and $\dot{V}(x(t)) = \dot{V}(x_t(0))$ is defined to be

$$\dot{V}(x(t)) = \dot{V}(x_t(0)) = \limsup_{h\to 0+} \frac{1}{h}\left[V(x(t+h)) - V(x(t))\right]$$
$$= \limsup_{h\to 0+} \frac{1}{h}\left[V\left(x_t(0) + \int_t^{t+h} f(s,x_s)ds\right) - V(x_t(0))\right] \quad \text{(A.3)}$$

for $t \geq t_0$, where $x(t+h) = x_{t+h}(0) = x_t(0) + \int_t^{t+h} f(s,x_s)ds$.

*Remark A.1:* It should be noted that notation: $\dot{x}(t) = f(t,x_t)$ denotes a very general type of equation and includes ordinary differential equations, differential difference equations, integro-differential equations and much more general equations; see, for example, [1] and [4, p. 37]–[6].

*Theorem A.1:* Let $V(x) = x^T P x$, where $x \in R^n$ and $P > 0 \in R^{n\times n}$. Then: 1) the equilibrium $x^* \equiv 0$ of system (A.1) is globally uniformly stable if along the solution $x(t_0,\phi)(t)$ of system (A.1) through any $(t_0,\phi) \in R \times C_n$

$$\dot{V}(y_t(0)) \leq 0 \qquad \forall y_t \in S_{\text{sta.}}\left(\overline{V}_{t_0}\right) \quad \text{(A.4)}$$

whenever $V(x_t(0)) = \overline{V}_{t_0}$ on $t \geq t_0$, where $S_{\text{sta.}}(\overline{V}_{t_0})$ is defined as

$$S_{\text{sta.}}\left(\overline{V}_{t_0}\right) = \left\{ y_t \in C_n \,\middle|\, \begin{array}{c} V(y_t(0)) = \overline{V}_{t_0} \\ \left\|P^{\frac{1}{2}} y_t(\theta)\right\|^2 = \left\|P^{\frac{1}{2}} y_t(0)\exp\left\{\frac{1}{2}\theta_X\right\}\right\|^2 \\ \theta \in [-\tau, 0] \quad \theta_X \in (-\infty, 0] \end{array} \right\}. \quad \text{(A.5)}$$

2) The equilibrium $x^* \equiv 0$ of (A.1) is globally uniformly asymptotically stable if there is a positive $\gamma > 0$ such that along the solution $x(t_0,\phi)(t)$ of (A.1) through any $(t_0,\phi) \in R \times C_n$

$$\dot{V}(y_t(0)) \leq -\gamma V(y_t(0)) \qquad \forall y_t \in S_{a.sta.}(L_t(\theta)) \quad \text{(A.6)}$$

whenever $V(x_t(0)) = \overline{V}_{t_0}\exp\{-\gamma(t-t_0)\}$ on $t \geq t_0$, where $S_{a.sta.}(L_t(\theta))$ is defined as

$$S_{\text{a.sta.}}(L_t(\theta)) = \left\{ y_t \in C_n \,\middle|\, \begin{array}{c} L_t(\theta) = L(t+\theta) = \overline{V}_{t_0}\exp\{-\gamma(t+\theta-t_0)\} \\ V(y_t(0)) = L_t(0) \\ \left\|P^{\frac{1}{2}} y_t(\theta)\right\|^2 = \left\|P^{\frac{1}{2}} y_t(0)\exp\left\{-\frac{1}{2}\times\gamma(\theta-\theta_X)\right\}\right\|^2 \\ \theta \in [-\tau, 0], \quad _X \in (-\infty, 0] \end{array} \right\}. \quad \text{(A.7)}$$

*Proof:* For the proof of this theorem, see [1].

