

# Robust stability analysis for uncertain delay systems with output feedback controller

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**Abstract:** A systematic approach is given in this paper for analyzing the robust stability of uncertain time-delay systems controlled by output feedback. By checking the eigenvalues of a Hamiltonian matrix, the stability of nominal systems can be examined first. Then, for the nominally stable uncertain systems with multiple time delays, a new method using structured singular value technique is proposed for finding a set of uncertain parameters within which the systems remain stable. Moreover, an illustrative example is given to show the usefulness of the proposed approach.

**Keywords:** Time-delay system; output feedback; Hamiltonian matrix; robust stability; structured singular value.

## 1. Introduction

For control systems with both time delays and uncertainties, the stability analysis problem is usually complicated. Recently, several robust stability test methods are proposed for such systems with various types of uncertainties [2, 9, 12, 13, 16]. However, to consider uncertain time-delay systems controlled by output feedback controller, these methods cannot be applied directly because when there are uncertainties in the input and output matrices of the open-loop system, multiplicative terms of uncertainties will appear in the resulting closed-loop system. For the case of cone bounded uncertainties, i.e. uncertainties of which norms are bounded by that of the state vector, Chou et al. [3] extend the results of [16] to deal with systems equipped with dynamical output feedback controller. For the case of structured uncertainties in the system matrix, of which the modulus matrix is bounded by a given constant matrix, a time domain approach is proposed by Yu et al. [15] to check the robust stability of systems with static output feedback controller. However, when there are linearly dependent parametric uncertainties, the multiplicative terms turn out to be quadratic functions of the uncertain parameters, and these methods still lead to conservative results.

In this paper, the robust stability problem of the case with quadratically coupled parametric uncertainties is investigated by means of the Lyapunov functional approach [7]. For uncertain systems with multiple time delays, we start by deriving a sufficient condition to check whether the uncertain system is nominally stable. If the nominal systems are stable, we proceed to study the derivative of a Lyapunov functional candidate of the uncertain systems. We use a transformation method which is equivalent to the completion of square in the associated Lyapunov derivative and a method based on the structured singular value (henceforth denoted as  $\mu$ ) technique [1, 4] to find a set of uncertainties within which the derivative is negative definite. Then, an example is given to illustrate the usefulness of the approach.

The notations adopted are as follows. Let  $h$  be a positive constant.  $x_t(\sigma) = x(t + \sigma)$  for  $-h \leq \sigma \leq 0$ .  $\|x\|_f = \sup_{-h \leq \sigma \leq 0} \|x(\sigma)\|$ , where  $\|\cdot\|$  is the Euclidean norm.  $C_h$  is the Banach space of continuous vector functions  $\eta: [-h, 0] \rightarrow \mathbb{R}^n$  with  $\|\cdot\|_f$ .  $\otimes$  refers to the Kronecker product, and  $I_\delta$  is the identity matrix of dimension  $\delta$ .

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**2. Stability analysis of uncertain output feedback systems with multiple time delays**

Consider the uncertain system ( $\Sigma_m$ ) described by the following functional differential equation and controlled by output feedback

$$\begin{aligned} \dot{x}(t) &= \left( A_0 + \sum_{i=1}^m k_i A_i \right) x(t) + \left( D_0 + \sum_{i=1}^m k_i D_i \right) x(t-h) \\ &\quad + \sum_{l=1}^q \sum_{i=1}^m k_i F_{il} x(t-h_l(t)) + \left( B_0 + \sum_{i=1}^m k_i B_i \right) u(t), \\ y(t) &= \left( C_0 + \sum_{i=1}^m k_i C_i \right) x(t), \\ u(t) &= Ly(t), \\ x(t) &= \phi(t) \quad \forall t \in [-h_{\max}, 0], \end{aligned}$$

