

$$+\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\frac{\sigma^2}{\lambda}\sum_{t=n}^{T}\left[\left(\frac{1-e^{-\lambda(t-n+1)}}{\lambda}\right) - \left(\frac{e^{-\lambda(T-t)} - e^{-\lambda(T+t-2n+2)}}{2\lambda}\right)\right]$$

$$+\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)(\rho\sigma\sigma_S)\sum_{t=n}^{T}\left(\frac{1-e^{-\lambda(t-n+1)}}{\lambda}\right)$$

$$= \sigma\left(\frac{1-e^{-\lambda\delta}}{\lambda\delta}\right)\sum_{t=n}^{T}\left[\left(\frac{1-e^{-\lambda(t-n+1)}}{\lambda}\right)\left(\frac{\sigma}{\lambda}+\rho\sigma_S\right)\right.$$

$$\left.+\frac{\sigma}{2\lambda^2}\left(e^{-\lambda(T+t)}+e^{-\lambda(t-n+1)}-e^{-\lambda(t+n-1)}-e^{-\lambda(T-t)}\right)\right]. \quad \square$$

As $\lambda \to 0$, the volatility term in the HJM model, $\sigma_f(t,u)$, goes to a constant σ and the numerical results derived from Proposition 3 should be almost equal to the results from Proposition 1, keeping all the other basic parameter estimates unchanged. To demonstrate this, let σ =0.01, λ =0.00001 and keeping all the other basic parameter estimates unchanged. Under these settings, $V_0(I)$ =0.32111, which is close to the value derived from Proposition 1, 0.32106. Table 2 compares the values derived by Proposition 3 and the values by simulation.[6] The tiny difference indicates that our explicit formula in Proposition 3 for the type-I guarantee is accurate.

[6] The simulation procedure is almost the same as described in Section 4. We adopt expression (11) to simulate the guaranteed rate at each record date.

Table 2. Fair Values of the Type-I Guarantee: $\sigma_f(t,u) = \sigma \cdot e^{-\lambda(u-t)} > 0$

T	$V_0(I)$				
	Formula (A)	Simulation (B)	Absolute Diff. (C)=(A)-(B)	Relative Diff. (D)	S.E. (E)
20	0.14875	0.14856	0.00019	0.00127	0.00068
25	0.20600	0.20454	0.00146	0.00708	0.00092
30	**0.26883**	**0.26813**	**0.00070**	**0.00260**	**0.00119**
35	0.33666	0.33507	0.00159	0.00472	0.00148
40	0.40908	0.40779	0.00129	0.00316	0.00179

Notes: We let σ = 0.01, λ = 0.1, and keep all the other basic parameter estimates (except *T*) unchanged. Column (A) lists the values derived by the formula in Proposition 3. Column (B) lists the values derived by simulation based on 50,000 paths. Column (C) describes the absolute difference between (A) and (B), and column (D) describes the relative difference, which is equal to the value in (C) divided by the value in (A). Column (E) reports the standard error of the simulation estimates.

6. Conclusions

We have derived the closed-form formulae to price the interest rate guarantees embedded in DC pension plans. Different from previous studies, we consider the relative guarantees linked to the δ-year spot rates. The guarantee design mixes several features commonly observed in exotic options, such as exchange options, path-dependent options and forward start options. Given that the volatility term in the HJM model is constant, the formulae for two types of guarantees, maturity guarantee and multi-period guarantee, are respectively described in Propositions 1 and 2. In addition, we obtain the pricing formula for the type-I guarantee in Proposition 3 under

the assumption that the volatility term in the HJM model is exponentially decaying. Numerical results and sensitivity analyses are also presented. We compare the values derived by the formulae with those derived by simulation, and justify that our explicit formulae are accurate and efficient.

This paper could be further extended. First, guarantee providers may care about how to hedge the risks from providing the relative guarantees linked to the δ-year spot rates. It is essential to explore suitable hedging strategies. Second, we assume that the volatility term in the HJM model is constant or exponentially decaying, which may not be the case in the real world. It is worth analyzing how to value interest rate guarantees under different term structure models. Third, we overlook the case that the participant changes the investment portfolio during the accumulation phase. An interesting extension is to incorporate investment strategies into this study and investigate how these strategies affect the terminal values of both guarantees.

References

Amin, K.I. and Morton, A.J. 1994. "Implied Volatility Functions in Arbitrage-free Term Structure Models." *Journal of Financial Economics* 35: 141-180.

Bouaziz, L., Briys, E. and Crouhy, M. 1994. "The Pricing of Forward-Starting Asian Options." *Journal of Banking and Finance* 18: 823-839.

Boyle, P.P. and Hardy, M.R. 1997. "Reserving for Maturity Guarantees; Two Approaches." *Insurance: Mathematics and Economics* 21: 113–127.

Boyle, P.P. and Schwartz, E.S. 1977. "Equilibrium Prices of Guarantees Under Equity-Linked Contracts." *Journal of Risk and Insurance* 44(4): 639-660.

Brennan, J.M. and Schwartz, E.S. 1976. "The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee." *Journal of Financial Economics* 3: 195-213.

Ekern, S. and Persson, S. 1996. "Exotic Unit-Linked Life Insurance Contracts." *The Geneva Papers on Risk and Insurance Theory* 21: 35-63.

Flesaker, B. 1993. "Testing the Heath-Jarrow-Morton/Ho-Lee Model of Interest Rate Contingent Claims Pricing." *Journal of Financial and Quantitative Analysis*, 28 (4): 483-495.

Glasserman, P. 2004. *Monte Carlo Methods in Financial Engineering*. Springer-Verlag, New York, USA.

Grosen, A. and Jørgensen, P.L. 1997. “Valuation of Early Exercisable Interest Rate Guarantees.” *Journal of Risk and Insurance* 64(3): 481-503.

Grosen, A. and Jørgensen, P.L. 2000. “Fair Valuation of Life Insurance Liabilities: The Impact of Interest Rate Guarantees, Surrender Options, and Bonus Policies.” *Insurance: Mathematics and Economics* 26: 37-57.

Hansen, M. and Miltersen, K.R. 2002. “Minimum Rate of Return Guarantees: the Danish Case.” *Scandinavian Actuarial Journal* 4: 280-318.

Harrison, M. and Kreps, D. 1979. “Martingales and Arbitrage in Multiperiod Securities Markets.” *Journal of Economics Theory* 20: 381-408.

Heath, D., Jarrow, R. and Morton, A. 1992. “Bond Pricing and the Term Structure of Interest Rates: A New Methodology for Contingent Claims Valuation.” *Econometrica* 60(1): 77-105.

Ho, T.S. and Lee, S.B. 1986. “Term Structure Movements and Pricing Interest Rate Contingent Claims.” *Journal of Finance* 41: 1011-1029.

Hull, J.C. 2000. *Options, Futures, & Other Derivatives*. Prentice-Hall, New Jersey, USA.

Kuo, I. and Paxson, D.A. 2006. “Multifactor Implied Volatility Functions for HJM Models.” *Journal of Futures Markets* 26: 809-833.

Lindset, S. 2004. “Relative Guarantees.” *The Geneva Papers on Risk and Insurance Theory* 29: 187-209.

Margrabe, W. 1978. “The Value of an Option to Exchange One Asset for Another.” *Journal of Finance* 33: 177-186.

Miltersen, K.R. and Persson, S.A. 1999. “Pricing Rate of Return Guarantee in a Heath-Jarrow-Morton Framework.” *Insurance: Mathematics and Economics* 25: 307-325.

Pennacchi, G. G. 1999. “The Value of Guarantees on Pension Fund Returns.” *Journal of Risk and Insurance* 66(2): 219-237.

Persson, S.A. and Aase, K.K. 1997. “Valuation of the Minimum Guaranteed Return Embedded in Life Insurance Products.” *Journal of Risk and Insurance* 64(4): 599-617.

Schrager, D. F. and Pelsser, A. A. 2004. “Pricing Rate of Return Guarantees in Regular Premium Unit Linked Insurance.” *Insurance: Mathematics and Economics* 35: 369-398.

Tsao, C.Y., Chang, C.C. and Lin, C.G. 2003. “Analytic Approximation Formulae for Pricing Forward-Starting Asian Options.” *Journal of Futures Markets* 23: 487-516.

Appendix

(A) Prove that $Var_0^Q\left(\sum_{t=n}^{T}\sigma_f W_t - \sigma_f\int_0^T W_u du\right) = \frac{\sigma_f^2}{3}\cdot[T + n(n-1)(n-2)]$.

Proof:

First, $Var_0^Q\left(\sum_{t=n}^{T}\sigma_f W_t - \sigma_f\int_0^T W_u du\right)$

$$= Var_0^Q\left[\sum_{t=n}^{T}\sigma_f W_t - \sigma_f\int_0^T (T-u)dW_u\right]$$

$$= \sigma_f^2\cdot Var_0^Q\left(\sum_{t=n}^{T} W_t\right) + \sigma_f^2\cdot Var_0^Q\left[\int_0^T (T-u)dW_u\right]$$

$$- 2\sigma_f^2\cdot Cov_0^Q\left[\sum_{t=n}^{T} W_t, \int_0^T (T-u)dW_u\right]. \qquad \text{(A.1)}$$

The second equation comes from the fact that $\int_0^t W_u du = \int_0^t (t-u)dW_u$, which can be easily shown by Ito's Lemma.

It is easy to derive that $Var_0^Q\left[\int_0^T (T-u)dW_u\right] = \int_0^T (T-u)^2 du = \frac{T^3}{3}$. Now we just need to calculate $Var_0^Q\left(\sum_{t=n}^{T} W_t\right)$ and $Cov_0^Q\left[\sum_{t=n}^{T} W_t, \int_0^T (T-u)dW_u\right]$. $\sum_{t=n}^{T} W_t$ can be expressed as follows:

$$\sum_{t=n}^{T} W_t = W_n + W_{n+1} + \ldots + W_T$$

$$= W_n + [W_n + (W_{n+1} - W_n)] + \ldots + [W_n + (W_{n+1} - W_n) + \ldots + (W_T - W_{T-1})]$$

$$= (T-n+1)W_n + (T-n)(W_{n+1} - W_n) + \ldots + 1\cdot(W_T - W_{T-1}).$$

Therefore,

$$Var_0^Q\left(\sum_{t=n}^{T} W_t\right)$$

$$= Var_0^Q[(T-n+1)W_n + (T-n)(W_{n+1}-W_n) + ... + 1\cdot(W_T - W_{T-1})]$$

$$= (T-n+1)^2 \cdot n + (T-n)^2 + ... + 1^2$$

$$= n(T-n+1)^2 + \frac{(T-n)(T-n+1)(2T-2n+1)}{6}.$$

In addition,

$$Cov_0^Q\left[\sum_{t=n}^{T} W_t, \int_0^T (T-u)dW_u\right]$$

$$= \sum_{t=n}^{T} Cov_0^Q\left[\int_0^t dW_u, \int_0^T (T-u)dW_u\right]$$

$$= \sum_{t=n}^{T}\int_0^t (T-u)du = \sum_{t=n}^{T}(Tt - \frac{t^2}{2})$$

$$= \frac{T(T-n+1)(T+n)}{2} - \frac{T(T+1)(2T+1) - n(n-1)(2n-1)}{12}.$$

Substitute these expressions into equation (A.1) and rearrange, we have

$$Var_0^Q\left(\sum_{t=n}^{T}\sigma_f W_t - \sigma_f\int_0^T W_u du\right) = \frac{\sigma_f^2}{3}\cdot[T + n(n-1)(n-2)]. \quad \square$$

(B) Prove that $-\int_{t-1}^{t} r_u du + g_1(t) + \sigma_f W_t = g_3(t) + \sigma_f\int_{t-1}^{t}(u-t+1)dW_u$,

where $g_3(t) = -\int_{t-1}^{t} f(0,u)du - \frac{\sigma_f^2}{6}(3t^2 - 3t + 1) + g_1(t)$.

Proof:

$$-\int_{t-1}^{t} r_u du + g_1(t) + \sigma_f W_t$$

$$= -\int_{t-1}^{t} f(0,u)du - \frac{\sigma_f^2}{2}\int_{t-1}^{t} u^2 du - \int_{t-1}^{t}\sigma_f W_u du + g_1(t) + \sigma_f W_t$$

$$= -\int_{t-1}^{t} f(0,u)du - \frac{\sigma_f^2}{6}(3t^2 - 3t + 1) - \int_{t-1}^{t}\sigma_f W_u du + g_1(t) + \sigma_f W_t .$$

Let $g_3(t) = -\int_{t-1}^{t} f(0,u)du - \frac{\sigma_f^2}{6}(3t^2 - 3t + 1) + g_1(t)$. Furthermore,

$$-\int_{t-1}^{t}\sigma_f W_u du + \sigma_f W_t$$

$$= \sigma_f\left(-\int_0^t W_u du + \int_0^{t-1} W_u du + W_t\right)$$

$$= \sigma_f\left[-\int_0^t (t-u)dW_u + \int_0^{t-1}(t-1-u)dW_u + W_t\right]$$

$$= \sigma_f\left[-tW_t + \int_{t-1}^{t} u dW_u + (t-1)W_{t-1} + W_t\right]$$

$$= \sigma_f\left[\int_{t-1}^{t} u dW_u - (t-1)(W_t - W_{t-1})\right] = \sigma_f\int_{t-1}^{t}(u-t+1)dW_u .$$

Then we obtain the equation. □

(C) Prove that $\tilde{\sigma}_t^2 \cdot t = \frac{\sigma_f^2}{3} + \sigma_S^2 - \rho\sigma_f\sigma_S > 0$.