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# Division Controllers for Homogeneous Dyadic Bilinear Systems

Yean-Ren Hwang, Min-Shin Chen, and Tzuyin Wu

***Abstract*—When a homogeneous bilinear system is *dyadic*, it permits the use of the so-called division controller for global stabilization. However, due to singularities in the control law, a dead-zone must be cascaded with the division controller in practical implementation. The contribution of this note is to show that the division controller, when cascaded with a dead-zone, can asymptotically drive the bilinear system state into a bounded set around the origin with the size of the bounded set proportional to the size of dead-zone. This note further proposes a $C^0$ modification and a $C^\infty$ modification of the conventional discontinuous dead-zone so that the control signals from the modified division controllers are continuous and smooth, respectively.**

***Index Terms*—Dead-zone, division controller, dyadic bilinear system, practical stability singularity.**

## I. Introduction

The control designs for bilinear systems have been attracting more and more attention because bilinear models arise in many physical phenomenon [1], [2]. Although bilinear systems can be regarded as the simplest class of nonlinear systems, control design for such systems is still a problem not completely solved. Consider, for example, a simple bilinear system with a multiplicative control input (the so-called *homogeneous* bilinear system)

$$\dot{x} = Ax + Nxu \quad (1)$$

where $x \in R^n$ is the state vector, $u \in R$ is a single control input, $A \in R^{n\times n}$ and $N \in R^{n\times n}$ are constant matrices. For the system (1), a quadratic state feedback control [3]–[6] has been proposed to achieve global *asymptotical* stability. A *normalized* quadratic state feedback control [7] is recently suggested to achieve global *exponential* stability for better performance and robustness. However, the controls in [3]–[5]

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and [7] are applicable only when the open-loop dynamics of (1) is (neutrally) stable. Hence, bilinear system control design for the open-loop unstable case still needs further investigations.

The purpose of this note is to show that when the system matrix $N$ in (1) is of rank one, i.e., $N = bc$ with $b$ and $c^T$ being column vectors, stabilization of open-loop unstable system (1) is achievable. In this case, the bilinear system (1) is said to be *dyadic* [1]

$$\dot{x} = Ax + byu, \qquad y = cx. \tag{2}$$

For a dyadic bilinear system, one can employ the so-called *division controller*

$$u = -\frac{kx}{y} \tag{3}$$

where the state feedback gain $k \in R^{1\times n}$ is chosen such that $A - bk$ is a stable matrix. With the division controller (3), the closed-loop system becomes linear

$$\dot{x} = (A - bk)x. \tag{4}$$

However, singularity takes place in the division control controller(3) at $y = cx = 0$. To avoid the singularity problem, one can superimpose a dead-zone on the division controller (3)

$$u = \begin{cases} -\frac{1}{y}kx, & |y| > \epsilon \\ 0, & |y| \le \epsilon \end{cases} \tag{5}$$

where $\epsilon$ is the size of the dead-zone. The closed-loop system then becomes nonlinear

$$\dot{x} = \begin{cases} (A - bk)x, & |y| > \epsilon \\ Ax, & |y| \le \epsilon. \end{cases} \tag{6}$$

It is mentioned that the design of combining a division controller with a dead-zone is first proposed in [6] for a dyadic bilinear system whose control input is both multiplicative and additive (the so-called *inhomogeneous* bilinear system)

$$\dot{x} = Ax + b(y + d_0)u, \qquad y = cx.$$

where constant $d_0 \ne 0$. When the system is inhomogeneous, the origin $x = 0$ is not inside the region defined by the dead-zone $|y + d_0| \le \epsilon$. Global asymptotic stability can therefore be established for the division controller in [6] under a geometric condition. However, when the bilinear system is homogeneous as in (2), the origin $x = 0$ is contained in the region defined by the dead-zone $|y| \le \epsilon$. In this case, achieving asymptotic stability by the division controller (5) becomes impossible when the open-loop dynamics is unstable.

This note aims for two goals. First, in Section II, a stability theorem will be established for the *homogeneous* bilinear system (2) with the proposed division controller (5). This new stability theorem will describe how the size of dead-zone affects the asymptotic behavior of the closed-loop system. Second, note that the division controller (5) creates discontinuous control signal at $|y| = \epsilon$. To avoid such discontinuity, it is proposed in Section III to replace the discontinuous dead-zone in (5) by its continuous approximations so as to ensure smoothness of the control signal. Finally, Section IV states the conclusions.

## II. Stability Analysis of Division Controller

The objective of this section is to establish the stabilizing property of the division controller (5) for the homogeneous dyadic system (2). Different from the geometric assumption in [6] for the inhomogeneous case, the only assumption imposed on the homogeneous bilinear system (2) is the controllability condition as stated in the following theorem.