where  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^p$  and  $y \in \mathbb{R}^r$  are the state, input, and output vectors, respectively,  $L \in \mathbb{R}^{p \times r}$  is the output feedback gain matrix,  $h > 0$  is the constant delay time,  $h_l(t)$ ,  $l = 1, \dots, q$ , are time-varying bounded delay times satisfying  $0 \leq h_l(t) \leq h_l < \infty$  and  $\dot{h}_l(t) \leq d_l < 1$ ,  $\phi(t) \in C_{h_{\max}}$ ,  $h_{\max} = \max\{h, \max_l h_l\}$ ,  $d_{\max} = \max_l d_l$ , and  $k_i \in \mathbb{R}$ ,  $i = 1, \dots, m$ , are the time-varying uncertain parameters satisfying  $\max_i |k_i| < \gamma$ . In addition, the matrices  $A_i$ ,  $B_i$ ,  $C_i$ ,  $D_i$ , and  $F_{il}$  of suitable dimensions are the structural influence matrices of  $k_i$ , and are assumed to be known. Thus, the closed-loop system can be written as

$$\begin{aligned} \dot{x}(t) &= \left( \bar{A}_0 + \sum_{i=1}^m k_i E_i + \sum_{i,j=1}^m k_i k_j E_{ij} \right) x(t) + \left( D_0 + \sum_{i=1}^m k_i D_i \right) x(t-h) \\ &\quad + \sum_{i=1}^q \sum_{i=1}^m k_i F_{il} x(t-h_l(t)), \end{aligned} \tag{1}$$

where  $\bar{A}_0 \triangleq A_0 + B_0 LC_0$ ,  $E_i \triangleq B_0 LC_i + B_i LC_0$ , and  $E_{ij} \triangleq B_i LC_j$ . The objectives of the following discussions are to derive a stability condition for the nominal system, and to find how large  $\gamma$  can be if the stability of the closed-loop uncertain system is to be preserved.

Consider the nominal system ( $\Sigma_0$ ),

$$\dot{x}(t) = \bar{A}_0 x(t) + D_0 x(t-h),$$

which is the system ( $\Sigma_m$ ) with  $k_i = 0$  for all  $i$ . We have the following theorem.

**Theorem 1.** *If  $\bar{A}_0$  is a stable matrix and for some  $\alpha > 0$*

$$H = \begin{bmatrix} \bar{A}_0 & D_0 D_0^T \\ -(\alpha + 1)I_n & -\bar{A}_0^T \end{bmatrix} \tag{2}$$

*has no eigenvalues on the  $j\omega$ -axis, then the system  $\Sigma_0$  is asymptotically stable.*

**Proof.** From the assumptions and Lemma 4 in [5], it is known that the algebraic Riccati equation (ARE)

$$\bar{A}_0^T P + P \bar{A}_0 + P D_0 D_0^T P + (\alpha + 1)I_n = 0 \tag{3}$$

has a positive-definite solution  $P$  for some  $\alpha > 0$ . Consider the Lyapunov functional candidate

$$V(t, x_t) = x^T(t) P x(t) + \int_{t-h}^t x^T(\sigma) x(\sigma) d\sigma. \tag{4}$$

Then the Lyapunov derivative  $\dot{V}(t, x_t)$  is

$$\begin{aligned} & x^T(t)(\bar{A}_0^T P + P\bar{A}_0 + I_n)x(t) + 2x(t)^T P D_0 x(t-h) - x^T(t-h)x(t-h) \\ &= [x^T(t) \ x^T(t-h)] \begin{bmatrix} \bar{A}_0^T P + P\bar{A}_0 + I_n & P D_0 \\ D_0^T P & -I_n \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix} \\ &= [x^T(t) \ x^T(t-h)] \begin{bmatrix} I_n & -P D_0 \\ 0 & I_n \end{bmatrix} \\ &\quad \times \begin{bmatrix} \bar{A}_0^T P + P\bar{A}_0 + P D_0 D_0^T P + I_n & 0 \\ 0 & -I_n \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -D_0^T P & I_n \end{bmatrix} \begin{bmatrix} x(t) \\ x(t-h) \end{bmatrix}. \end{aligned}$$

Since  $P$  is the solution of ARE (3), we have

$$\dot{V}(t, x_t) \leq -\alpha \|x(t)\|^2.$$

Thus, we conclude this theorem by using the Lyapunov stability theorem [7, p. 105].