Proof:

According to the definition, $\tilde{\sigma}_t^2 \cdot t$ can be expressed as $Var_0^Q[\ln(X_{1t} / X_{2t})] =$

$Var_0^Q(\ln X_{1t}) + Var_0^Q(\ln X_{2t}) - 2\cdot Cov_0^Q(\ln X_{1t}, \ln X_{2t})$, where

$$Var_0^Q(\ln X_{1t}) = Var_0^Q\left[\sigma_f\int_{t-1}^{t}(u-t+1)dW_u\right] = \sigma_f^2\int_{t-1}^{t}(u-t+1)^2 du = \frac{1}{3}\sigma_f^2,$$

$$Var_0^Q(\ln X_{2t}) = Var_0^Q(\sigma_S\int_{t-1}^{t} dZ_u) = \sigma_S^2, \text{ and}$$

$$Cov_0^Q(\ln X_{1t}, \ln X_{2t}) = Cov_0^Q\left[\sigma_f\int_{t-1}^{t}(u-t+1)dW_u, \sigma_S\int_{t-1}^{t} dZ_u\right]$$

$$= Cov_0^Q\left[\sigma_f\int_{t-1}^{t}(u-t+1)dW_u, \sigma_S\int_{t-1}^{t}\rho dW_u\right]$$

$$= \rho\sigma_f\sigma_S \cdot \int_{t-1}^{t}(u-t+1)du = \frac{1}{2}\rho\sigma_f\sigma_S .$$

Hence,

$$\tilde{\sigma}_t^2 \cdot t = \frac{\sigma_f^2}{3} + \sigma_S^2 - \rho\sigma_f\sigma_S$$

$$= \frac{1}{3}(\sigma_f^2 + 3\sigma_S^2 - 3\sigma_f\sigma_S)$$

$$= \frac{1}{3}[(\sigma_f - \sqrt{3}\sigma_S)^2 + (2\sqrt{3}-3)\sigma_f\sigma_S)] > 0 . \qquad \square$$

(D) Details of Reducing Discretization Error

We follow the concept in Glasserman (2004) to reduce discretization error when simulating investment portfolio price by equation (3) and discount factor, $\exp\left(-\int_0^T r_u du\right)$. From equation (7), we know that given $\sigma_f(t,u) = \sigma_f$, the stochastic differential equation for short rates is

$$dr_t = \left(\frac{\partial f(0,t)}{\partial t} + \sigma_f^2 t\right)dt + \sigma_f dW_t .$$

Also rewrite equation (3) as $d\ln S_t = \left(r_t - \frac{1}{2}\sigma_S^2\right)dt + \sigma_S dZ_t$. By equation (6.38) in Glasserman (2004), we have

$$r_{t+\Delta} = r_t + \left(\frac{\partial f(0,t)}{\partial t} + \sigma_f^2 t\right)\Delta + \sigma_f\left(W_{t+\Delta} - W_t\right) + \frac{1}{2}\left(\frac{\partial^2 f(0,t)}{\partial t^2} + \sigma_f^2\right)\Delta^2 , \qquad \text{(D.1)}$$

$$\ln S_{t+\Delta} = \ln S_t + \left(r_t - \frac{1}{2}\sigma_S^2\right)\Delta + \left[\rho\sigma_S\left(W_{t+\Delta} - W_t\right) + \sqrt{1-\rho^2}\,\sigma_S\left(W'_{t+\Delta} - W'_t\right)\right]$$

$$+ \frac{1}{2}\left(\frac{\partial f(0,t)}{\partial t} + \sigma_f^2 t\right)\Delta^2 + \frac{1}{2}\sigma_f\left(W_{t+\Delta} - W_t\right)\Delta , \text{ or}$$

$$S_{t+\Delta} = S_t \cdot \exp\left\{\left(r_t - \frac{1}{2}\sigma_S^2\right)\Delta + \left[\rho\sigma_S\left(W_{t+\Delta} - W_t\right) + \sqrt{1-\rho^2}\,\sigma_S\left(W'_{t+\Delta} - W'_t\right)\right]\right.$$
$$\left. + \frac{1}{2}\left(\frac{\partial f(0,t)}{\partial t} + \sigma_f^2 t\right)\Delta^2 + \frac{1}{2}\sigma_f\left(W_{t+\Delta} - W_t\right)\Delta\right\}. \quad \text{(D.2)}$$

Let $D_t = \exp\left(-\int_0^t r_u du\right)$, we have

$$D_{t+\Delta} = D_t\left\{1 - r_t\Delta + \frac{1}{2}\left[r_t^2 - \left(\frac{\partial f(0,t)}{\partial t} + \sigma_f^2 t\right)\right]\Delta^2 - \frac{1}{2}\sigma_f\left(W_{t+\Delta} - W_t\right)\Delta\right\}, \quad \text{(D.3)}$$

where Δ is time interval and is equal to 1/52 year in our simulation setting. Equations (D.1)-(D.3) are used to simulate short rates, asset prices and discount factor under the assumption that $\sigma_f(t,u) = \sigma_f$.

If $\sigma_f(t,u) = \sigma \cdot e^{-\lambda(u-t)} > 0$, we follow the next expressions to simulate short rates, portfolio prices and discount factor:

$$r_{t+\Delta} = r_t + \alpha_t\Delta + \sigma\left(W_{t+\Delta} - W_t\right) + \frac{1}{2}\left(\lambda \cdot \frac{\partial f(0,t)}{\partial t} + \frac{\partial^2 f(0,t)}{\partial t^2} + \sigma^2 e^{-2\lambda t} - \lambda\alpha_t\right)\Delta^2$$
$$-\frac{1}{2}\lambda\sigma\left(W_{t+\Delta} - W_t\right)\Delta,$$
$$S_{t+\Delta} = S_t \cdot \exp\left\{\left(r_t - \frac{1}{2}\sigma_S^2\right)\Delta + \left[\rho\sigma_S\left(W_{t+\Delta} - W_t\right) + \sqrt{1-\rho^2}\,\sigma_S\left(W'_{t+\Delta} - W'_t\right)\right]\right.$$
$$\left. + \frac{1}{2}\alpha_t\Delta^2 + \frac{1}{2}\sigma\left(W_{t+\Delta} - W_t\right)\Delta\right\}, \text{ and}$$
$$D_{t+\Delta} = D_t\left\{1 - r_t\Delta + \frac{1}{2}\left[r_t^2 - \alpha_t\right]\Delta^2 - \frac{1}{2}\sigma\left(W_{t+\Delta} - W_t\right)\Delta\right\},$$

where

$$\alpha_t = -\lambda\left[r_t - f(0,t)\right] + \frac{\partial f(0,t)}{\partial t} + \frac{\sigma^2\left(1 - e^{-2\lambda t}\right)}{2\lambda}.$$

(E) Prove that $Var_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right)\right]=\sigma^2\cdot\sum_{i=n}^{T}\sum_{j=n}^{T}\frac{1}{2\lambda}\cdot e^{-\lambda(i+j)}\left(e^{2\lambda\min(i,j)}-1\right).$

Proof:

$$
\begin{aligned}
&Var_0^Q\left[\sum_{t=n}^{T}\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right]\\
&=\sigma^2\cdot\sum_{i=n}^{T}\sum_{j=n}^{T}Cov_0^Q\left[\int_0^i e^{-\lambda(i-v)}dW_v,\int_0^j e^{-\lambda(j-v)}dW_v\right]\\
&=\sigma^2\cdot\sum_{i=n}^{T}\sum_{j=n}^{T}\int_0^{\min(i,j)} e^{-\lambda(i+j-2v)}dv\\
&=\sigma^2\cdot\sum_{i=n}^{T}\sum_{j=n}^{T}\frac{1}{2\lambda}\cdot e^{-\lambda(i+j)}\left(e^{2\lambda\min(i,j)}-1\right). \qquad \square
\end{aligned}
$$

(F) Prove that $Var_0^Q\left[\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]=\frac{\sigma^2}{\lambda^2}\left[T-\frac{2}{\lambda}(1-e^{-\lambda T})+\frac{1}{2\lambda}(1-e^{-2\lambda T})\right].$

Proof:

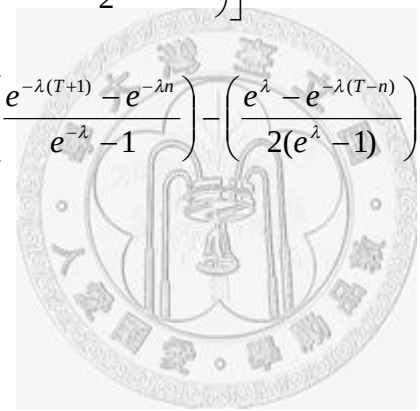
$$
\begin{aligned}
&Var_0^Q\left[\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]\\
&=Var_0^Q\left[\sigma\int_0^T\int_v^T e^{-\lambda(u-v)}du dW_v\right]=Var_0^Q\left[\frac{\sigma}{\lambda}\int_0^T\left(1-e^{-\lambda(T-v)}\right)dW_v\right]\\
&=\frac{\sigma^2}{\lambda^2}\int_0^T\left(1-e^{-\lambda(T-v)}\right)^2dv\\
&=\frac{\sigma^2}{\lambda^2}\int_0^T\left(1-2e^{-\lambda(T-v)}+e^{-2\lambda(T-v)}\right)dv\\
&=\frac{\sigma^2}{\lambda^2}\left[T-\frac{2}{\lambda}(1-e^{-\lambda T})+\frac{1}{2\lambda}(1-e^{-2\lambda T})\right]. \qquad \square
\end{aligned}
$$

(G) Prove that $Cov_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right),\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right]$

$$=\frac{\sigma^2}{\lambda^2}\left\{(T-n+1)-\left(\frac{e^{-\lambda(T+1)}-e^{-\lambda n}}{e^{-\lambda}-1}\right)-\left(\frac{e^{\lambda}-e^{-\lambda(T-n)}}{2(e^{\lambda}-1)}\right)+\left(\frac{e^{-\lambda(2T+1)}-e^{-\lambda(T+n)}}{2(e^{-\lambda}-1)}\right)\right\}.$$

Proof:

$$
\begin{aligned}
& Cov_0^Q\left[\sum_{t=n}^{T}\left(\sigma\int_0^t e^{-\lambda(t-v)}dW_v\right),\sigma\int_0^T\int_0^u e^{-\lambda(u-v)}dW_v du\right] \\
&= \sum_{t=n}^{T} Cov_0^Q\left[\sigma\int_0^t e^{-\lambda(t-v)}dW_v,\frac{\sigma}{\lambda}\int_0^T\left(1-e^{-\lambda(T-v)}\right)dW_v\right] \\
&= \frac{\sigma^2}{\lambda}\sum_{t=n}^{T} Cov_0^Q\left[\int_0^t e^{-\lambda(t-v)}dW_v,\int_0^T\left(1-e^{-\lambda(T-v)}\right)dW_v\right] \\
&= \frac{\sigma^2}{\lambda}\sum_{t=n}^{T}\int_0^t\left(e^{-\lambda(t-v)}-e^{-\lambda(T+t-2v)}\right)dv \\
&= \frac{\sigma^2}{\lambda}\sum_{t=n}^{T}\left[\left(\frac{1-e^{-\lambda t}}{\lambda}\right)-\left(\frac{e^{-\lambda(T-t)}-e^{-\lambda(T+t)}}{2\lambda}\right)\right] \\
&= \frac{\sigma^2}{\lambda^2}\sum_{t=n}^{T}\left[1-e^{-\lambda t}-\left(\frac{e^{-\lambda(T-t)}-e^{-\lambda(T+t)}}{2}\right)\right] \\
&= \frac{\sigma^2}{\lambda^2}\left\{(T-n+1)-\left(\frac{e^{-\lambda(T+1)}-e^{-\lambda n}}{e^{-\lambda}-1}\right)-\left(\frac{e^{\lambda}-e^{-\lambda(T-n)}}{2(e^{\lambda}-1)}\right)+\left(\frac{e^{-\lambda(2T+1)}-e^{-\lambda(T+n)}}{2(e^{-\lambda}-1)}\right)\right\}.\square
\end{aligned}
$$



Impacts of Investment Strategies and Participants' Behaviors on Guarantee Costs and Retirement Benefits in Defined Contribution Pension Plans

Abstract

This paper investigates how investment strategies and behaviors affect guarantee costs and retirement benefits in defined contribution plans that provide interest rate guarantees. Several investment strategies are considered, including both a buy-and-hold strategy without portfolio modifications and those that allow frequent modifications of the portfolio during the accumulation phase. According to the numerical results offered by simulation techniques, a participant chasing the highest expected retirement benefits should adopt the buy-and-hold strategy and distribute all contributions in the portfolio with high expected rate of return and high volatility in our guarantee designs. After incorporating frequent modification behaviors, averages and risk measures of guarantee costs may be higher than when the participant always holds the portfolio with high expected rate of return and volatility. Therefore, when the pension plan provides rate of return guarantees, the plan provider cannot ignore the impact of the participant's frequent modification behaviors.

1. Introduction

Defined contribution (DC) pension plans are becoming more and more popular in many countries. With a DC plan, retirement benefits depend on the accumulated account value at retirement. The participant usually has the right to select investment portfolios but also must bear the downside risk of poor investment performance. To avoid the inadequacy of pension benefits, minimum rate of return guarantees often appear in DC plans. If the actual accumulated account value is less than the guaranteed account value, the participant withdraws the guaranteed amount, and in some scenarios, the related guarantee costs might be substantial. We investigate how the participant's investment strategies and behaviors affect these guarantee costs and retirement benefits in DC plans that contain interest rate guarantees.

Mitchell and Utkus (2004) link behavioral finance research to pension plan design and analyze how participants make decisions to save and invest. Dowd and Blake (2006) note that insurance policies with long horizons increase the pressure associated with taking policyholder behavior into account. These authors imply that the pension plan provider should consider how participants behave and incorporate possible behaviors into its risk management. Participants might adopt a variety of strategies to allocate their contributions during the accumulation phase or modify portfolios, such that the

portfolios at retirement date are not necessary the same as those at inception. To estimate guarantee costs accurately, it therefore is essential to consider a participant's allocation and modification strategies, especially when the plan has long horizons. The failure to do so can lead to misestimated guarantee costs and thus great harm for the guarantee provider.