*Theorem 1:* [8], [9]: The dyadic bilinear system (2) is controllable (strongly locally controllable as referred to in [1]) if and only if $(A, b)$ is controllable and $(A, c)$ is observable.

The controllability assumption in Theorem 1 ensures that arbitrary eigenvalue assignment [11] of the matrix $A - bk$ in (6) is achievable by a proper choice of the state feedback gain$k$ in the division controller (5).

To establish the stabilizing property of the closed-loop system (6), one relies on an important observation: a scaling of the coordinate

$$x = \epsilon z \tag{7}$$

results in a transformed system that is *independent of* $\epsilon$

$$\dot{z} = \begin{cases} (A - bk)z, & |y_z| > 1 \\ Az, & |y_z| \le 1 \end{cases} \tag{8}$$

where its output $y_z$ is defined as

$$y_z = cz.$$

This observation will reveal how the size of dead-zone $\epsilon$ affects the asymptotic behavior of the original $x$ system (6), as will be seen in Theorem 3.

The following definitions will facilitate the analysis of the $z$ system (8).

*Definition:* The state–space $R^n$ is divided into two sets $\Omega^-$, $\Omega^+$ and their boundary $\Omega^0$

$$\Omega^- = \{z \in R^n \,||y_z(t)| \le 1\}, \; \Omega^+ = \{z \in R^n \,||y_z(t)| > 1\}$$
$$\Omega^0 = \{z \in R^n \,||y_z(t)| = 1\}.$$

Further, let $\{t_i\}$ be a nondecreasing time sequence, where $t_{2i}$'s denote the time instants when the state crosses the boundary $\Omega^0$ from $\Omega^+$ into $\Omega^-$, and $t_{2i+1}$'s the time instants when the state crosses the boundary $\Omega^0$ from $\Omega^-$ into $\Omega^+$. The time durations staying in $\Omega^-$ and $\Omega^+$ are, therefore, given, respectively, by

$$\Delta_i^- = t_{2i+1} - t_{2i} \quad \text{and} \quad \Delta_i^+ = t_{2i+2} - t_{2i+1}.$$

In other words, $z(t) \in \Omega^-$ when $t \in [t_{2i}, t_{2i+1}]$, and $z(t) \in \Omega^+$ when $t \in (t_{2i+1}, t_{2i+2})$. Both $z(t_{2i})$ and $z(t_{2i+1})$ are called crossing points; they are on the boundary $\Omega^0$ and satisfy

$$|y_z(t_i)| = 1, \qquad \text{for all } i. \tag{9}$$

*Lemma 1:* If $cb = 0$, the existence of solution for the nonlinear system (8) is guaranteed.

*Proof:* Infinite switching on the boundary $\Omega^0$ will take place when the flow of state trajectory at $\Omega^0$ points toward $\Omega^+$ and the flow at neighboring points in $\Omega^+$ points toward $\Omega^-(\supset \Omega^0)$. In this case, the sign of the inner product $(c, \dot{z})|_{z\in\Omega^+}$ is different from that of $(c, \dot{z})|_{z\in\Omega^-}$; that is, $(c, \dot{z})|_{z\in\Omega^+} \cdot (c, \dot{z})|_{z\in\Omega^-} < 0$.

However, under the assumption of this lemma, $cb = 0$, one has $(c, \dot{z})|_{z\in\Omega^+} = c(A - bk)z = cAz - cb \cdot kz = cAz = (c, \dot{z})|_{z\in\Omega^-}$. Hence, $cb = 0$ rules out the possibility of $(c, \dot{z})|_{z\in\Omega^+} \cdot (c, \dot{z})|_{z\in\Omega^-} < 0$ and, hence, the possibility of infinite switching at $\Omega^0$. Since the system (8) satisfies the global Lipschitz condition, and is piecewise continuous with no infinite switching, the existence of solution for (8) is guaranteed by [10, App. B, Th. 6]. □

*Lemma 2:* The system state in (8) can grow or decay only exponentially

$$\|z(\tau)\|e^{-q(t-\tau)} \le \|z(t)\| \le \|z(\tau)\|e^{q(t-\tau)} \qquad \forall\, t \ge \tau \tag{10}$$

where $q = \|A\| + \|bk\|$.