From Theorem 1, it is known that the stability of the nominal system  $\Sigma_0$  can be guaranteed if ARE (3) has a positive-definite solution. However, in the presence of uncertainties and time-varying delays, it is intuitively clear that the stability condition would be more stringent. To proceed, we assume the nominal system satisfies the assumptions of Theorem 1 for  $\alpha > 1$ . Thus the nominal system is asymptotically stable, and ARE (3) has a positive-definite solution  $P$  for some  $\alpha > 1$ . This is equivalent to assuming that the ARE

$$\bar{A}_0^T P + P\bar{A}_0 + P D_0 D_0^T P + (\beta + 2)I_n = 0, \quad \beta > 0 \tag{5}$$

has a positive-definite solution  $P$ . In other words, we make a little stronger assumption to cope with the additional terms in the system dynamics. We then turn to our objective: the robust stability of the uncertain system.

For the uncertain system ( $\Sigma_m$ ), consider the Lyapunov functional candidate

$$V(t, x_t) = x^T(t) P x(t) + \int_{t-h}^t x^T(\sigma) x(\sigma) d\sigma + \frac{1}{q} \sum_{i=1}^q \int_{t-h_i(t)}^t x^T(\sigma) x(\sigma) d\sigma. \tag{6}$$

Then, with the satisfaction ARE (5), it is easy to show that the Lyapunov derivative can be written as

$$\begin{aligned} \dot{V}(t, x_t) &= x^T(t) \left[ P\bar{A}_0 + \bar{A}_0^T P + 2I_n + \sum_{i=1}^m k_i (P E_i + E_i^T P) + \sum_{i,j=1}^m k_i k_j (P E_{ij} + E_{ij}^T P) \right] x(t) \\ &\quad + 2x^T(t) P \left( D_0 + \sum_{i=1}^m k_i D_i \right) x(t-h) + 2x^T(t) P \sum_{i=1}^q \sum_{i=1}^m k_i F_{i1} x(t-h_i(t)) \\ &\quad - x^T(t-h)x(t-h) - \frac{1}{q} \sum_{i=1}^q [1 - \dot{h}_i(t)] x^T(t-h_i(t)) x(t-h_i(t)) \\ &\leq x^T(t) T Q T^T x(t), \end{aligned} \tag{7}$$

where  $x(t) = [x^T(t) \ x^T(t-h) \ x^T(t-h_1(t)) \ \cdots \ x^T(t-h_q(t))]^T$ ,

$$T = \begin{bmatrix} I_n & -P(D_0 + \sum_{i=1}^m k_i D_i) & -P \sum_{i=1}^m k_i F_{i1} & \cdots & -P \sum_{i=1}^m k_i F_{iq} \\ 0 & I_n & 0 & \cdots & 0 \\ 0 & 0 & I_n & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & \dots & & 0 & I_n \end{bmatrix},$$

$$Q = \begin{bmatrix} S(k) & 0 & 0 & \dots & 0 \\ 0 & -I_n & 0 & & \vdots \\ \vdots & & -\frac{1-d_{\max}}{q}I_n & & \\ & & & \ddots & 0 \\ 0 & 0 & \dots & 0 & -\frac{1-d_{\max}}{q}I_n \end{bmatrix},$$

and

$$S(k) = -\beta I_n + \sum_{i=1}^m k_i [PE_i + E_i^T P + P(D_0 D_i^T + D_i D_0^T)P] + \sum_{i,j=1}^m k_i k_j \left[ PE_{ij} + E_{ij}^T P + \frac{1}{2}P(D_i D_j^T + D_j D_i^T)P + \frac{q}{2(1-d_{\max})} \sum_{l=1}^q P(F_{il} F_{jl}^T + F_{jl} F_{il}^T)P \right] \tag{8}$$

$$= -\beta \left[ I_n + \sum_{i=1}^m k_i \bar{E}_i + \sum_{i,j=1}^m k_i k_j \bar{E}_{ij} \right], \tag{9}$$

with

$$\bar{E}_i \triangleq -\frac{1}{\beta} [PE_i + E_i^T P + P(D_0 D_i^T + D_i D_0^T)P],$$