Many papers discuss how to determine the fair value of a contract embedded with various guarantees, including Brennan and Schwartz (1976), Sherris (1995), Ekern and Persson (1996), Boyle and Hardy (1997), Persson and Aase (1997), Miltersen and Persson (1999), Pennacchi (1999), Hansen and Miltersen (2002), and Lindset (2004). These articles derive fair values without incorporating any investment strategy, whereas other articles focus on investment strategies in a DC pension plan. For example, Blake et al. (2001) discuss how value-at-risk estimates, which measure the risk that participants face in DC plans, vary with allocation strategies. Vigna and Haberman (2001) study which investment strategies minimize the cost incurred by the deviation of the actual pension fund level from the target level. In addition, Boulier et al. (2001) explore optimal portfolio composition in a plan that contains a guarantee, which relates to the level of the interest rate at retirement, and the evolution of short rates is described by Vasicek (1977) model. Menoncin (2002) considers background

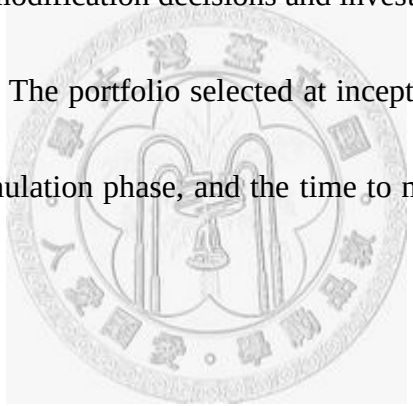
risk and inflation risk to derive the optimal portfolio allocation. Battocchio and Menoncin (2004) deal with stochastic interest rates, salary risk, and inflation risk in an optimal portfolio allocation problem. In the latter three articles, the objective is to maximize the expected value of the terminal utility function with specific constraints.

In this research, we attempt to fill an existing gap in the literature by taking investment strategies into account and thereby estimate the costs of guarantees provided in DC plans. We consider seven strategies, each of which describes different investment behavior, and investigate how these strategies affect guarantee costs and retirement benefits. In the simplest strategy, known as the "*buy-and-hold strategy*," the participant neither changes portfolio allocation methods nor sells his or her portfolios during the accumulation phase.[1] We consider several allocation approaches, and the proportion of contributions allocated to each portfolio is always invariant. Benartzi and Thaler (2001) document that participants tend to allocate contributions evenly among the assets available in the plan and call it the "1*/n heuristic*."[2] We therefore take this allocation approach into account in the buy-and-hold strategy.

[1] Zweig (1998) asks several economists how they put their money on the TIAA-CREF, the largest DC plan in the world. William F. Sharpe, the professor known for the Sharpe index, says that: "I am strictly a buy-and-hold investor. In fact, I have never sold an equity mutual fund."

[2] For example, if there are 10 available assets in the pension plan, the participant spreads his or her contributions evenly across all 10 assets. Huberman and Jian (2006) and Agnew (2006) call this strategy the "*framing (effect) 1/n heuristic*." Huberman and Jian (2006) show that many participants follow the "*conditional 1/n rule*," in which case they do not use all assets available in the plan but instead distribute contributions evenly among the assets they use, typically no more than three or four.

Another consideration we address is that participants may modify their investment portfolios irregularly. Fund management companies and administrators periodically disclose the performance of all portfolios available in the plan through analyses of to-date performance that employ a variety of indexes, such as the realized rate of return or the Sharpe index. We assume that some participants always track these performance indexes and prefer the portfolio with the best past performance.[3] If the past performance of the current portfolio is not the best among the available portfolios, the participant may make modification decisions and invest all savings in the portfolio with the best performance. The portfolio selected at inception therefore is not always the same during the accumulation phase, and the time to modify the portfolio cannot be predicted ex ante.



Furthermore, Barber and Odean (2003) show that investors prefer to realize gains rather than losses in their tax-deferred accounts, consistent with the "*disposition effect*," as first labeled by Shefrin and Statman (1985).[4] Dhar and Zhu (2006) also

[3] Mitchell and Utkus (2004) summarize that the "representativeness heuristic" and "availability heuristic" can account for reliance on past performance. The representativeness heuristic means that when making decisions, people tend to be affected by some descriptions or patterns, even if those descriptions are uninformative or the patterns are random; the availability heuristic implies that people are prone to rely on the most easily or cheaply available information. Readers interested in these two heuristics could refer to Tversky and Kahneman (1974). Moreover, Voronkova and Bohl (2005) demonstrate that Polish pension fund investors (institutional investors) tend to purchase stocks with excellent past performance. Pension funds under their analysis are classified into defined contribution plans. During the period they investigate, these plans provide minimum guaranteed rates of return, which are announced quarterly.

[4] The central concept of the disposition effect is that the investor tends to sell winners (portfolios with gains) too soon and hold losers (portfolios with losses) too long. As the portfolio appreciates and its

show that the disposition effect exists among the investors in individual retirement accounts (IRA) and Keogh accounts. Therefore, we consider strategies related to the disposition effect herein.

This research deals with pension plans that include interest rate guarantees. We analyze two types, type-I (maturity) and type-II (multi-period) guarantees, as we introduced in the first essay. The dynamics of interest rates are described by the CIR (1985) model. It is difficult to obtain explicit formulae for fair values of these two guarantees after incorporating modification behaviors. We therefore adopt a Monte Carlo simulation approach in our analysis. By simulating modification behaviors, guaranteed rates, and actual rates of return at all record dates in the real world, we can derive the statistics of guarantee costs and retirement benefits at retirement date.

Our numerical results offer several significant findings. First, guarantee costs and retirement benefits are both affected by investment strategies. Low retirement benefits ensue when participants put all their contributions in a portfolio with low expected rate of return and low volatility. This conservative allocation approach costs more for

value moves above a reference point (e.g., purchase price), the investor becomes more risk averse and sells the winner as soon as possible to either feel pride or avoid regret if the value declines. If the portfolio depreciates and the value falls below the reference point, the investor becomes risk seeking and maintains the loser so that he or she does not have to concede a wrong investment decision or miss the buoyancy in the future. Many authors have demonstrated the existence of the disposition effect in financial markets, including Shefrin and Statman (1985), Odean (1998), Grinblatt and Keloharju (2001), Genesove and Mayer (2001), Locke and Mann (2005), and Shu et al. (2005).

the type-I guarantee provider but less for the type-II guarantee provider in our basic parameter settings. Second, the buy-and-hold allocation strategy with higher retirement benefits may cause lower type-I guarantee costs but higher type-II guarantee costs. Therefore, the type-II guarantee provider has an incentive to impose restrictions on allocation methods to reduce its risks, but such restrictions deter the participant from reaping the highest possible retirement benefits. Third, frequent modifications of the portfolio may not do good to the growth of pension benefits, regardless of whether transaction costs exist. This finding is consistent with Barber and Odean's (2000) conclusion that frequent trading hurts the investment performance. Frequent modifications also may increase guarantee costs in some guarantee designs. If the guarantee provider neglects the effect of frequent modifications, its guarantee costs likely will be much higher than its ex ante estimates.

The remainder of this article is organized as follows. Section 2 describes the characteristics of guarantees, the financial models, and the strategies under analysis. Section 3 provides the numerical results and presents how guarantee costs and retirement benefits vary with strategies. The implications emerging from the numerical results are also provided. Section 4 concludes the paper.

2. Structures of Guarantees, Financial Models, and Strategies

2.1 Structures of Guarantees

Financial institutions or governments provide pension plans with minimum guaranteed rates of return. Two types of guarantees in the previous essay are considered here. Firstly assume that the participant allocates contributions to only one portfolio. Under the type-I guarantee (maturity guarantee), the retirement benefits received at time *T* are expressed as

$$\Gamma_T(I) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot Max\left[\prod_{t=n}^{T} \exp(\overline{R}(t,t+\delta)), \prod_{t=n}^{T} \exp(R_t) \right], \quad (1)$$

and the costs paid by the guarantee provider at time *T* are

$$V_T(I) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot Max\left[\prod_{t=n}^{T} \exp(\overline{R}(t,t+\delta)) - \prod_{t=n}^{T} \exp(R_t), 0 \right]. \quad (2)$$

With the type-II guarantee (multi-period guarantee), the retirement benefits and the guarantee costs at time *T* are

$$\Gamma_T(II) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \prod_{t=n}^{T} Max\left[\exp(\overline{R}(t,t+\delta)), \exp(R_t)\right], \text{ and} \quad (3)$$

$$V_T(II) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \left\{ \prod_{t=n}^{T} Max\left[\exp(\overline{R}(t,t+\delta)), \exp(R_t)\right] - \prod_{t=n}^{T} \exp(R_t) \right\}, \quad (4)$$

respectively. $\overline{R}(t,t+\delta)$ is the guaranteed rate at time *t*, which is equal to the time-*t* spot rate for maturity $t+\delta$, and R_t is the actual rate of return in the *t*-th year. The derivations of $\overline{R}(t,t+\delta)$ and R_t are described subsequently. The meanings of other symbols can be read in Section 2 of the previous essay.

We emphasize that equations (1) - (4) hold if the participant selects only one portfolio. Assume the number of portfolios available in the plan is $N \geq 1$ and the proportion of contributions that the participant allocates to portfolio *i* is always fixed at $p^{(i)}$, i = 1, 2, …, *N*. Then, the amount contributed in the *n*-th year changes to $Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \sum_{i=1}^{N}\left[p^{(i)} \cdot \prod_{t=n}^{T} \exp(R_t^{(i)}) \right]$ at time *T*, and the total actual benefits accumulated from inception to the retirement date are $\sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \sum_{i=1}^{N}\left[p^{(i)} \cdot \prod_{t=n}^{T} \exp(R_t^{(i)}) \right]$, where $R_t^{(i)}$ is the actual rate of return on portfolio *i* in the *t*-th year. Following this concept, $\Gamma_T(I)$, $\Gamma_T(II)$, $V_T(I)$, and $V_T(II)$ are redefined as, respectively,

$$\Gamma_T(I) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \left\{ \sum_{i=1}^{N} p^{(i)} \cdot Max\left[\prod_{t=n}^{T} \exp(\overline{R}(t,t+\delta)), \prod_{t=n}^{T} \exp(R_t^{(i)}) \right] \right\}, \quad (5)$$

$$V_T(I) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \left\{ \sum_{i=1}^{N} p^{(i)} \cdot Max\left[\prod_{t=n}^{T} \exp(\overline{R}(t,t+\delta)) - \prod_{t=n}^{T} \exp(R_t^{(i)}), 0 \right] \right\}, \quad (6)$$

$$\Gamma_T(II) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \left\{ \sum_{i=1}^{N} p^{(i)} \cdot \prod_{t=n}^{T} Max\left[\exp(\overline{R}(t,t+\delta)), \exp(R_t^{(i)}) \right] \right\}, \text{ and} \quad (7)$$

$$V_T(II) = \sum_{n=1}^{T} Y_0 \cdot c \cdot (1+k)^{n-1} \cdot \left\{ \sum_{i=1}^{N} p^{(i)} \cdot \left\{ \prod_{t=n}^{T} Max\left[\exp(\overline{R}(t,t+\delta)), \exp(R_t^{(i)}) \right] - \prod_{t=n}^{T} \exp(R_t^{(i)}) \right\} \right\}$$

. (8)

2.2 Interest Rate and Portfolio Price Dynamics

We assume that the stochastic process for short rates is given by

$$dr_t = \kappa(\theta - r_t)dt + \sigma_r \sqrt{r_t}\, dW_t,$$

where κ is the speed-of-adjustment parameter, θ is the long-term mean of interest rates, σ_r is the diffusion factor of interest rates, and $\{W_t : t \geq 0\}$ is a standard

one-dimensional Brownian motion. κ, θ, and σ_r are all positive constants. This term structure model appears in Cox et al. (1985). The time-*t* spot rate for maturity $t+\delta$ is

$$\overline{R}(t,t+\delta)=\frac{-\ln P(t,t+\delta)}{\delta}, \tag{9}$$

where $P(t,t+\delta)=\exp(A(\delta)+B(\delta)\cdot r_t)$ is the time-*t* price of a zero-coupon bond maturing at time $t+\delta$, r_t is the short rate at time *t*,

$$A(\delta)=\frac{2\kappa\theta}{\sigma_r^2}\{\ln[2\gamma\cdot\exp((\gamma+\kappa)\cdot\delta/2)]-\ln[(\gamma+\kappa)\cdot(e^{\gamma\cdot\delta}-1)+2\gamma]\},$$

$$B(\delta)=2(1-e^{\gamma\cdot\delta})/[(\gamma+\kappa)(e^{\gamma\cdot\delta}-1)+2\gamma], \text{ and } \gamma=\sqrt{\kappa^2+2\sigma_r^2}.$$

The price of portfolio *i* per share follows a lognormal distribution, and its diffusion process is:

$$dS_t^{(i)}=\mu^{(i)}S_t^{(i)}dt+\sigma_S^{(i)}S_t^{(i)}dZ_t^{(i)},$$

where $\mu^{(i)}$ is the expected annual rate of return on portfolio *i*, $\sigma_S^{(i)}$ is the non-negative constant annualized volatility of portfolio *i*, and $\{Z_t^{(i)}:t\geq 0\}$ is a standard one-dimensional Brownian motion that influences the change of the price of portfolio *i*. For simplicity, we assume that $Cov(W_t, Z_t^{(i)})=0$ for any *t* and *i*, and $Cov(Z_t^{(i)}, Z_t^{(i')})=0$ for any *t* and $i\neq i'$. Given that time-*t* price of portfolio *i* is $S_t^{(i)}$ and the time interval is Δ, we can express $S_{t+\Delta}^{(i)}$ as

$$S_{t+\Delta}^{(i)}=S_t^{(i)}\cdot\exp\left[\mu^{(i)}\Delta-\frac{1}{2}\left(\sigma_S^{(i)}\right)^2\Delta+\sigma_S^{(i)}(Z_{t+\Delta}^{(i)}-Z_t^{(i)})\right]. \tag{10}$$

2.3 Investment Strategies and Participants' Behaviors

Some participants always maintain the same portfolio(s) during the accumulation phase, but others will sell and pursue the portfolio with the best past performance. We therefore consider seven strategies, each of which describes different investment behavior. In strategy S1, the participant allocates contributions to one or more than one portfolio and does not sell the portfolio(s) until retirement. By contrast, in strategies S2-S7, the participant selects only one portfolio in which to invest and deliberates about whether to modify the current portfolio every six months. We refer to the date on which modification is allowed as "*the modification date*." The modification decision depends on the past performance of each available portfolio. The differences among strategies are summarized in Figure 1, and the details of each strategy are as follows:

(i) Strategy S1. The proportion of contributions allocated to each portfolio is always fixed. If there are N available portfolios in the pension plan and the participant spreads contributions evenly among all of these portfolios, we say that the "framing 1/N strategy" is adopted.

(ii) Strategy S2. At each modification date, the participant calculates the actual rate of return on each portfolio since inception. If the current portfolio is not the most profitable, the participant sells it and allocates all savings to the most profitable

portfolio. At time *t*, the actual rate of return on portfolio *i* since inception is computed as $\hat{R}_t^{(i)} \equiv \ln(S_t^{(i)} / S_0^{(i)})$.