*Proof:* Note that the state in (8) satisfies the inequality $|d/dt(\|z(t)\|)| \le \|\dot{z}(t)\| \le q\|z(t)\|$, for all $t \ge 0$. Integrating the inequalities yields (10). □

The following lemma shows that under the observability condition of $(A, c)$, which is guaranteed by the controllability assumption in Theorem 1, bounded output $(y_z)$ implies bounded state $(z)$ for the case $|y_z| \leq 1$ in (8).

*Lemma 3:* Consider the closed-loop system (8). If $|y_z(t)| \leq 1$ for all $t \in [t_0, t_f]$, then

$$\|z(t_0)\|^2 \leq \frac{t_f - t_0}{\underline{\sigma}}\left[W_{t_f - t_0}\right]$$

where $W_{t_f-t_0} \in R^{n \times n}$ is the observability Grammian of $(A, c)$

$$W_{t_f-t_0} = \int_{t_0}^{t_f} e^{A^T(t-t_0)}c^T c e^{A(t-t_0)}dt = \int_0^{t_f-t_0} e^{A^T t}c^T c e^{At}dt. \quad (11)$$

*Proof:* Since the dyadic bilinear system (2) is controllable, it follows from Theorem 1 that $(A, c)$ is observable. Therefore, the observability Grammian $W_{t_f-t_0}$ in (11) is positive definite [10] and, hence, $\underline{\sigma}[W_{t_f-t_0}] > 0$ for any $t_f > t_0$. When $|y_z(t)| \leq 1$, the closed-loop state satisfies $\dot{z}(t) = Az(t)$ and, hence, $y_z(t) = cz(t) = ce^{A(t-t_0)}z(t_0)$. One can deduce from $|y_z(t)| \leq 1$ for all $t \in [t_0, t_f]$ that $(t_f - t_0) \geq \int_{t_0}^{t_f} \|y_z(t)\|^2 dt = z^T(t_0)W_{t_f-t_0}z(t_0) \geq \underline{\sigma}[W_{t_f-t_0}]\|z(t_0)\|^2$. The last inequality proved the result. □

*Theorem 2:* Under the controllability assumption in Theorem 1 and $cb = 0$, the transformed state $z(t)$ in (8) will converge asymptotically to a bounded set $D_z$ around the origin.

*Proof:* Assume the contrary; that is, the state $z(t)$ escapes to infinity asymptotically. This may happen in three different situations as discussed later. It will be proved that none of the situations will take place; hence, $z(t)$ in fact remains uniformly bounded.

*Case 1:* $z(t)$ escapes to infinity while remaining inside $\Omega^-$.

Under the hypothesis of this case, $|y_z(t)| \leq 1$ for all $t$ greater than some $T$. Choose a time sequence $\{t_k\}$ with $t_0 = T$, and $t_{k+1} - t_k = \Delta t$ being any positive real number. Quoting Lemma 3 with $t_0 = t_k$ and $t_f = t_{k+1}$ gives

$$\|z(t_k)\|^2 \leq \frac{\Delta t}{\underline{\sigma}[W_{\Delta t}]} \qquad \forall\, k \geq 0 \quad (12)$$

where $W_{\Delta t} > 0$ is as defined in (11). The uniform boundedness result in (12) contradicts the hypothesis of this case that $z(t)$ escapes to infinity. Therefore, this case will not take place.

*Case 2:* $z(t)$ escapes to infinity while remaining inside $\Omega^+$.

Since in $\Omega^+$ the closed-loop dynamics is exponentially stable by the control design, the state will not escape to infinity in $\Omega^+$. Hence, this case will not take place.

*Case 3:* $z(t)$ escapes to infinity by switching back and forth between $\Omega^-$ and $\Omega^+$.

There are three subcases in Case 3, in which Cases 3a and 3b assume uniformly bounded crossing points, and Case 3c assumes asymptotically exploding crossing points.

*Case 3a:* The crossing points $z(t_i)'s \in \Omega^0$ remain uniformly bounded, and there exist a sequence of points in $\Omega^-$ that explode to infinity.