$$\bar{E}_{ij} \triangleq -\frac{1}{\beta} [PE_{ij} + E_{ij}^T P + \frac{1}{2}P(D_i D_j^T + D_j D_i^T)P] - \frac{q}{2\beta(1-d_{\max})} \sum_{l=1}^q P(F_{il} F_{jl}^T + F_{jl} F_{il}^T)P,$$

and

$$k \triangleq [k_1 \dots k_m].$$

Note that basically the transformation procedure in relation (7) is completing the square and transferring the Lyapunov derivative into a quadratic form. Consequently, it is readily seen that  $\dot{V}(t, x_t)$  is negative-definite if  $S(k)$  is negative-definite for all  $|k_i| < \gamma, i = 1, \dots, m$ . This is true if and only if

$$\begin{aligned} \det \left( I_n + \sum_{i=1}^m k_i \bar{E}_i + \sum_{i,j=1}^m k_i k_j \bar{E}_{ij} \right) &= \det [I_n + (k \otimes I_n)^T \bar{E}_e + (k \otimes I_n)^T \bar{E}_{ee} (k \otimes I_n)] \\ &= \det \left\{ I_{(m+1)n} + \begin{bmatrix} (k \otimes I_n)^T & 0 \\ 0 & k \otimes I_n \end{bmatrix} \begin{bmatrix} \bar{E}_e & \bar{E}_{ee} \\ -I_n & 0 \end{bmatrix} \right\} \\ &\neq 0, \end{aligned} \tag{10}$$

where

$$\bar{E}_e = [\bar{E}_1^T \ \bar{E}_2^T \ \dots \ \bar{E}_m^T]^T \quad \text{and} \quad \bar{E}_{ee} = \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \dots & \bar{E}_{1m} \\ \bar{E}_{21} & \bar{E}_{22} & & \bar{E}_{2m} \\ \vdots & & \dots & \vdots \\ \bar{E}_{m1} & \bar{E}_{m2} & \dots & \bar{E}_{mm} \end{bmatrix},$$

since  $S(k) = -\beta I_n$  is negative-definite when  $k = 0$ , and the eigenvalues of  $S(k)$  cannot become positive when  $|k_i|$ 's vary from 0 to  $\gamma$ . It is easy to check that inequality (10) holds if and only if

$$\det \left\{ I_{(m+1)n} + \sum_{i=1}^m k_i U_i \bar{E} \right\} \neq 0, \tag{11}$$

where  $U_i = (e_1 e_i^T + e_{i+1} e_{m+1}^T) \otimes I_n$ ,  $e_j$  is the  $j$ th column of  $I_{m+1}$ , and

$$\bar{E} = \begin{bmatrix} \bar{E}_e & \bar{E}_{ee} \\ -I_n & 0 \end{bmatrix}.$$

Thus, we can obtain the following theorem.

**Theorem 2.** *The uncertain delay system  $(\Sigma_m)$  is asymptotically stable if*

$$\gamma < \mu^{-1}(M), \tag{12}$$

where

$$M = \begin{bmatrix} U_1 \bar{E} \\ \vdots \\ U_m \bar{E} \end{bmatrix} [I_{(m+1)n} \dots I_{(m+1)n}], \tag{13}$$

and  $\mu(\cdot)$  denotes the structured singular value [4] with respect to the set of real uncertainty matrices of the form  $\bar{A} = \text{diag}[k_1 I_{(m+1)n}, \dots, k_m I_{(m+1)n}]$ .

**Proof.** Using the definition of the structured singular value, we know that inequality (10) holds if and only if inequality (12) holds. Thus,  $S(k)$  is negative-definite with the satisfaction of inequality (12). Hence, there exist an  $\varepsilon > 0$  such that  $\dot{V}(t, x_t) \leq -\varepsilon \|x(t)\|^2$ . This concludes the proof.

Note that if all  $F_{ii}$ 's are absent, it is possible to use the Lyapunov functional (4) and the assumption of Theorem 1 instead of (6) and that of Theorem 2 to show the robust asymptotic stability of  $(\Sigma_m)$ . This reduces some conservativeness because sometimes the solution of ARE (3) exists while that of ARE (5) does not.