(iii) Strategy S3. This is similar to S2, but in this case the participant selects the portfolio with the highest one-year-to-date actual rate of return at each modification date. At time $t \geq 1$, the one-year-to-date actual rate of return on portfolio i is defined as $\widetilde{R}_{t-1\sim t}^{(i)} \equiv \ln(S_t^{(i)} / S_{t-1}^{(i)})$; at time $t = 0.5$ (the first modification date), we calculate the half-a-year-to-date actual rate of return as $\widetilde{R}_{0\sim 0.5}^{(i)} \equiv \ln(S_{0.5}^{(i)} / S_0^{(i)})$.

(iv) Strategy S4. Unlike S2 and S3, here the participant selects the portfolio with the highest one-year-to-date Sharpe index, defined as $SI_t^{(i)} \equiv (\widetilde{R}_{t-1\sim t}^{(i)} - r_{t-1\sim t}) / \sigma_{t-1\sim t}^{(i)}$, where $r_{t-1\sim t}$ is the arithmetic average of the one-month spot rates at the beginning of the past 12 months, and $\sigma_{t-1\sim t}^{(i)}$ is the annualized standard deviation of monthly rates of return on portfolio *i* over the past 12 months. At time $t = 0.5$, the Sharpe index is expressed as $SI_{t=0.5}^{(i)} \equiv (\widetilde{R}_{0\sim 0.5}^{(i)} - r_{0\sim 0.5}) / \sigma_{0\sim 0.5}^{(i)}$, where $r_{0\sim 0.5}$ and $\sigma_{0\sim 0.5}^{(i)}$ are derived from simulated data in the first six months.

(v) Strategy S5. This strategy and the following two apply the disposition effect. The participant sets up a "reference price" to measure the gains and losses of the current portfolio.[5] At each modification date, if the rate of return with respect to

[5] In principle, the participant regards the purchase price of the current portfolio as the reference price to measure gains and losses, according to Odean (1998). But some exceptions exist, such as if the portfolio does not change for several years, it would not be reasonable to view the purchase price as

the reference price moves above the stop-gain level, $G > 0$, the participant has an incentive to sell the portfolio and realize gains. However, if the rate of return with respect to the reference price falls below the stop-loss level, $L < 0$, the participant may sell. We assume that $|G| < |L|$, in line with the spirit of the disposition effect, namely, that the investor tends to sell winners too soon and hold losers too long. If the rate of return measured with respect to the reference price reaches G or L by the modification date, the participant selects a new portfolio with the highest actual rate of return since inception, as described for S2. If the stop-gain or stop-loss level does not occur, or the current portfolio is the most profitable, the participant does not make any modification.

(vi) Strategy S6. The setting is analogous to S5, but the participant selects the portfolio with the highest one-year-to-date actual rate of return at the modification date, as described by S3.

(vii) Strategy S7. Similar to S5, but in this case the participant chooses the portfolio with the highest one-year-to-date Sharpe index, as described by S4.

---

the reference price. We illustrate how to set up the reference price in this case and other details of strategies S5-S7 in Appendix (A).

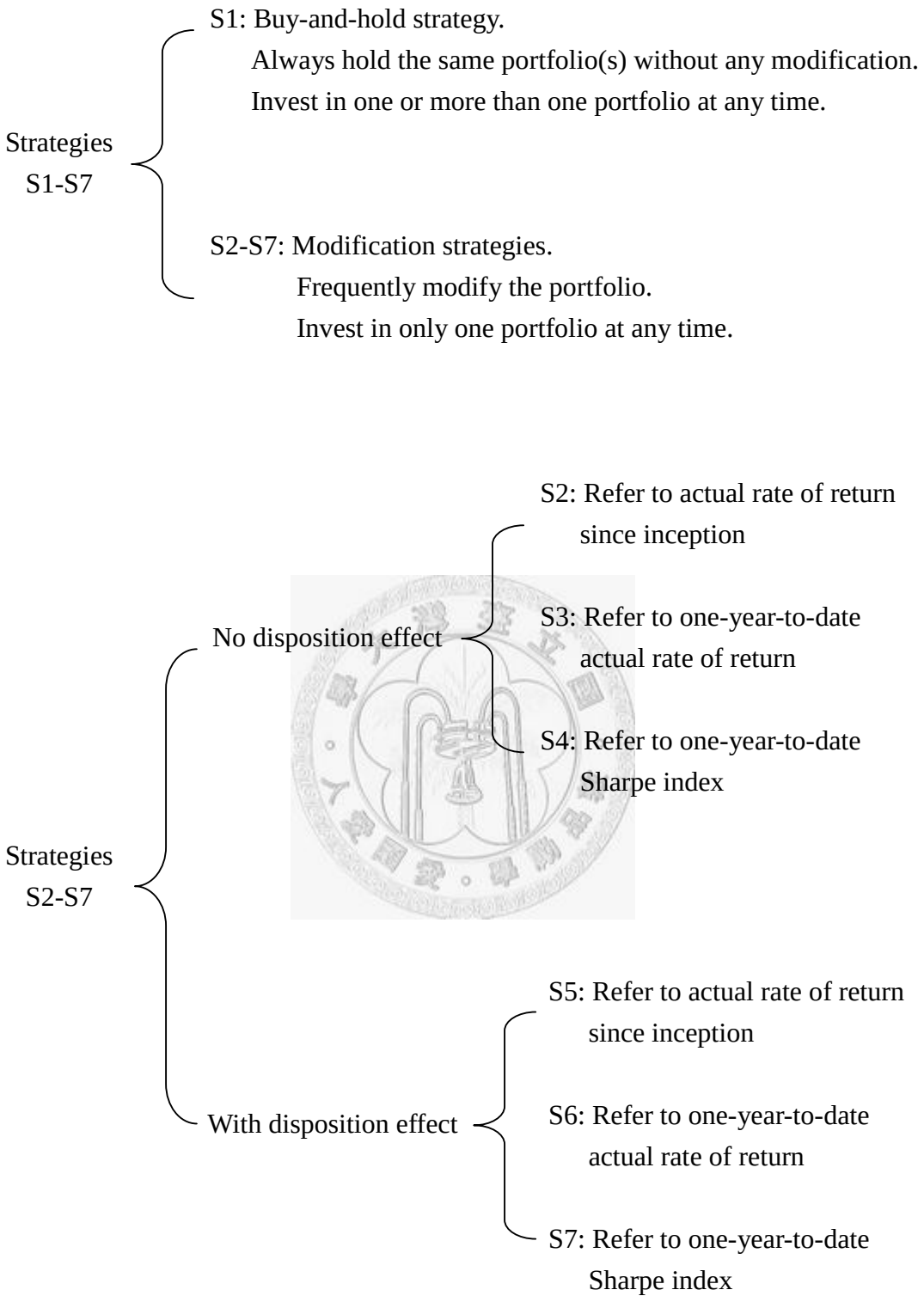


Figure 1. Differences among Strategies

2.4 Actual Rate of Return and Income Replacement Ratio

Assume that the participant pays transaction costs if he or she sells the portfolio. Without loss of generality, the transaction cost is proportional to the selling price, and the proportion is equal to *f*. For strategy S1, in which the participant does not sell the portfolio(s) until retirement, the actual rate of return on portfolio *i* recorded at time *t* is given by

$$R_t^{(i)} = \begin{cases} \ln(S_t^{(i)} / S_{t-1}^{(i)}) & if \quad t = 1,2,...,T-1 \\ \ln[(1-f)S_t^{(i)} / S_{t-1}^{(i)}] & if \quad t = T \end{cases}. \tag{11}$$

For strategies S2-S7, we consider four cases to calculate the actual rate of return, assuming the participant invests all savings in portfolio *i* in the beginning of the *t*-th year:

(i) In the middle of the *t*-th year, the participant sells portfolio *i* and buys portfolio i', then sells portfolio i' at the end of the *t*-th year. In this case,

$$R_t = \ln[(1-f)S_{t-0.5}^{(i)} / S_{t-1}^{(i)}] + \ln[(1-f)S_t^{(i')} / S_{t-0.5}^{(i')}]. \tag{12}$$

(ii) In the middle of the *t*-th year, the participant sells portfolio *i* and buys portfolio i' but does not sell portfolio i' at the end of the *t*-th year. In this case,

$$R_t = \ln[(1-f)S_{t-0.5}^{(i)} / S_{t-1}^{(i)}] + \ln(S_t^{(i')} / S_{t-0.5}^{(i')}). \tag{13}$$

(iii) In the *t*-th year, the participant does not sell portfolio *i* until the end of the year. In this case,

$$R_t = \ln[(1-f)S_t^{(i)} / S_{t-1}^{(i)}]. \quad (14)$$

(iv) The participant does not sell portfolio *i* in the *t*-th year. In this case,

$$R_t = \ln(S_t^{(i)} / S_{t-1}^{(i)}). \quad (15)$$

We next present how to calculate the income replacement ratio (IRR) at retirement date, which is a measure of the adequacy of retirement life compared with life before retirement. Assume that the participant lives until the beginning of year $x+T$ (i.e., the last time to withdraw the annuity is the beginning of year $x+T$). At time T, the retirement benefits received by the participant are Γ_T, the present value of the annuity paid once a year since time T is $\ddot{a} \equiv \sum_{j=0}^{x} P(T, T+j)$, and the yearly wages just before retirement are $Y_0 \cdot c \cdot (1+k)^{T-1}$. The IRR then is derived by $\frac{\Gamma_T / \ddot{a}}{Y_0 \cdot c \cdot (1+k)^{T-1}}$. For simplicity, we neglect mortality rates here.

3. Numerical Results

This section discusses how guarantee costs and retirement benefits vary with different investment strategies. We apply statistics and risk measures, including average, standard error (S.E.), value-at-risk (VaR), and the conditional tailed expectation value (CTE). To focus on the impact of the investment strategy, we assume that the participant remains alive at the retirement date and that the guarantee provider never

goes out of business. We simulate 50,000 paths to obtain the numerical results.[6] The basic parameter settings in this simulation, which we refer to as the "*standard scenario*," are as follows:

(i) T =30, $Y_0 = 1$, $k = 0.03$, $c = 0.06$, and $x = 25$.

(ii) $\delta = 2$, $\kappa = 0.18$, $\theta = 0.05$, $\sigma_r = 0.14$, and $r_0 = 0.03$.[7]

(iii) G = 0.2, L = -0.4, and f = 0. [8]

(iv) Three portfolios are available in the pension plan, with expected annual rates of return and annualized volatility of (A) $\mu^{(A)}$ = 0.04, $\sigma_S^{(A)}$ = 0.02 (Portfolio A); (B) $\mu^{(B)}$ = 0.08, $\sigma_S^{(B)}$ = 0.1 (Portfolio B); and (C) $\mu^{(C)}$ = 0.12, $\sigma_S^{(C)}$ = 0.2 (Portfolio C).[9] For strategies S2-S7, the participant selects portfolio C at inception. We also set $S_0 = 1$ for all portfolios.

3.1 Results for S1 (Buy-and-Hold Strategy)

For this strategy, we consider 10 allocation mixes: (0, 0, 1), (0, 0.25, 0.75), (0, 0.5, 0.5), (0.06, 0.63, 0.31), (0.25, 0.5, 0.25), (0.435, 0.38, 0.185), (0.62, 0.26, 0.12), (0.81, 0.13, 0.06), (1, 0, 0), and (1/3, 1/3, 1/3). The (0, 0, 1) mix represents the case when the

[6] The simulation procedure is described in Appendix (B).

[7] We describe how to derive the estimates of κ, θ, and σ_r in Appendix (C). We follow the approach proposed by Scott (1996) to simulate short rates in the CIR model. The details appear in Glasserman (2004).

[8] The values of *G*, *L*, and *f* are arbitrary in the standard scenario. We analyze how average guarantee costs and retirement benefits change with different *G*, *L*, and *f* subsequently.

[9] Appendix (D) illustrates why we set these expected rates of return and volatility.

participant always allocates contributions to portfolio C; (1, 0, 0) indicates investment only in portfolio A; and (1/3, 1/3, 1/3) means that the participant adopts the framing 1/*N* strategy. We derive the first nine mixes from the efficient portfolio frontier (Figure 2) and order them according to their expected annual rates of return.

The average guarantee costs and average retirement benefits for strategy S1 with a type-I guarantee appear in Figure 3; those for the type-II guarantee are in Figure 4. In addition, Tables 1 and 2 respectively list the type-I and type-II guarantee costs for S1 with different allocation proportion mixes. As Figure 3 and Table 1 show, in the standard scenario, the lowest average costs for the type-I guarantee emerge from the (0, 0, 1) mix. The high expected rate of return from this mix leads to high actual rates of return, so the guarantee provider rarely must pay out significant compensations because of this guarantee. The highest average costs for the type-I guarantee exist for the (1, 0, 0) mix, because portfolio A features the lowest expected rate of return, and the actual benefits are very low. The harm that portfolio A causes to the type-I guarantee provider is emphasized by the risk measures in Table 1. Allocating all contributions to portfolio A results in the highest 99% and 95% quantile risk measures of type-I guarantee costs.

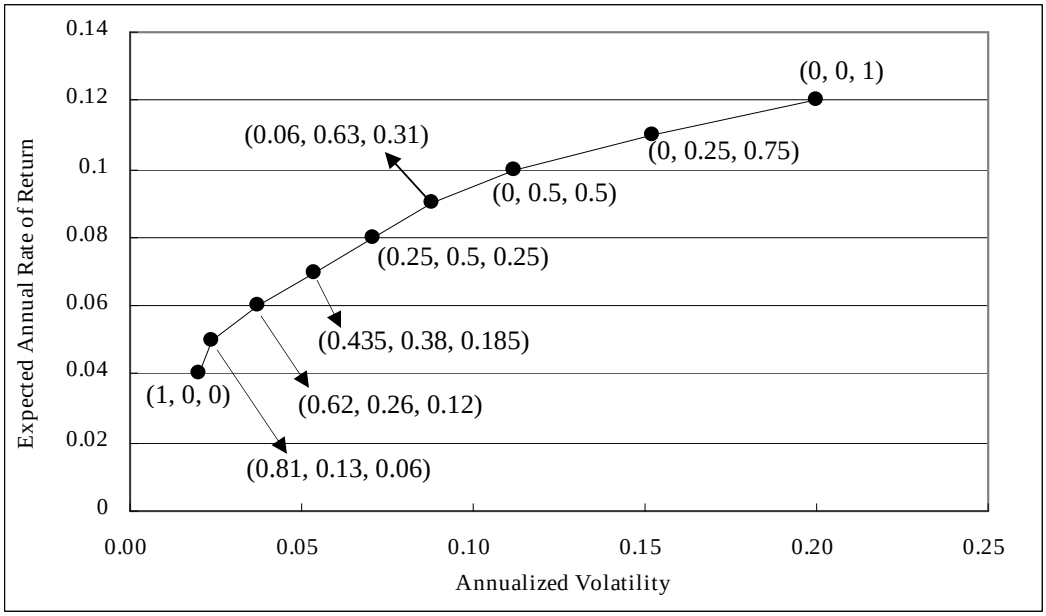


Figure 2. Efficient Frontier Generated by Portfolios A, B, and C

Notes: The efficient frontier is generated by assuming that the weight of each portfolio is not negative.