Between two consecutive crossing of the boundary, let $t_{2i+1/2}$ denote the time instant corresponding to the maximum norm in $\Omega^-$; that is, $\|z(t_{2i+1/2})\| = sup\|z(t)\|$ for all $t \in (t_{2i},\ t_{2i+1})$. Since the state escapes to infinity, this sequence of peak points $z(t_{2i+1/2})$ satisfies

$$\lim_{i\to\infty} z\left(t_{2i+\frac{1}{2}}\right) = \infty. \quad (13)$$

By definition, $z(t_{2i+1/2})$ is the *exploding* peak point in $\Omega^-$, and $z(t_{2i+1})$ is the *bounded* crossing point on $\Omega^0$. Since the state can only decay exponentially fast from Lemma 2, (13), and the uniform boundedness hypothesis of $z(t_{2i+1})$ suggest that $\lim_{i\to\infty}(t_{2i+1} - t_{2i+1/2}) = \infty$. Hence, there exists a positive $\delta t$ such that $t_{2i+1} - t_{2i+1/2} > \delta t$ for sufficiently large $i$. Applying Lemma 3 with $t_0 = t_{2i+1/2}$ and $t_f = t_{2i+1/2} + \delta t$, one concludes that $z(t_{2i+1/2})$ is uniformly bounded; contradicting (13). Therefore, this case will not take place.

*Case 3b:* The crossing points $z(t_i)$'s remain uniformly bounded, and there exist a sequence of points in $\Omega^+$ that explode to infinity.

Between two consecutive crossing of the boundary $\Omega^0$, let $t_{2i-1/2}$ denote the time instant corresponding to the maximum norm in $\Omega^+$; that is, $\|z(t_{2i-1/2})\| = sup\|z(t)\|$ for all $t \in (t_{2i-1},\ t_{2i})$. From the hypothesis of this subcase, one must have that $z(t_{2i-1/2})$ explodes to infinity asymptotically. By a similar argument as in Case 3a, one can also deduce that $\lim_{i\to\infty}(t_{2i-1/2} - t_{2i-1}) = \infty$. Recall that the system dynamics in $\Omega^+$ is exponentially stable, hence, the state norm satisfies the inequality $\|z(t_{2i-1/2})\| \leq \|z(t_{2i-1})\|\beta e^{-\alpha(t_{2i-1/2}-t_{2i-1})}$ for some positive constants $\beta$ and $\alpha$. The previous conclusion $\lim_{i\to\infty}(t_{2i-1/2} - t_{2i-1}) = \infty$ then suggests that $\|z(t_{2i-1/2})\|$ approaches zero as $i$ approaches infinity given bounded crossing point $z(t_{2i-1})$; contradicting the previous statement that $z(t_{2i-1/2})$ asymptotically explodes to infinity. Hence, one concludes that this case will not take place.

*Case 3c:* The crossing points approach infinity, $\lim_{i\to\infty}\|z(t_i)\| = \infty$.

There are two subcases in Case 3c, depending on whether $\lim_{i\to\infty}\Delta_i^+ = 0$ or otherwise (there exists a subsequence $\Delta_{i_k}^+$ of $\Delta_i^+$ such that $\Delta_{i_k} \geq m^+ > 0$ for some lower bound $m^+$, in which $\Delta_i^+ = t_{2i+2} - t_{2i+1}$ denotes the $i'$th time span over which the state stays inside $\Omega^+$). However, according to the proof of Lemma 1, there is no infinite switching. Hence, the first subcase will not take place.