The use of  $\mu$ -analysis technique to treat the negative-definiteness preservation problem of a perturbed matrix is also adopted in [14] to handle the robust stability analysis problems of nondelayed uncertain systems. In fact such approach can be generalized to process the so-called rank preservation problem, and applied to many problems in robust control theory. Although the numerical computation of  $\mu$  with respect to real uncertainties is a difficult problem [6, 10], the well-developed  $\mu$ -analysis software [1] for complex uncertainties can be used to provide an upper bound. As we shall see in the subsequent example, in spite of the possible existence of conservativeness which is not caused by the proposed approach, but by the adopted computation algorithm, the application results are quite satisfactory.

### 3. An example

Consider the uncertain system  $(\Sigma_m)$  with the following data:  $m = 2, q = 1, d_{\max} = 0.5$ ,

$$A_0 = \begin{bmatrix} -1 & 5 \\ 1 & -1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C_0^T = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0.5 & 0 \\ 0.75 & 1 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C_1^T = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_{11} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$A_2, B_2, C_1, D_2$  and  $F_{21}$  are zero matrices of suitable dimensions, and the output feedback gain  $L = -4$ . Thus, we have

$$\bar{A}_0 = \begin{bmatrix} -3 & 1 \\ -1 & -5 \end{bmatrix},$$

which is a stable matrix. By using Theorem 1, it is found that ARE (3) has a positive-definite solution

$$P = \begin{bmatrix} 0.3324 & 0.0195 \\ 0.0195 & 0.2112 \end{bmatrix}$$

at  $\alpha = 1$ . Hence, the nominal system is asymptotically stable. This agrees with the result in [8]. In addition, using Theorem 2 and the  $\mu$ -analysis package [1], we know that ARE (5) has a positive-definite solution

$$P = \begin{bmatrix} 0.5020 & 0.0325 \\ 0.0325 & 0.3237 \end{bmatrix}$$

at  $\beta = 1$ , and get  $\mu^{-1}(M) = 0.1600$ . As a result, we know that the system  $(\Sigma_m)$  remains asymptotically stable provided  $\max_i |k_i| < 0.1600$ , even if  $k_i$ 's are time-varying. Figure 1 shows the region of negative-definite  $S(k)$  in the parameter plane for  $\beta = 1$  (so the stability region guaranteed by the specific Lyapunov functional), which is the region to the right of the dotted line, the boundary on which  $\det[S(k)] = 0$ . The dotted line is obtained by letting  $k_2 = \rho k_1$  with a constant  $\rho$ , solving the equation  $\det[S(k)] = 0$  for the only variable  $k_1$ , and repeating the procedure for various values of  $\rho$ . The square in the center is the boundary of the derived uncertainty range, of which the lower left corner almost touches the boundary. This shows that a nonconservative result is obtained in this example, although  $\mu$ -analysis algorithm for complex uncertainties is applied. Moreover, by using Theorem 2.1 of [11], it is known that if there exists an  $\beta^*$  such that ARE (5) has a positive-definite solution, then it has positive-definite solutions for all  $\beta \in (0, \beta^*]$ . Since  $\mu^{-1}(M)$  depends on  $M$ , which in turns depends on  $P$  and hence  $\beta$ , better results may be found if a simple one-dimensional search over  $\beta$  is carried out. In this example, it is found that the largest  $\beta^* \in [10.4, 10.5)$ . Thus the plot of  $\mu^{-1}[M(\beta)]$  is depicted in Figure 2 for  $0.1 \leq \beta \leq 10.4$  with increment 0.5. Indeed, a better bound 0.3367 is found around  $\beta = 6.1$ .