In contrast, Figure 4 and Table 2 reveal that the highest average and risk measures of type-II guarantee costs come from the (0, 0, 1) mix. Therefore, the portfolio with the highest expected rate of return and volatility appears to cause the highest type-II guarantee costs. We explain this finding on the basis of the payoff structure. From the multi-period binding property and equation (8), we know that high actual rates of return in some periods will magnify the guarantee obligation. Even though only one record date exists such that $\exp(\overline{R}(t,t+\delta)) > \exp(R_t^{(i)})$, the total guarantee costs will be very great if the other $R_t^{(i)}$'s are very high. Hence, portfolio C, with its high expected rates of return, usually leads to high guarantee costs. Such a magnifying effect does not exist in the type-I guarantee. Equation (6) indicates that low

guaranteed rates in some periods diminish the effect of high guaranteed rates in other periods, and high actual rates of return reduce the shortfall made up for by the guarantee provider. Thus, the (0, 0, 1) mix does not produce high type-I guarantee costs.

Tables 3 and 4 report the statistics related to retirement benefits and income replacement ratios for S1. The (0, 0, 1) mix has the highest average retirement benefits and (1, 0, 0) has the lowest, regardless of the type of guarantee. The guarantee mechanism transfers the downside risk of poor investment performance to the guarantee provider, so investing all contributions in portfolio C takes advantage of the high expected rate of return and seldom involves downside risk. Portfolio A, with the lowest expected rate of return, counteracts wealth accumulation. The last rows in Tables 3 and 4 show that the framing 1/*N* strategy is not necessarily good for the participant, because it cannot create the highest retirement benefits.

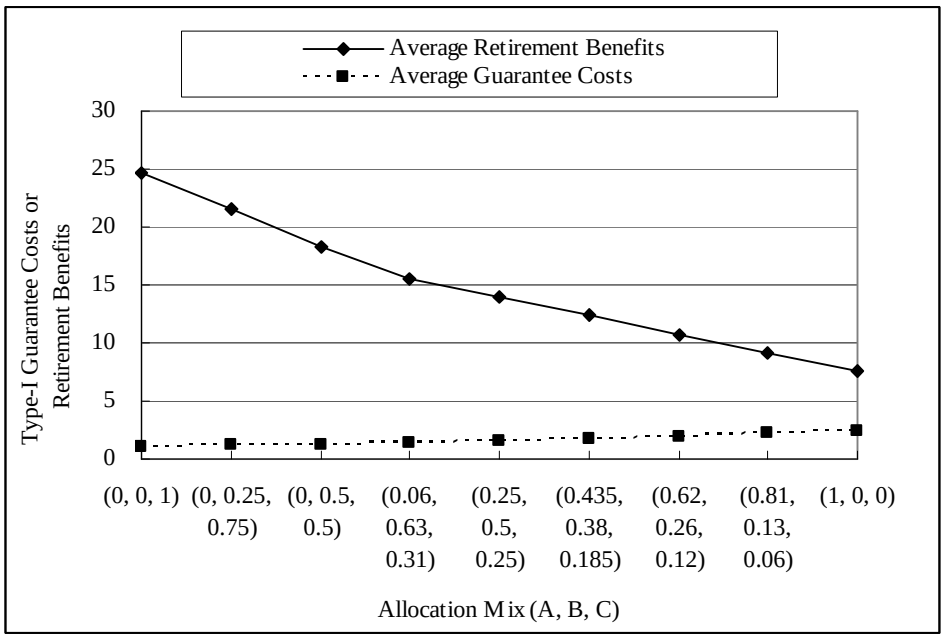


Figure 3. Average Retirement Benefits and Guarantee Costs under a Type-I Guarantee in S1 and the Standard Scenario

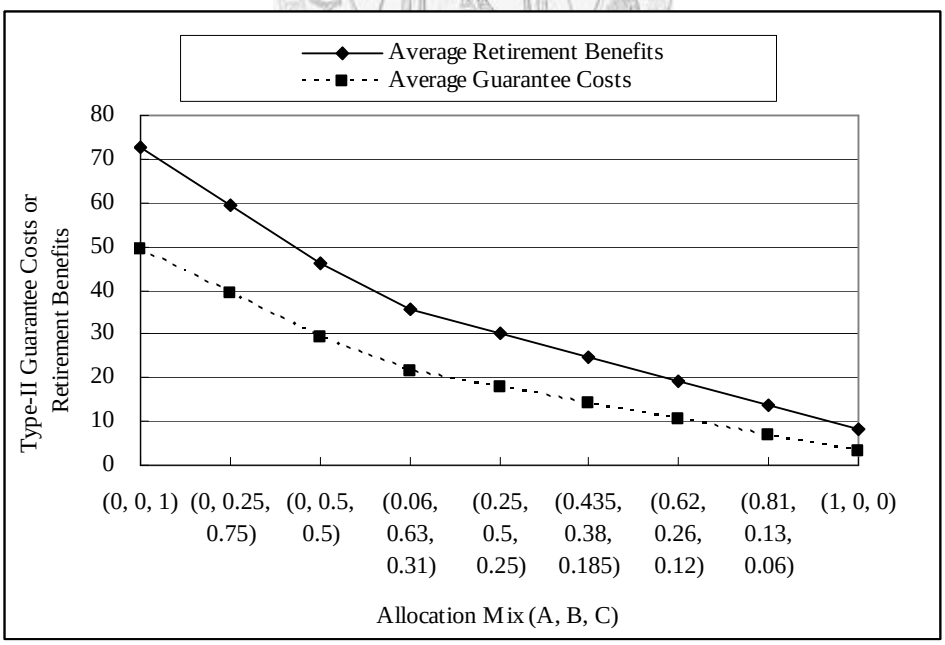


Figure 4. Average Retirement Benefits and Guarantee Costs under a Type-II Guarantee in S1 and the Standard Scenario

Table 1.

Summary Statistics of Type-I Guarantee Costs in S1 and the Standard Scenario

Allocation Mix (A, B, C)	Average	S.E.	VaR(99%)	VaR(95%)	CTE(99%)	CTE(95%)
(0, 0, 1)	1.106	0.043	17.604	4.128	50.084	16.349
(0, 0.25, 0.75)	1.165	0.043	17.971	4.131	50.892	16.570
(0, 0.5, 0.5)	1.223	0.043	18.693	4.470	52.105	17.269
(0.06, 0.63, 0.31)	1.330	0.044	20.251	5.130	53.660	18.291
(0.25, 0.5, 0.25)	1.541	0.044	21.448	5.959	55.013	19.336
(0.435, 0.38, 0.185)	1.749	0.045	22.522	6.843	56.411	20.430
(0.62, 0.26, 0.12)	1.956	0.045	23.783	7.759	57.847	21.563
(0.81, 0.13, 0.06)	2.167	0.046	25.032	8.656	59.317	22.733
(1, 0, 0)	2.379	0.047	26.460	9.582	60.845	23.929
(1/3, 1/3, 1/3)	1.608	0.044	21.175	6.098	54.808	19.341

Table 2.

Summary Statistics of Type-II Guarantee Costs in S1 and the Standard Scenario

Allocation Mix (A, B, C)	Average	S.E.	VaR(99%)	VaR(95%)	CTE(99%)	CTE(95%)
(0, 0, 1)	49.198	0.219	220.085	124.101	351.732	195.763
(0, 0.25, 0.75)	39.222	0.175	172.582	97.204	280.732	154.576
(0, 0.5, 0.5)	29.245	0.132	126.767	70.274	211.536	113.945
(0.06, 0.63, 0.31)	21.300	0.101	92.973	49.983	159.914	83.136
(0.25, 0.5, 0.25)	17.758	0.089	79.106	41.981	139.545	71.083
(0.435, 0.38, 0.185)	14.046	0.077	65.541	33.607	118.465	58.500
(0.62, 0.26, 0.12)	10.334	0.066	51.816	25.115	98.052	46.237
(0.81, 0.13, 0.06)	6.791	0.056	39.882	17.460	79.575	35.190
(1, 0, 0)	3.248	0.048	27.840	10.789	62.785	25.321
(1/3, 1/3, 1/3)	20.580	0.100	92.543	49.588	158.451	82.724

Table 3.

Summary Statistics of Retirement Benefits and IRR

under a Type-I Guarantee in S1 and the Standard Scenario

Allocation Mix (A, B, C)	Retirement Benefits			Income Replacement Ratio	
	Average	S.E.	VaR (10%)	Average	S.E.
(0, 0, 1)	24.734	0.118	7.407	0.684	0.004
(0, 0.25, 0.75)	21.509	0.093	8.146	0.596	0.003
(0, 0.5, 0.5)	18.284	0.070	8.511	0.509	0.003
(0.06, 0.63, 0.31)	15.575	0.056	8.307	0.436	0.002
(0.25, 0.5, 0.25)	13.984	0.053	7.781	0.393	0.002
(0.435, 0.38, 0.185)	12.351	0.050	7.221	0.350	0.002
(0.62, 0.26, 0.12)	10.718	0.048	6.621	0.306	0.002
(0.81, 0.13, 0.06)	9.127	0.047	5.946	0.263	0.002
(1, 0, 0)	7.537	0.047	4.964	0.221	0.002
(1/3, 1/3, 1/3)	14.701	0.058	7.576	0.413	0.002

Table 4.

Summary Statistics of Retirement Benefits and IRR

under a Type-II Guarantee in S1 and the Standard Scenario

Allocation Mix (A, B, C)	Retirement Benefits			Income Replacement Ratio	
	Average	S.E.	VaR (10%)	Average	S.E.
(0, 0, 1)	72.826	0.288	25.377	2.050	0.010
(0, 0.25, 0.75)	59.566	0.224	23.183	1.679	0.008
(0, 0.5, 0.5)	46.306	0.162	20.706	1.307	0.006
(0.06, 0.63, 0.31)	35.545	0.118	17.894	1.005	0.005
(0.25, 0.5, 0.25)	30.201	0.101	15.542	0.855	0.004
(0.435, 0.38, 0.185)	24.648	0.085	13.161	0.700	0.004
(0.62, 0.26, 0.12)	19.095	0.070	10.746	0.545	0.003
(0.81, 0.13, 0.06)	13.751	0.057	8.294	0.395	0.003
(1, 0, 0)	8.407	0.048	5.444	0.245	0.002
(1/3, 1/3, 1/3)	33.673	0.118	15.862	0.953	0.005

Several implications emerge from Figures 3 and 4 and Tables 1-4. First, with the type-I guarantee, both the participant and the guarantee provider suffer from a conservative allocation strategy. Only investing in portfolio A, the (1, 0, 0) mix, creates low retirement benefits and high type-I guarantee costs. This strategy seems inadequate for the participant but may be adopted in the real world. Thaler and Johnson (1990) indicate that prior losses often reduce people's willingness to take risks, so a participant who has experienced great losses by investing in volatile portfolios may tend to allocate contributions to safer portfolios and thus place a heavier financial burden on type-I guarantee providers. However, such a conservative allocation strategy entails the least type-II guarantee costs and does the least harm to the type-II guarantee provider. This finding clearly shows that whether an allocation strategy is good or bad for the guarantee provider depends on the guarantee structure.

Perhaps the (1, 0, 0) mix is too extreme. We now explain the possibility of the other proportion mixes in the context of a type-I guarantee. Shefrin (2000) points out that hope and fear affect how investors evaluate alternatives. The participant hopes for a comfortable retirement life but fears sudden downtrends that depreciate the accumulated benefits. If this participant invests all contributions in portfolio A, the low actual benefits result in inadequate retirement life. But if he or she invests all

contributions in portfolio C, the participant faces greater risk from a downtrend. Therefore, to balance hope and fear, the participant may simultaneously select more than one portfolio, which should create sufficient retirement benefits and still protect the participant from a more negative situation. Table 3 demonstrates this scenario. Although the (0, 0.5, 0.5) mix does not produce the highest average retirement benefits, its lower 10% quantile is the highest, which supports the possibility that the participant may not allocate all contributions to portfolio C.

Second, from the participant's viewpoint, the allocation strategy of investing all contributions in portfolio C results in the highest expected retirement benefits. But for the type-II guarantee provider, the (1, 0, 0) mix entails the lowest guarantee costs, whereas (0, 0, 1) is the least favorite mix. If the guarantee does not get charged and the provider bears these costs, a conflict of interest arises between the participant and the type-II guarantee provider. To diminish guarantee costs, the type-II guarantee provider imposes limitations on the allocation proportion, such as that the participant may not allocate all savings to portfolio C, which prevents the participant from gaining the highest retirement benefits. Moreover, the type-II guarantee provider has a strong incentive to offer only portfolios with low expected rates of return and volatility to reduce risks.

We also report the average costs and retirement benefits under type-I and type-II guarantees with different θ and σ_r in Tables 5-12. All values increase with the estimate of θ (or σ_r), because high θ (or σ_r) tends to have high guaranteed rates. Furthermore, the implications mentioned previously still hold for different θ and σ_r. Tables 9 and 10 also indicate that the type-I guarantee provider and the participant both prefer portfolio C and have consistent interests only when σ_r is high. When σ_r = 0.05, the lowest average costs and the highest average retirement benefits under the type-I guarantee exist for the mixes of (0, 0.5, 0.5) and (0, 0, 1), respectively. From the expression of $\overline{R}(t, t+\delta)$, we know that low σ_r does not usually produce extremely high or low guaranteed rates. The mix of (0. 0.5, 0.5) has a high expected rate of return and moderately high volatility; therefore, the actual rates of return are usually high, and savings do not often depreciate significantly. Although the (0, 0, 1) mix offers the highest expected rate of return, its high volatility increases the probability that actual rates of return will be significantly lower than guaranteed rates. This implies that the conflict of interest between the participant and the guarantee provider may emerge under some term structures.