One now discusses the second subcase. From (9), one has

$$|y\,(t_{2i_k+1})| = 1 \quad (14)$$

and from the hypothesis of this case

$$\lim_{i_k\to\infty}\|z\,(t_{2i_k+1})\| = \infty. \quad (15)$$

For all $t \in [t_{2i_k+1}, t_{2i_k+2}] = [t_{2i_k+1}, t_{2i_k+1} + \Delta_{i_k}^+]$, the system dynamics is linear and stable

$$\dot{z}(t) = (A - bk)z(t) \quad y_z(t) = cz(t).$$

Discretizing the aforementioned system with a sampling time $T = m^+/n$, where $m^+$ is the lower bound of $\Delta_{i_k}^+$, and $n$ the system dimension, one obtains, for $h = 0, 1, \ldots, n-1$

$$\begin{aligned} z\,(t_{2i_k+1} + (h+1)T) &= Fz\,(t_{2i_k+1} + hT) \\ y_z\,(t_{2i_k+1} + hT) &= cz\,(t_{2i_k+1} + hT) \end{aligned} \quad (16)$$

where $F = e^{(A-bk)T}$. From (16), one can deduce

$$\begin{bmatrix} y_z\,(t_{2i_k+1}) \\ y_z\,(t_{2i_k+1} + T) \\ \vdots \\ y_z\,(t_{2i_k+1} + (n-1)T) \end{bmatrix} = \begin{bmatrix} c \\ cF \\ \vdots \\ cF^{n-1} \end{bmatrix} z\,(t_{2i_k+1}). \quad (17)$$

Without loss of generality, one can assume that the state feedback gain $k$ is chosen such that $(A - bk, c)$ is observable and $A - bk$ has no eigenvalues with the same real parts. Then the discretized pair $(F, c)$ is also observable [11, Th. D-1], and the observability matrix on the right-hand side of (17) is full rank. Equations (16) and (18) then imply that at least one element on the left-hand side of (17) must approach infinity as $i_k$ approaches infinity; that is

$$\lim_{i_k\to\infty}|y_z\,(t_{2i_k+1} + l_{i_k}T)| = \infty \quad (18)$$

for some integer $0 \le l_{i_k} \le n-1$. Comparing (14) and (18), one observes that the output signal explodes from $|y_z(t_{2i_k+1})| = 1$ to $|y_z(t_{2i_k+1} + l_{i_k}T)| = \infty$ within a finite time $l_{i_k}T (\le (n-1)T)$ as $i_k \to \infty$. This contradicts Lemma 2 that the system state can grow only exponentially fast, and cannot have finite-time explosion phenomena. With this contradiction, one concludes that this case will not take place.

Since, in no case, the state approaches infinity, one concludes the boundedness of the closed-loop state (8). □

The second theorem, which is the main result of this note, establishes the stability property of the original $x$ system (6).

*Theorem 3:* Under the same conditions as in Theorem 2, the controlled dyadic bilinear system (2) and (5) is *practically* stable [12] in the sense that the state will asymptotically converge to a bounded set $D_x$ around the origin, and the size of the bounded set $D_x$ is proportional to the dead-zone size $\epsilon$.

*Proof:* From Theorem 2, $\lim_{t\to\infty} z(t) \in D_z$. The relation (7) immediately suggests that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \epsilon z(t) \in \epsilon D_z$; that is, $D_x = \epsilon D_z$, where the size of $D_z$ is independent of $\epsilon$ because (8) is independent of $\epsilon$. □

*Remark:* Theorem 3 shows that the division controller (5) not only stabilizes the homogeneous dyadic bilinear system, but can also force the system state to converge as closely to the origin as desired by the choice of a sufficiently small dead-zone size $\epsilon$.

## III. Continuous Division Controller

The division controller (5) studied in the previous section has a disadvantage that it creates discontinuous control signals at $|y| = \epsilon$. This is due to the discontinuity of the dead-zone. Since discontinuous control signals are not acceptable in many practical applications, two *continuous* division controllers are suggested in this section to ensure continuity and smoothness of the control signal.

The first continuous division controller is a $C^0$ controller

$$u = \begin{cases} -\frac{1}{y}kx, & |y| > \epsilon \\ -\frac{y}{\epsilon^2}kx, & |y| \le \epsilon \end{cases}. \quad (19)$$

The control signal in (20) is now continuous at $|y| = \epsilon$, but its time derivative is not continuous at $|y| = \epsilon$.

The second *continuous* division controller is a $C^\infty$ controller

$$u = -\frac{y}{y^2 + \epsilon^2}kx. \quad (20)$$

The control signal in (20) not only is continuous but has continuous derivatives up to any order at $|y| = \epsilon$.