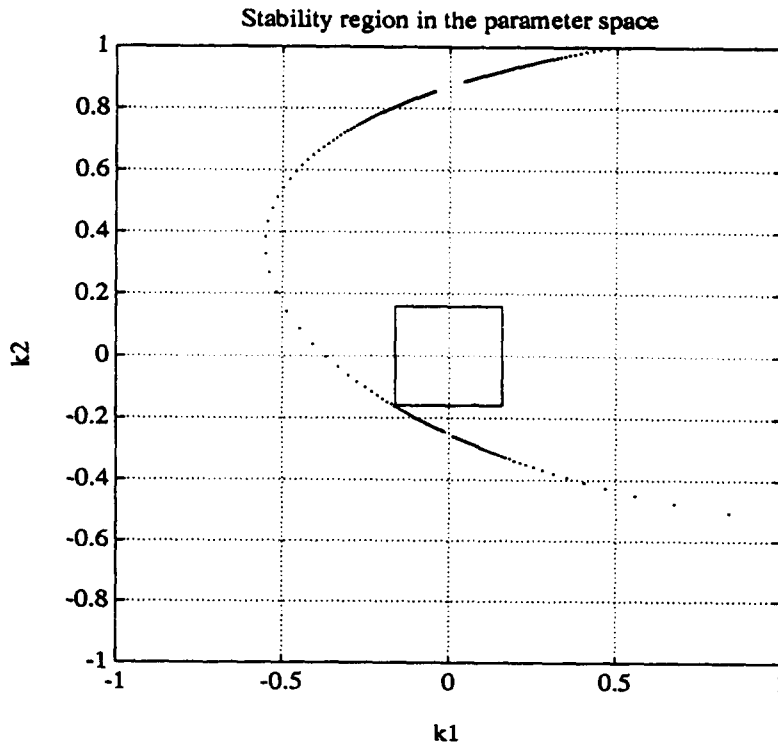


Fig. 1. The region of negative definite  $S(k)$  for  $\beta = 1$ .

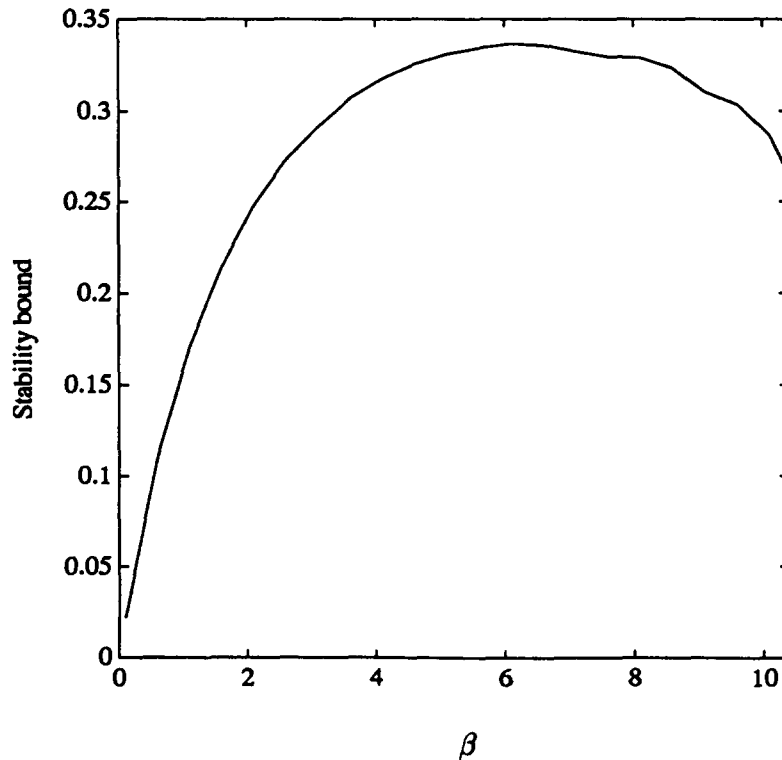


Fig. 2. The plot of  $\mu^{-1}[M(\beta)]$ .

#### 4. Conclusion

In this paper, we propose a new approach to deal with the robust stability problem of the uncertain time-delay systems controlled by static output feedback. To check whether the uncertain systems are nominally stable, we prove that it suffices to check locations of the eigenvalues of an associated Hamiltonian matrix. For the uncertain systems which are nominally stable and possibly have multiple noncommensurate time-varying delays, a transform method is given to transfer the Lyapunov derivative of the system into a quadratic form. Then it is shown that with the  $\mu$ -analysis technique we can find a set of uncertainties within which the uncertain time-delay systems keep asymptotic stability. A numerical example is provided to illustrate the use of the proposed approach, and a one-dimensional search is suggested for improving the final results.

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