Table 5.

Average Type-I Guarantee Costs, Given Different θ

Allocation Mix (A, B, C)	θ				
	0.03	0.04	0.05	0.06	0.07
(0, 0, 1)	0.370	0.654	1.106	1.802	2.854
(0, 0.25, 0.75)	0.376	0.678	1.165	1.916	3.049
(0, 0.5, 0.5)	0.381	0.703	1.223	2.029	3.244
(0.06, 0.63, 0.31)	0.406	0.759	1.330	2.208	3.518
(0.25, 0.5, 0.25)	0.473	0.886	1.541	2.527	3.964
(0.435, 0.38, 0.185)	0.539	1.010	1.749	2.841	4.403
(0.62, 0.26, 0.12)	0.604	1.134	1.956	3.155	4.842
(0.81, 0.13, 0.06)	0.671	1.261	2.167	3.474	5.287
(1, 0, 0)	0.738	1.387	2.379	3.794	5.733
(1/3, 1/3, 1/3)	0.500	0.931	1.608	2.617	4.074

Notes: All parameter estimates except for θ are the same as those in the standard scenario.

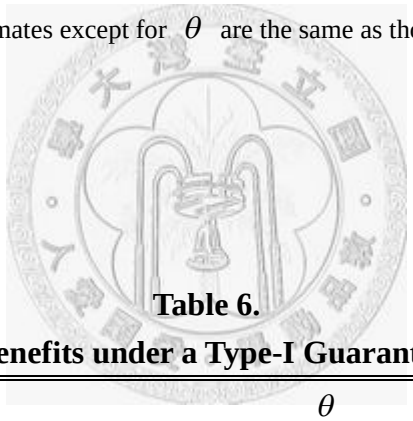


Table 6.

Average Retirement Benefits under a Type-I Guarantee, Given Different θ

Allocation Mix (A, B, C)	θ				
	0.03	0.04	0.05	0.06	0.07
(0, 0, 1)	23.998	24.282	24.734	25.430	26.482
(0, 0.25, 0.75)	20.720	21.023	21.509	22.260	23.393
(0, 0.5, 0.5)	17.442	17.763	18.284	19.090	20.304
(0.06, 0.63, 0.31)	14.651	15.004	15.575	16.452	17.763
(0.25, 0.5, 0.25)	12.916	13.329	13.984	14.970	16.407
(0.435, 0.38, 0.185)	11.141	11.612	12.351	13.443	15.005
(0.62, 0.26, 0.12)	9.366	9.895	10.718	11.916	13.603
(0.81, 0.13, 0.06)	7.631	8.220	9.127	10.434	12.247
(1, 0, 0)	5.896	6.546	7.537	8.952	10.891
(1/3, 1/3, 1/3)	13.593	14.024	14.701	15.710	17.167

Notes: All parameter estimates except for θ are the same as those in the standard scenario.

Table 7.

Average Type-II Guarantee Costs, Given Different θ

Allocation Mix (A, B, C)	θ				
	0.03	0.04	0.05	0.06	0.07
(0, 0, 1)	37.200	42.724	49.198	56.769	65.748
(0, 0.25, 0.75)	29.376	33.898	39.222	45.478	52.933
(0, 0.5, 0.5)	21.551	25.072	29.245	34.187	40.118
(0.06, 0.63, 0.31)	15.331	18.049	21.300	25.192	29.906
(0.25, 0.5, 0.25)	12.588	14.930	17.758	21.169	25.334
(0.435, 0.38, 0.185)	9.711	11.662	14.046	16.956	20.545
(0.62, 0.26, 0.12)	6.834	8.393	10.334	12.742	15.756
(0.81, 0.13, 0.06)	4.091	5.275	6.791	8.720	11.184
(1, 0, 0)	1.348	2.156	3.248	4.697	6.612
(1/3, 1/3, 1/3)	14.817	17.434	20.580	24.357	28.949

Notes: All parameter estimates except for θ are the same as those in the standard scenario.

Table 8.

Average Retirement Benefits under a Type-II Guarantee, Given Different θ

Allocation Mix (A, B, C)	θ				
	0.03	0.04	0.05	0.06	0.07
(0, 0, 1)	60.828	66.352	72.826	80.397	89.376
(0, 0.25, 0.75)	49.720	54.242	59.566	65.822	73.277
(0, 0.5, 0.5)	38.612	42.132	46.306	51.247	57.178
(0.06, 0.63, 0.31)	29.576	32.293	35.545	39.436	44.151
(0.25, 0.5, 0.25)	25.031	27.373	30.201	33.612	37.777
(0.435, 0.38, 0.185)	20.314	22.264	24.648	27.558	31.147
(0.62, 0.26, 0.12)	15.596	17.155	19.095	21.504	24.518
(0.81, 0.13, 0.06)	11.051	12.235	13.751	15.680	18.144
(1, 0, 0)	6.506	7.315	8.407	9.856	11.770
(1/3, 1/3, 1/3)	27.910	30.527	33.673	37.450	42.042

Notes: All parameter estimates except for θ are the same as those in the standard scenario.

Table 9.

Average Type-I Guarantee Costs, Given Different σ_r

Allocation Mix	σ_r					
(A, B, C)	0.05	0.12	0.13	0.14	0.15	0.16
(0, 0, 1)	0.311	0.715	0.882	1.106	1.419	1.897
(0, 0.25, 0.75)	0.287	0.748	0.927	1.165	1.490	1.982
(0, 0.5, 0.5)	0.263	0.781	0.973	1.223	1.562	2.066
(0.06, 0.63, 0.31)	0.291	0.867	1.069	1.330	1.679	2.194
(0.25, 0.5, 0.25)	0.433	1.066	1.275	1.541	1.896	2.414
(0.435, 0.38, 0.185)	0.570	1.261	1.476	1.749	2.108	2.631
(0.62, 0.26, 0.12)	0.707	1.456	1.677	1.956	2.321	2.848
(0.81, 0.13, 0.06)	0.849	1.655	1.883	2.167	2.537	3.068
(1, 0, 0)	0.991	1.854	2.088	2.379	2.753	3.289
(1/3, 1/3, 1/3)	0.505	1.139	1.345	1.608	1.959	2.474

Notes: All parameter estimates except for σ_r are the same as those in the standard scenario.

Table 10.

Average Retirement Benefits under a Type-I Guarantee, Given Different σ_r

Allocation Mix	σ_r					
(A, B, C)	0.05	0.12	0.13	0.14	0.15	0.16
(0, 0, 1)	23.939	24.343	24.510	24.734	25.047	25.525
(0, 0.25, 0.75)	20.631	21.092	21.272	21.509	21.835	22.326
(0, 0.5, 0.5)	17.323	17.842	18.033	18.284	18.622	19.127
(0.06, 0.63, 0.31)	14.536	15.111	15.314	15.575	15.924	16.438
(0.25, 0.5, 0.25)	12.876	13.509	13.717	13.984	14.339	14.857
(0.435, 0.38, 0.185)	11.172	11.863	12.078	12.351	12.710	13.233
(0.62, 0.26, 0.12)	9.469	10.217	10.439	10.718	11.082	11.609
(0.81, 0.13, 0.06)	7.809	8.615	8.843	9.127	9.497	10.028
(1, 0, 0)	6.149	7.013	7.247	7.537	7.911	8.447
(1/3, 1/3, 1/3)	13.598	14.232	14.438	14.701	15.052	15.567

Notes: All parameter estimates except for σ_r are the same as those in the standard scenario.

Table 11.

Average Type-II Guarantee Costs, Given Different σ_r

Allocation Mix	σ_r					
(A, B, C)	0.05	0.12	0.13	0.14	0.15	0.16
(0, 0, 1)	44.050	47.230	48.114	49.198	50.500	52.246
(0, 0.25, 0.75)	34.819	37.531	38.289	39.222	40.349	41.867
(0, 0.5, 0.5)	25.588	27.832	28.465	29.245	30.199	31.488
(0.06, 0.63, 0.31)	18.236	20.112	20.644	21.300	22.111	23.213
(0.25, 0.5, 0.25)	14.956	16.682	17.163	17.758	18.493	19.495
(0.435, 0.38, 0.185)	11.519	13.086	13.516	14.046	14.703	15.602
(0.62, 0.26, 0.12)	8.083	9.490	9.869	10.334	10.913	11.709
(0.81, 0.13, 0.06)	4.802	6.060	6.389	6.791	7.295	7.991
(1, 0, 0)	1.522	2.629	2.908	3.248	3.677	4.273
(1/3, 1/3, 1/3)	17.566	19.431	19.946	20.580	21.358	22.416

Notes: All parameter estimates except for σ_r are the same as those in the standard scenario.

Table 12.

Average Retirement Benefits under a Type-II Guarantee, Given Different σ_r

Allocation Mix	σ_r					
(A, B, C)	0.05	0.12	0.13	0.14	0.15	0.16
(0, 0, 1)	67.678	70.858	71.742	72.826	74.128	75.874
(0, 0.25, 0.75)	55.163	57.875	58.634	59.566	60.694	62.211
(0, 0.5, 0.5)	42.648	44.892	45.525	46.306	47.259	48.548
(0.06, 0.63, 0.31)	32.481	34.357	34.888	35.545	36.356	37.457
(0.25, 0.5, 0.25)	27.399	29.124	29.606	30.201	30.936	31.938
(0.435, 0.38, 0.185)	22.122	23.688	24.118	24.648	25.305	26.204
(0.62, 0.26, 0.12)	16.844	18.252	18.631	19.095	19.675	20.470
(0.81, 0.13, 0.06)	11.762	13.020	13.349	13.751	14.255	14.951
(1, 0, 0)	6.681	7.787	8.067	8.407	8.835	9.431
(1/3, 1/3, 1/3)	30.659	32.524	33.039	33.673	34.451	35.509

Notes: All parameter estimates except for σ_r are the same as those in the standard scenario.

3.2 Results for Strategies S2-S7 (with Modification)

Frequent modification influences both guarantee costs and retirement benefits. A participant chasing the highest expected retirement benefits distributes all savings to portfolio C, so the value derived from the allocation mix of (0, 0, 1) in S1 serves as the benchmark and we rename this allocation strategy as "All-C." Participants who adopt strategies S2-S7 select portfolio C at time 0 and then change to the other portfolio according to their criteria, as outlined in Section 2.3. Even though portfolios A and B have lower expected rates of return, participants with strategies S2-S7 may select one of these portfolios when it features the best past performance. The statistical summaries of guarantee costs and retirement benefits for different strategies are listed in Table 13 for the type-I guarantee and Table 14 for the type-II guarantee.

Consider the type-I guarantee first. The average guarantee costs for S2-S7 are all higher than those for the All-C strategy, and the risk measures of guarantee costs, including the 99% and 95% VaR and CTE, for S2-S7 are all not less than those for the All-C strategy. In addition, the average retirement benefits for S2-S7 are all lower than those for the All-C strategy. Recall that the participant may select portfolio A or B in S2-S7, which usually leads to lower actual rates of return because portfolios A and B both have lower expected rates of return and volatility. Because the guaranteed

rates are the same for all strategies, lower actual rates of return lead to lower retirement benefits and higher type-I guarantee costs.

With regard to the type-II guarantee, the average guarantee costs, risk measures of guarantee costs, and average retirement benefits for S2-S7 are all lower than those for the All-C strategy. Participants with S2-S7 may select the portfolio with lower expected rate of return and volatility, but low actual rates of return account for low actual benefits and retirement benefits. Furthermore, the magnifying effect from the type-II guarantee structure (see Section 3.1) in S2-S7 is less obvious than in the All-C strategy because the actual rates of return tend to be lower. Thus, retirement benefits decrease more than actual benefits, and guarantee costs in turn are lower for S2-S7.

It is worthwhile to discuss the implications from Tables 13 and 14 further. Some research notes that participants in DC plans seldom trade over the period under investigation. For example, Agnew et al. (2003) show that the average number of trades per year is 0.26 for a 401(k) plan. Nevertheless, the guarantee provider should not overlook the effect of trading behaviors, because even though the modifications in S5-S7 are not very frequent, the average and risk measures of type-I guarantee costs are all higher than those for the All-C strategy. Take S6 as example. The difference in

average type-I guarantee costs between strategies All-C and S6 is 0.15. If 10,000 participants join the plan with a type-I guarantee and adopt S6, and their initial yearly wages are $10,000, the difference increases to $15 million — a significant additional fiscal burden to the type-I guarantee provider. Because the All-C strategy bears the highest retirement benefits (including averages and lower 10% quantiles), a type-I guarantee provider can persuade the participant not to engage in frequent modifications. In contrast, the type-II guarantee provider hopes the participant does not maintain portfolio C because frequent modification behavior reduces guarantee costs. This again shows the conflict of interest between the participant and the type-II guarantee provider.

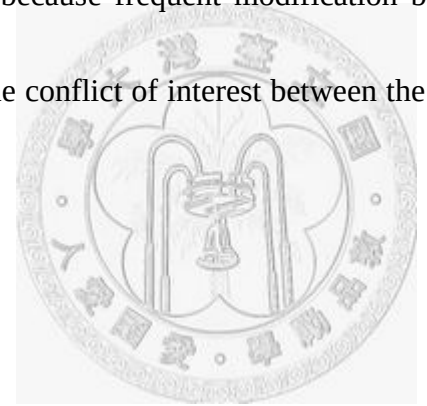


Table 13. Summary Statistics for Different Strategies in the Standard Scenario:

Type-I Guarantee

Strategy		All-C	S2	S3	S4	S5	S6	S7
Guarantee Costs	Average	1.106	1.212	1.351	1.293	1.205	1.256	1.311
	S.E.	0.043	0.043	0.044	0.043	0.043	0.043	0.044
	VaR(99%)	17.604	19.220	20.594	20.000	18.850	19.319	19.239
	VaR(95%)	4.128	4.804	5.488	5.134	4.694	4.897	5.312
	CTE(99%)	50.084	52.356	54.295	53.009	51.878	52.695	52.879
	CTE(95%)	16.349	17.650	18.895	18.148	17.474	17.843	18.179
Retirement Benefits	Average	24.734	20.609	15.909	12.431	21.125	20.027	11.900
	S.E.	0.118	0.104	0.071	0.056	0.107	0.096	0.056
	VaR(10%)	7.407	6.756	6.388	5.777	6.801	6.631	5.700
IRR	Average	0.684	0.572	0.445	0.353	0.586	0.556	0.338
	S.E.	0.004	0.003	0.003	0.002	0.003	0.003	0.002
Average Actual Benefits		23.628	19.397	14.558	11.138	19.919	18.771	10.589
Percentage of AP for A		0	2.8	14.0	31.6	1.4	5.1	34.1
Percentage of AP for B		0	31.8	35.0	32.8	26.7	22.9	28.8
Percentage of AP for C		100	65.4	50.9	35.6	71.9	72.0	37.1
Average Modification Frequency		0	5.032	24.890	24.434	1.182	2.164	3.454

Notes: Average actual benefits are derived by subtracting average guarantee costs from average retirement benefits. Percentage of AP for A (B, C) represents the percentage of the accumulation phase to hold portfolio A (B, C). Average modification frequency means the average number of times the portfolio is modified during the accumulation phase.