Notice that for controller (19), the discontinuous dead-zone $F_1(s)$ is approximated by a $C^0$ curve $F_2(s)$, and for controller (20), the discontinuous dead-zone is approximated by a $C^\infty$ curve $F_3(s)$ as shown in Fig. 1, where

$$F_1(y) = \begin{cases} 1, & |y| > \epsilon \\ 0, & |y| \le \epsilon \end{cases}$$

$$F_2(y) = \begin{cases} 1, & |y| > \epsilon \\ \frac{y^2}{\epsilon^2}, & |y| \le \epsilon \end{cases}$$

$$F_3(y) = \frac{y^2}{y^2 + \epsilon^2}.$$

Both approximation curves $F_2(y)$ and $F_3(y)$ become closer and closer to the original discontinuous curve $F_1(y)$ as $\epsilon$ becomes smaller and smaller. One can establish the same stability result as in Theorem 3 for the previous two continuous division controllers (19) and (20) for small $\epsilon$, however, the required stability analysis is much more tedious and is omitted here.

*Example:* Consider an open-loop unstable dyadic system (2) with

$$A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad c = [\,1\ 1\,]$$

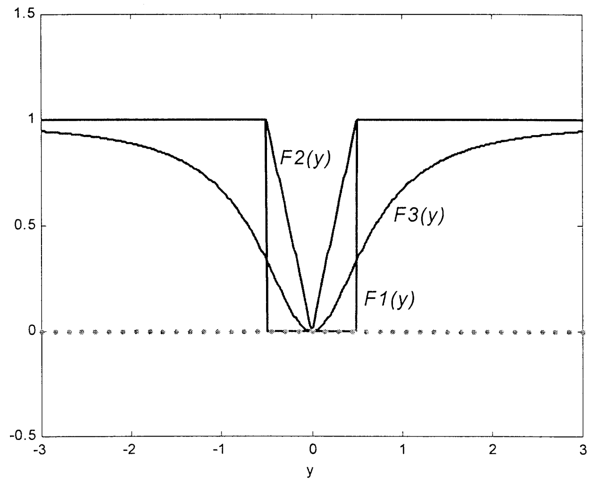


Fig. 1. Functions $F_1(y)$, $F_2(y)$, and $F_3(y)$.

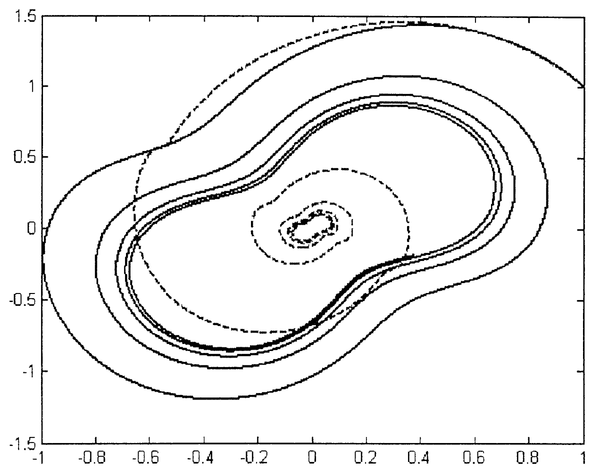


Fig. 2. Phase portrait for division controller (20).

and the initial condition $x^T(0) = [1, 1]$. One tests the $C^\infty$ division controller (20) with the state feedback gain $k = [\,3.25\ \ 1.25\,]$, which places eigenvalues of $A - bk$ at $-0.5 \pm 2j$. The value of $\epsilon$ is chosen to be 0.5 and 0.05, respectively. Fig. 2 shows the phase portrait, where the solid line is the trajectory with $\epsilon = 0.5$, and the dash line with $\epsilon = 0.05$. It is seen that the state trajectories approach limit cycles, and the size of the limit cycle is indeed proportional to $\epsilon$.

## IV. Conclusion

Note that when the dyadic bilinear systems [(2)] are open-loop (neutrally) stable, they can also be stabilized a quadratic state feedback control [3]–[6]. The quadratic control has the advantage that it guarantees asymptotic stability, but its decay rate becomes extremely slow asymptotically. In comparison, the division control in this note can result in arbitrarily fast decay rate by eigenvalue assignment of $A - bk$. Furthermore, it can apply to open-loop unstable systems. However, it guarantees only practical stability instead of asymptotic stability because of the use of a dead-zone.