Table 14. Summary Statistics for Different Strategies in the Standard Scenario: Type-II Guarantee

Strategy		All-C	S2	S3	S4	S5	S6	S7
Guarantee Costs	Average	49.198	34.133	21.531	14.552	35.717	31.851	14.185
	S.E.	0.219	0.190	0.118	0.121	0.199	0.170	0.128
	VaR(99%)	220.085	184.091	106.972	102.173	194.556	169.563	113.919
	VaR(95%)	124.101	98.564	55.036	44.384	103.915	88.337	47.351
	CTE(99%)	351.732	292.212	183.911	184.937	307.646	268.251	207.927
	CTE(95%)	195.763	160.518	94.143	86.810	168.921	146.018	96.459
Retirement Benefits	Average	72.826	53.530	36.089	25.690	55.636	50.622	24.773
	S.E.	0.288	0.252	0.148	0.137	0.264	0.226	0.147
	VaR(10%)	25.377	14.958	14.236	9.590	15.175	15.501	7.829
IRR	Average	2.050	1.509	1.022	0.745	1.569	1.430	0.718
	S.E.	0.010	0.009	0.005	0.005	0.009	0.008	0.006
Average Actual Benefits		23.628	19.397	14.558	11.138	19.919	18.771	10.589
Percentage of AP for A		0	2.8	14.0	31.6	1.4	5.1	34.1
Percentage of AP for B		0	31.8	35.0	32.8	26.7	22.9	28.8
Percentage of AP for C		100	65.4	50.9	35.6	71.9	72.0	37.1
Average Modification Frequency		0	5.032	24.890	24.434	1.182	2.164	3.454

Notes: Average actual benefits are derived by subtracting average guarantee costs from average retirement benefits. Percentage of AP for A (B, C) represents the percentage of the accumulation phase to hold portfolio A (B, C). Average modification frequency means the average number of times the portfolio is modified during the accumulation phase.

Before completing this section, we analyze how average guarantee costs and retirement benefits vary with different θ, σ_r, stop-gain levels (*G*), stop-loss levels (*L*), and transaction costs (*f*), while keeping all the other parameter estimates in the standard scenario unchanged. The numerical results appear in Tables 15-19. We have several findings:

(i) As shown in Tables 15 and 16, all values are increasing functions of θ (or σ_r), because high θ (or σ_r) tends to produce high guaranteed rates.

(ii) The high level of *G* has several effects: (i) The participant modifies the portfolio infrequently and holds portfolio C often, leading to high actual rates of return, and (ii) realized huge losses in some periods are not easily offset by infrequently realized gains (even if their accumulation is great), which leads to low actual benefits. These effects work together and counteract each other; thus, guarantee costs and retirement benefits are not very sensitive to *G*, as Table 17 reports.

(iii) Many effects emerge from lower *L*, but primarily, the participant with lower *L* modifies the portfolio less frequently and realizes gains more often than losses, given that $G = 0.2$. This tendency is helpful to wealth accumulation and leads to high actual benefits and retirement benefits. Therefore, we observe in Table 18 that as *L* deceases (becomes more negative), retirement benefits increase, and type-I guarantee costs decrease; for the type-II guarantee, the magnifying effect

makes retirement benefits increase more so that guarantee costs also increase.

(iv) As Table 19 shows, when *f* increases, type-I guarantee costs also increase but retirement benefits decrease. From equations (11)-(15), we know that higher *f* causes lower actual rates of return and lower actual benefits. The multi-period binding property in the type-II guarantee makes retirement benefits decrease more, so guarantee costs decrease.

Table 15.

Average Guarantee Costs and Retirement Benefits, Given Different θ and Strategies

Type-I Guarantee							
θ	Average Guarantee Costs in Each Strategy						
	All-C	S2	S3	S4	S5	S6	S7
0.03	0.370	0.392	0.436	0.411	0.388	0.411	0.411
0.04	0.654	0.708	0.789	0.753	0.703	0.738	0.759
0.05	1.106	1.212	1.351	1.293	1.205	1.256	1.311
0.06	1.802	1.988	2.208	2.115	1.980	2.045	2.146
0.07	2.854	3.156	3.483	3.326	3.143	3.224	3.371
θ	Average Retirement Benefits in Each Strategy						
	All-C	S2	S3	S4	S5	S6	S7
0.03	23.998	19.789	14.993	9.951	20.307	19.182	9.227
0.04	24.282	20.105	15.347	11.113	20.622	19.509	10.465
0.05	24.734	20.609	15.909	12.431	21.125	20.027	11.900
0.06	25.430	21.385	16.766	14.023	21.900	20.817	13.564
0.07	26.482	22.553	18.041	15.943	23.062	21.995	15.605
Type-II Guarantee							
θ	Average Guarantee Costs in Each Strategy						
	All-C	S2	S3	S4	S5	S6	S7
0.03	37.200	25.355	15.130	7.316	26.570	23.382	6.697
0.04	42.724	29.378	18.033	10.535	30.761	27.245	9.980
0.05	49.198	34.133	21.531	14.552	35.717	31.851	14.185
0.06	56.769	39.735	25.755	19.533	41.558	37.321	19.383
0.07	65.748	46.449	30.888	25.714	48.555	43.913	25.887
θ	Average Retirement Benefits in Each Strategy						
	All-C	S2	S3	S4	S5	S6	S7
0.03	60.828	44.752	29.688	16.856	46.490	42.153	15.513
0.04	66.352	48.775	32.591	20.895	50.680	46.016	19.686
0.05	72.826	53.530	36.089	25.690	55.636	50.622	24.773
0.06	80.397	59.132	40.313	31.442	61.478	56.092	30.801
0.07	89.376	65.846	45.446	38.331	68.474	62.684	38.121

Table 16.

Average Guarantee Costs and Retirement Benefits, Given Different σ_r and Strategies

Type-I Guarantee							
σ_r	Average Guarantee Costs in Each Strategy						
	All-C	S2	S3	S4	S5	S6	S7
0.12	0.715	0.791	0.903	0.852	0.787	0.834	0.862
0.13	0.882	0.972	1.098	1.043	0.967	1.015	1.054
0.14	1.106	1.212	1.351	1.293	1.205	1.256	1.311
0.15	1.419	1.540	1.693	1.632	1.533	1.584	1.656
0.16	1.897	2.034	2.201	2.137	2.024	2.077	2.166
σ_r	Average Retirement Benefits in Each Strategy						
	All-C	S2	S3	S4	S5	S6	S7
0.12	24.343	20.187	15.461	12.304	20.707	19.605	11.776
0.13	24.510	20.369	15.656	12.346	20.887	19.786	11.819
0.14	24.734	20.609	15.909	12.431	21.125	20.027	11.900
0.15	25.047	20.937	16.251	12.643	21.452	20.356	12.082
0.16	25.525	21.430	16.759	13.011	21.944	20.849	12.442
Type-II Guarantee							
σ_r	Average Guarantee Costs in Each Strategy						
	All-C	S2	S3	S4	S5	S6	S7
0.12	47.230	32.552	20.222	13.787	34.072	30.265	13.319
0.13	48.114	33.260	20.808	14.100	34.808	30.978	13.715
0.14	49.198	34.133	21.531	14.552	35.717	31.851	14.185
0.15	50.500	35.182	22.425	15.177	36.807	32.900	14.845
0.16	52.246	36.611	23.643	16.135	38.299	34.342	15.840
σ_r	Average Retirement Benefits in Each Strategy						
	All-C	S2	S3	S4	S5	S6	S7
0.12	70.858	51.948	34.780	25.239	53.991	49.036	24.233
0.13	71.742	52.657	35.366	25.403	54.727	49.749	24.480
0.14	72.826	53.530	36.089	25.690	55.636	50.622	24.773
0.15	74.128	54.578	36.983	26.187	56.726	51.671	25.270
0.16	75.874	56.008	38.201	27.009	58.219	53.114	26.116

Table 17.

Average Guarantee Costs and Retirement Benefits,
Given Different Stop-Gain Levels (*G*) and Strategies

		Average Guarantee Costs			Average Retirement Benefits		
	G	S5	S6	S7	S5	S6	S7
	0.1	1.203	1.236	1.291	21.059	19.872	12.001
Type-I	0.15	1.206	1.257	1.305	21.096	19.967	11.870
Guarantee	0.2	1.205	1.256	1.311	21.125	20.027	11.900
	0.25	1.208	1.264	1.312	21.159	20.109	12.039
	0.3	1.211	1.282	1.334	21.246	20.134	12.169
	G	S5	S6	S7	S5	S6	S7
	0.1	35.429	31.506	14.024	55.285	50.142	24.734
Type-II	0.15	35.562	31.609	13.920	55.452	50.319	24.485
Guarantee	0.2	35.717	31.851	14.185	55.636	50.622	24.773
	0.25	35.833	32.189	14.746	55.783	51.034	25.473
	0.3	36.143	32.254	15.120	56.178	51.105	25.956

Table 18.

Average Guarantee Costs and Retirement Benefits,
Given Different Stop-Loss Levels (*L*) and Strategies

		Average Guarantee Costs			Average Retirement Benefits		
	L	S5	S6	S7	S5	S6	S7
	-0.3	1.216	1.310	1.343	20.753	18.642	11.544
Type-I	-0.35	1.210	1.284	1.331	20.934	19.438	11.751
Guarantee	-0.4	1.205	1.256	1.311	21.125	20.027	11.900
	-0.45	1.200	1.237	1.296	21.272	20.476	12.023
	-0.5	1.195	1.219	1.286	21.379	20.837	12.123
	L	S5	S6	S7	S5	S6	S7
	-0.3	34.408	27.088	12.833	53.945	44.419	23.033
Type-II	-0.35	35.078	29.714	13.557	54.802	47.868	23.977
Guarantee	-0.4	35.717	31.851	14.185	55.636	50.622	24.773
	-0.45	36.277	33.633	14.748	56.349	52.872	25.475
	-0.5	36.733	35.115	15.203	56.918	54.733	26.040

Table 19.

Average Guarantee Costs and Retirement Benefits,
Given Different Transaction Costs (*f*) and Strategies

Type-I Guarantee							
f	Average Guarantee Costs						
	All-C	S2	S3	S4	S5	S6	S7
0	1.106	1.212	1.351	1.293	1.205	1.256	1.311
0.002	1.108	1.222	1.397	1.335	1.212	1.264	1.321
0.004	1.111	1.231	1.443	1.380	1.218	1.271	1.327
0.006	1.113	1.240	1.491	1.426	1.222	1.279	1.337
0.008	1.116	1.249	1.540	1.473	1.226	1.286	1.344
0.01	1.118	1.259	1.591	1.523	1.231	1.293	1.357
f	Average Retirement Benefits						
	All-C	S2	S3	S4	S5	S6	S7
0	24.734	20.609	15.909	12.431	21.125	20.027	11.900
0.002	24.689	20.528	15.499	12.134	21.071	19.913	11.850
0.004	24.645	20.447	15.105	11.850	21.013	19.854	11.805
0.006	24.600	20.366	14.729	11.579	20.959	19.745	11.755
0.008	24.555	20.287	14.368	11.321	20.904	19.639	11.692
0.01	24.510	20.208	14.023	11.075	20.844	19.537	11.631
Type-II Guarantee							
f	Average Guarantee Costs						
	All-C	S2	S3	S4	S5	S6	S7
0	49.198	34.133	21.531	14.552	35.717	31.851	14.185
0.002	49.154	34.100	21.497	14.505	35.643	31.719	14.152
0.004	49.111	34.068	21.466	14.463	35.580	31.616	14.122
0.006	49.068	34.037	21.436	14.424	35.509	31.479	14.099
0.008	49.025	34.008	21.409	14.389	35.455	31.350	14.070
0.01	48.983	33.979	21.383	14.358	35.380	31.253	14.031
f	Average Retirement Benefits						
	All-C	S2	S3	S4	S5	S6	S7
0	72.826	53.530	36.089	25.690	55.636	50.622	24.773
0.002	72.735	53.406	35.599	25.304	55.503	50.368	24.681
0.004	72.644	53.284	35.128	24.933	55.376	50.199	24.600
0.006	72.554	53.164	34.674	24.578	55.246	49.945	24.518
0.008	72.464	53.045	34.237	24.237	55.133	49.703	24.418
0.01	72.375	52.928	33.815	23.910	54.993	49.497	24.306

4. Conclusions

This study analyzes whether a participant's investment strategies and behaviors influence guarantee costs and retirement benefits in defined contribution (DC) pension plans. Many previous articles show that participants in DC plans have special allocation methods and investment behaviors. We refer to those studies to establish seven strategies. The numerical results from our investigation indicate that guarantee costs and retirement benefits vary with investment strategies and that the participant and the guarantee provider may face a conflict of interest (depending on the payoff structure of the guarantee and the interest rate dynamics). In some scenarios, the provider has an incentive to offer portfolios with low expected rates of return and volatility to reduce risks. Such offerings cannot provide high retirement benefits to the participant. In addition, compared with the All-C strategy, frequent modifications hurt both the participant and the type-I guarantee provider because they cause low retirement benefits and high type-I guarantee costs.

We acknowledge the influence of several important assumptions in this paper. First, we assume the plan provider allows the participant to modify the investment portfolio during the accumulation phase, which is consistent with many DC plans in the real world. Second, a participant who adopts any modification strategy selects the new

portfolio with the best past performance among the alternative portfolios in our model. This selection approach makes sense, because the participant has incentive to pursue a portfolio with strong performance. Third, selling decisions in S5-S7 are controlled by the disposition effect, which requires further confirmation with regard to the DC plan embedded with rate of return guarantees, though Dhar and Zhu (2006) document that the disposition effect exists among the investors with IRA or Keogh accounts. However, this assumption does not reduce the contribution of this research. Our findings suggest that the guarantee provider must consider carefully how participants behave. Further research also might uncover different behavioral patterns, other than the disposition effect. Guarantee providers should incorporate these patterns as well when estimating their guarantee costs and risks.

References

Agnew, J. R. 2006. “Do Behavioral Biases Vary across Individuals? Evidence from Individual Level 401(k) Data.” *Journal of Financial and Quantitative Analysis* 41(4): 939-962.

Agnew, J., Balduzzi, P. and Sundén, A. 2003. “Portfolio Choice and Trading in a Large 401(k) Plan.” *The American Economic Review* 93(1): 193-215.

Barber, B.M. and Odean, T. 2000. “Trading Is Hazardous to Your Wealth: The Common Stock Investment Performance of Individual Investors.” *Journal of Finance* 55(2): 773-806.

Barber, B.M. and Odean, T. 2003. “Are Individual Investors Tax Savvy? Evidence from Retail and Discount Brokerage Accounts.” *Journal of Public Economics* 88: 419-442.

Battocchio, P. and Menoncin, F. 2004. “Optimal Pension Management in a Stochastic Framework.” *Insurance: Mathematics and Economics* 34: 79–95.

Benartzi, S. and Thaler, R.H. 2001. “Naïve Diversification Strategies in Defined Contribution Saving Plans.” *The American Economic Review* 91(1): 79-98.

Blake, D., Cairns, A.J.G. and Dowd, K. 2001. “Pensionmetrics: Stochastic Pension Plan Design and Value-at-Risk during the Accumulation Phase.” *Insurance: Mathematics and Economics* 29: 187-215.

Boulier, J., Huang, S. and Taillard, G. 2001. "Optimal Management under Stochastic Interest Rates: the Case of a Protected Defined Contribution Pension Fund." *Insurance: Mathematics and Economics* 28: 173–189.

Boyle, P.P. and Hardy, M.R. 1997. "Reserving for Maturity Guarantees; Two Approaches." *Insurance: Mathematics and Economics* 21: 113–127.

Brennan, J.M. and Schwartz, E.S. 1976. "The Pricing of Equity-Linked Life Insurance Policies with an Asset Value Guarantee." *Journal of Financial Economics* 3: 195-213.

Cox, J.C., Ingersoll, J.E. and Ross, S.A. 1985. "A Theory of the Term Structure of Interest Rates." *Econometrica* 53: 385-407.

Dhar, R. and Zhu, N. 2006. "Up Close and Personal: Investor Sophistication and the Disposition Effect." *Management Science* 52(5): 726-740.

Dowd, K. and Blake, D. 2006. "After VaR: The Theory, Estimation, and Insurance Applications of Quantile-Based Risk Measures." *Journal of Risk & Insurance* 73(2): 193-229.

Ekern, S. and Persson, S.A. 1996. "Exotic Equity-linked Life Insurance Contracts." *The Geneva Papers on Risk and Insurance Theory* 21: 35-63.

Genesove, D. and Mayer, C. 2001. "Loss Aversion and Seller Behavior: Evidence from the Housing Market." *Quarterly Journal of Economics* 116: 1233−1260.

Glasserman, P. 2004. *Monte Carlo Methods in Financial Engineering*. Springer-Verlag, New York, USA.

Grinblatt, M. and Keloharju, M. 2001. "What Makes Investors Trade?" *Journal of Finance* 56(2): 589-616.

Hansen, M. and Miltersen, K.R. 2002. "Minimum Rate of Return Guarantees: the Danish Case." *Scandinavian Actuarial Journal* 4: 280-318.

Hardy, M. 2003. *Investment Guarantees: Modeling and Risk Management for Equity-Linked Life Insurance*. John Wiley & Sons, Inc., New Jersey, USA.

Huberman, G. and Jiang, W. 2006. "Offering versus Choice in 401(k) Plans: Equity Exposure and Number of Funds." *Journal of Finance* 61(2): 763-801.

Kahneman, D. and Tversky, A. 1979. "Prospect Theory: An Analysis of Decision under Risk." *Econometrica* 47: 263-292.

Lindset, S. 2004. "Relative Guarantees." *The Geneva Papers on Risk and Insurance Theory* 29: 187-209.

Locke, P.R. and Mann, S.C. 2005. "Professional trader discipline and trade disposition." *Journal of Financial Economics* 76: 401-444.

Menoncin, F. 2002. "Optimal Portfolio and Background Risk: an Exact and an Approximated Solution." *Insurance: Mathematics and Economics* 31: 249–265.

Miao, W.C. 2003. “Quadratic Variation Estimators for Diffusion Models in Finance.” Working Paper.

Miltersen, K.R. and Persson, S.A. 1999. “Pricing Rate of Return Guarantee in a Heath-Jarrow-Morton Framework.” *Insurance: Mathematics and Economics* 25: 307-325.

Mitchell, O. and Smetters, K. 2003. *The Pension Challenge: Risk Transfers and Retirement Income Security*. Oxford University Press, New York, USA.

Mitchell, O.S. and Utkus, S.P. 2004. “Lessons from Behavioral Finance for Retirement Plan Design.” from: Mitchell, O.S. and Utkus, S.P. 2004. *Pension Design and Structure: New Lessons from Behavioral Finance*, Chapter 1. Oxford University Press, New York, USA.

Odean, T. 1998. “Are Investors Reluctant to Realize Their Losses?” *Journal of Finance* 53(5): 1775-1798.

Pennacchi, G. G. 1999. “The Value of Guarantees on Pension Fund Returns.” *Journal of Risk and Insurance* 66(2): 219-237.

Persson, S.A., and Aase, K.K. 1997. “Valuation of the Minimum Guaranteed Return Embedded in Life Insurance Products.” *Journal of Risk and Insurance* 64(4): 599-617.

Scott, L.O. 1996. “Simulating a Multi-factor Term Structure Model over Relatively Long Discrete Time Periods.” in *Proceedings of the IAFE First Annual Computational Finance Conference*, Graduate School of Business, Stanford University.

Sharpe, W.F. 1966. “Mutual Fund Performance.” *Journal of Business* 39(1): 119-138.

Shefrin, H. 2000. *Beyond Greed and Fear: Understanding Behavioral Finance and the Psychology of Investing*. Harvard Business School Press, Boston, USA.

Shefrin, H. and Statman, M. 1985. “The Disposition to Sell Winners Too Early and Ride Losers Too Long: Theory and Evidence.” *Journal of Finance* 40(3): 777-790.

Sherris, M. 1995. “The Valuation of Option Features in Retirement Benefits.” *Journal of Risk and Insurance* 62(3): 509-535.

Shu, P.G., Yeh, Y.H., Chiu, S.B. and Chen, H.C. 2005. “Are Taiwanese Individual Investors Reluctant to Realize Their Losses?” *Pacific-Basin Finance Journal* 13(2): 201-223.

Thaler, R.H. and Johnson, E.J. 1990. “Gambling with the House Money and Trying to Break Even: The Effect of Prior Outcomes on Risky Choice.” *Management Science* 36(6): 643-660.

Tversky, A. and Kahneman, D. 1974. “Judgment under Uncertainty: Heuristics and Biases.” *Science* 185: 1124-1131.

Vasicek, O. 1977. “An Equilibrium Characterization of Term Structure.” *Journal of Financial Economics* 5: 177-188.

Vigna, E. and Haberman, S. 2001. “Optimal Investment Strategy for Defined Contribution Pension Schemes.” *Insurance: Mathematics and Economics* 28, 233-262.

Voronkova, S. and Bohl, M.T. 2005. “Institutional Traders Behavior in an Emerging Stock Market: Empirical Evidence on Polish Pension Fund Investors.” *Journal of Business Finance and Accounting* 32(7&8): 1537-1560.

Zweig, J. 1998. “Five Investing Lessons from America’s Top Pension Fund.” *Money* 27(1): 115-18.

Appendix

(A) Details of Strategies S5-S7

In this appendix, we illustrate how participants who choose S5-S7 determine the reference price if the portfolio has not changed for a long time. We refer to the date the participant buys the current portfolio as "*the purchase date*." Suppose that the actual rate of return on the current portfolio *i* since the purchase date moves above the stop-gain level *G* or falls below the stop-loss level *L* at time τ, and portfolio *i* happens to have the best past performance. Even though the participant does not modify the current portfolio, he or she has experienced large gains or losses so a new reference price is reset as the time-τ price of portfolio *i*, or equivalently, $S_{\tau}^{(i)}$. Then, one of the sufficient conditions to modify the portfolio at time $\hat{\tau} > \tau$ is $\ln[(1-f)S_{\hat{\tau}}^{(i)} / S_{\tau}^{(i)}] > G > 0$, or $\ln[(1-f)S_{\hat{\tau}}^{(i)} / S_{\tau}^{(i)}] < L < 0$, where f is the parameter for transaction costs.

(B) Simulation Procedures

Step 1: Given the parameter estimates, simulate the two-year spot rate at the end of each year and the price of each portfolio at the end of each month, according to equations (9) and (10), respectively. For S4 and S7, also simulate the one-month spot rate at the beginning of each month to calculate the Sharpe

index. We do not need to consider the discretization error here.

Step 2: For strategies S2-S7, measure the past performance of each portfolio in the middle and at the end of each year. According to Section 2.3, past performance may be measured by actual rate of return since inception, one-year-to-date actual rate of return, or one-year-to-date Sharpe index. For strategy S1, we skip this step.

Step 3: The participant makes modification decisions and allocates all savings to the portfolio with the best past performance. However, the participant makes no modification when the current portfolio achieves the best past performance in S2-S7 or when the rate of return measured with respect to the reference price does not reach *G* or *L* in S5-S7. If modification occurs in the middle of the year, we record the actual rate of return during the past six months. For strategy S1, we skip this step.

Step 4: At the end of each year, compute the actual rate of return (R_t) according to equations (11)-(15), depending on whether modification occurs in each year.

Step 5: At time *T*, calculate $\Gamma_T(I)$, $V_T(I)$, $\Gamma_T(II)$, and $V_T(II)$ for this path.

Step 6: Repeat steps 1-5 for 50,000 paths. Calculate the arithmetic average of $\Gamma_T(I)$, $V_T(I)$, $\Gamma_T(II)$, and $V_T(II)$ over the 50,000 paths and derive the average values. Risk measures are also derived from these simulated data.

(C) Estimating Interest Rate Parameters

To estimate interest rate parameters, we employ the monthly rates of commercial papers maturing after 1-30 days traded in the primary market in Taiwan from May 1985 to April 2006. The data come from the *Taiwan Economic Journal* (*TEJ*) Database. We refer to Miao (2003) to estimate the interest rate parameters in the CIR model. Assume that the rates in the data set are r_1, r_2, …, r_{240}, then, the estimate of σ_r^2 is

$$\hat{\sigma}_r^2 = \frac{\sum_{j=1}^{239}(r_{j+1} - r_j)^2}{\Delta \cdot \sum_{j=1}^{239} r_j}$$

where Δ is the time interval, which we set to Δ=1/12. Taking the square root of $\hat{\sigma}_r^2$, we derive the estimate of σ_r.

To estimate κ and θ, we have to consider the stochastic differential equation in the CIR model, $dr_t = \kappa(\theta - r_t)dt + \sigma_r\sqrt{r_t}dW_t$. The discrete-time form of this equation is expressed as $r_{t+\Delta} - r_t = \kappa(\theta - r_t)\Delta + \sigma_r\sqrt{r_t}\cdot\varepsilon_t\sqrt{\Delta}$, or equivalently, $r_{t+\Delta}/\left(\sigma_r\cdot\sqrt{r_t\Delta}\right) = \kappa\theta\left(\left(\sqrt{\Delta/r_t}\right)/\sigma_r\right) + (1-\kappa\Delta)\left(\left(\sqrt{r_t/\Delta}\right)/\sigma_r\right) + \varepsilon_t$. Let $\beta_1 = \kappa\theta$ and $\beta_2 = 1-\kappa\Delta$. We apply an ordinary least squares regression to estimate β_1 and β_2. In the regression model, $\left(\sqrt{\Delta/r_t}\right)/\sigma_r$ and $\left(\sqrt{r_t/\Delta}\right)/\sigma_r$ represent the independent variables, and $r_{t+\Delta}/\left(\sigma_r\cdot\sqrt{r_t\Delta}\right)$ is the dependent variable. The intercept term is zero,

and ε_t is the residual. We derive estimates of β_1 and β_2, $\hat{\beta}_1 = \hat{\kappa}\hat{\theta}$ and $\hat{\beta}_2 = 1 - \hat{\kappa}\Delta$, which implies that $\hat{\kappa} = (1 - \hat{\beta}_2)/\Delta$ and $\hat{\theta} = \hat{\beta}_1 / \hat{\kappa} = \hat{\beta}_1 \Delta / (1 - \hat{\beta}_2)$.

(D) Estimating Expected Rate of Return and Volatility of Each Portfolio

We obtain data from the *TEJ* database and use the end-of-year values of open-ended funds that persist from 1995 to the end of 2005 to estimate the expected annual rate of return (μ) and annualized volatility (σ_S). The results show that the expected rates of return on the funds with low volatility lie between 0.038 and 0.04, and their volatility ranges from 0.018 to 0.021. We therefore set μ = 0.04 and σ_S = 0.02 as the expected rate of return and volatility for portfolio A, which represents a suitable portfolio for conservative participants. Furthermore, we find that one fund investing in Asian securities reveals an expected rate of return approximately equal to 0.12 and volatility roughly equal to 0.2. Therefore, we set μ = 0.12 and σ_S = 0.2 as the expected rate of return and volatility for portfolio C, the portfolio suitable for an aggressive participant. We also establish that for portfolio B, μ = 0.08 and σ_S = 0.1 to denote a portfolio with moderate return rate and volatility. Some other funds have very high volatility (about 0.4) and very high expected rates of return (about 0.2), but because these funds are too risky, we do not take them into account.