The dyadic bilinear systems studied in this note can also be stabilized by the feedback linearization control [13]. However, the feedback linearization control has the same singularity problem as the division control in this note because the dyadic bilinear system fails to have a well-defined relative degree [14].

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# Orthogonal Rational Functions for System Identification: Numerical Aspects

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***Abstract*—Recently, there has been a growing interest in the use of orthogonal rational functions (ORFs) in system identification. There are many advantages over more classical techniques. Probably due to a known explicit expression for the basis functions when the orthogonality weight is uniformly equal to 1 (the so called Malmquist basis), the attention has been on the development of methods using this basis. However, for some discrete identification problems, this choice of the orthogonality weight may still lead to serious numerical problems due to the ill conditioning of the linear system of equations to be solved. In this note, we give an algorithm based on a more general system of ORF to overcome the numerical problem and which allows for a fast-order update of the estimate.**

***Index Terms*—Least squares, orthogonal rational functions (ORFs), system identification.**

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## I. INTRODUCTION

The use of orthogonal rational functions (ORFs) in system identification, has been advertised by many authors in the recent literature. The advantages of representing a transfer function as a linear combination of ORF are numerous. However, in our opinion, some aspect has not been given proper attention. It concerns the choice of the appropriate inner product for the orthogonality. The ORF that are used in the literature are usually orthogonal with respect to a uniform measure on the complex unit circle or the real line.

For example, for discrete time systems, the polynomials orthogonal with respect to the Lebesgue measure on the unit circle are simply the monomials $\{z^k\colon\ k = 0, 1, \ldots\}$ (Fourier basis), while for a general measure they result in more general Szegő polynomials. The computation of the monomials is trivial since they are explicitly known. The general orthogonal polynomials with respect to an arbitrary measure are, however, also simple to compute. Given the moments of the orthogonality measure (e.g., the autocorrelation coefficients of the impulse response), they can be efficiently computed by the Levinson or Schur algorithm. The use of the Fourier basis is inappropriate in cases where the approximating model is computed for successive orders. It does not allow a recursive computation (updating an AR model of degree $n$ to degree $n+1$ requires the recomputation of all the coefficients with respect to the Fourier basis) and it may result in a very bad numerical conditioning of the problem, which is manifested by very big rounding errors that may overrule the exact results completely.

Exactly the same situation occurs in the case of ORF. If the Lebesgue measure is used, an explicit form for the ORF is known. These ORF are attributed by Walsh [1, p. 224] to Takenaka [2] and Malmquist [3] (called TM basis in the rest of this note). The advantage of knowing the basis functions explicitly can be completely overshadowed by the disadvantages from a computational and numerical viewpoint. Models of successive degrees are not nested, so that a recursive update is impossible, and there can be a considerable loss of accuracy due to rounding errors for certain problem settings like for example frequency data in a narrow band and/or a high-degree model.

In this note, we want to show that by choosing an appropriate weight function in discrete time frequency domain identification, the ORF can still be computed efficiently, while the numerical conditioning becomes theoretically optimal.

Of course, given the same poles of the system and given the same objective function, the optimal model that we compute will theoretically be exactly the same as the one computed by any other method. It may also be that with respect to other optimality criteria, our computed solution is worse than the one computed by another method. We do not want to enter a polemic of what an optimality criterion should be, or whether computations should be time or frequency domain based. Our main point is that there is no reason why, from a numerical point of view, one should restrict the discussion to the TM basis. Although the theoretical analysis of the more general basis can be considerably more complicated, the numerics are not, so if another orthogonality weight is more appropriate, then it should be used. We illustrate our point with a very simple case of discrete frequency domain identification, but the idea is applicable in many other situations as well.

We will consider a discrete-time system

$$y(t) = G_0(q)u(t) + v(t), \qquad t \in \{0, \pm 1, \pm 2, \ldots\} \tag{1}$$

where the input $u(t)$ and output $y(t)$ are supposed to be in $l^2$ and the noise $v(t)$ is filtered white noise: $v(t) = H(q)e(t)$. This setting is standard [4].