行政院國家科學委員會補助專題研究計畫成果報告

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REPORT ON NSC PROJECT 89-2115-M-002-012

COHOMOLOGY THEORY IN BIRATIONAL GEOMETRY II

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§0 Abstract

This research is originated from the study of the K-equivalence relation in a birational class of smooth complex projective varieties. The main goal of this note is to formulate the **Main Conjectures** in this theory. Three different proofs of the equivalence of Hodge structures will also be surveyed.

§1 Definitions

Let X be an n dimensional complex normal $\mathbf{Q}$-Gorenstein variety. Recall that X has (at most) terminal (resp. canonical, resp. log-terminal) singularities if there is a (hence for any) resolution $\phi : Y \to X$ such that in the canonical bundle relation

$$K_Y =_{\mathbf{Q}} \phi^* K_X + \sum a_i E_i,$$

we have that $a_i > 0$ (resp. $a_i \geq 0$, resp. $a_i > -1$) for all i. Here, the E_i's vary among the prime components of all the exceptional divisors. For two $\mathbf{Q}$-Gorenstein varieties X and X', we say that X and X' are K-equivalent, written as $X =_K X'$, if there is a smooth variety Y and a birational correspondence $(\phi, \phi') : X \leftarrow Y \to X'$, such that

$$\phi^* K_X =_{\mathbf{Q}} \phi'^* K_{X'}.$$

Notice that this property do not depend on the choice of Y.

The following are some typical geometric situations that lead to K-equivalence. They indicate the scope of applications of this theory.

Fact 1.1 *Any composite of flops is a K-equivalence.*

Fact 1.2 *Let $f: X \dashrightarrow X'$ be a birational map between two varieties with at most canonical singularities such that K_X (resp. $K_{X'}$) is nef along the exceptional locus $Z \subset X$ (resp. $Z' \subset X'$), then $X =_K X'$. Moreover, f extends to an isomorphism in codimension one if X and X' are terminal. This applies, in particular, if both X and X' are minimal models.*

Fact 1.3 *All cohomologically small resolutions of a singular variety, if they exist, are all K-equivalent to each other.*

Fact 1.1 follows directly from the definitin of flops. Fact 1.2 is proved in [Wang1]. Fact 1.3 was essentially proved by [Totaro], he actually showed that cohomological small resolutions are all relative minimal models. Then one may apply 1.2 to obtain 1.3.

§2 Main Conjectures

Just as in the case of birational minimal models, we expect that any K-equivalence can be decomposed into composite of some "nice flops". However, this "rigid decomposition" is too hard to deal with. Instead, we would like to state a series of conjectures on K-equivalent varieties. With the hope to eliminate the necessity of a rigid decomposition result.

Conjecture I (Canonical Isomorphism) Fix a birational map $f : X \dashrightarrow X'$ between two smooth proper complex varieties and let $T := \phi'_* \circ \phi^*$ be the cohomology correspondence induced from a birational correspondence $(\phi, \phi') : X \leftarrow Y \rightarrow X'$ which commutes with f and with smooth Y. If X and X' are K-equivalent, then T induces an canonical isomorphism on cohomologies

$$T : H^i(X, \mathbf{Q}) \rightarrow H^i(X', \mathbf{Q}),$$

which is compatible with the rational Hodge structures.

Notice that T is determined by the closure of the graph $\bar{\Gamma}_f \subset X \times X'$ through the Künneth formula, hence is independent of the choice of Y.

Conjecture II (Quantum Cohomology/Kähler moduli) Under Conjecture I, T also induces an isomorphism on the big quantum cohomology rings of X and X'.

Conjecture III (Birational Complex Moduli) K-equivalent smooth varieties have canonically isomorphic (at least local) moduli spaces. Moreover, the compactified polarized moduli spaces are again K-equivalent.

All these conjectures are known in dimension three through classification results on flops in the minimal model theory. More precisely, **I** is proved in [Kollár] with the help of Saito's theory of mixed Hodge modules. **III** is in [Kollár/Mori] where a theorem on the existence of simaultaneous flops of threefolds are established. **II** is announced recently in [Li/Ruan], where besides classification of three dimensional flops, they further proved a glueing formula of holomorphic curves in order to compute the Gromov-Ruan-Witten invariants that define the quantum cohomology ring. All these techniques involved are very unlikely to possibly be pushed toward higher dimensions. On the other hand, a result in [Wang1] shows that K-equivalent smooth varieties do have the same Hodge numbers in all dimensions.

Inspired by recent work of [Totaro] and [Wang3] on complex elliptic genera, we state the following even more ambitious conjecture:

Conjecture IV (Soft Decomposition Structure) K-equivalent smooth varieties admit symplectic deformations such that the K-equivalence relation deformed into copies of classical flops.

Basically, Totaro proved that "Complex bordism ring modulo classical flops = complex elliptic genera". The author generalizes this to "Complex bordism ring modulo K-equivalece = complex elliptic genera". That is, inside the complex cobordism ring of stably almost complex manifolds, the ideal generated by classical flops are indeed the same as the seemingly much larger ideal generated by all K-equivalent pairs. In other words, Conjecture IV is true up to cobordism. We believe that Conjecture IV will be the key to understand Conjectures **I**, **II** and **III**. In fact, results in [Wang3] already give strong evidence of **III**. Namely, $\chi(X, T_X) = \chi(X', T_{X'})$ for K-equivalent pairs X and X'.

§3 Noncanonical Equivalece of Hodge Structures

The first proof of this statement is essentially in [Wang1] using the theory of motivic integration developped by Denef and Loeser. This proof has been explained in my NSC report of last year [Wang2], so will not be repeated here.

The second proof is a continuation of [Wang1], which will appear in a joint work with J.-K. Yu. Basically in [Wang1] we already know that K-equivalent smooth varieties have the same local zata functions for almost all primes. These will also be the local zeta functions for Crystalline cohomologies by the argument of Katz and Messing. Now a standard density argument will implies that the étale cohomologies of X and X', when regarded as Galois representations of the defining (characteristic zero) field of X and X', have isomorphic semi-simplifications. A straightforward refinement of Faltings' resolution to Fontaine's conjectures, which explain Grothendieck's "mysterious functors" to equate p-adic étale cohomology and Crystalline cohomology after tensoring the Barsoti-Tate groups, we may coclude that the $\mathbf{Q}_p$ Hodge strcutures of the p-adic étale cohomologies of X and X' are isomorphic — basically because the category of pure Hodge structures are semi-simple. This proof does not gives equivalence of $\mathbf{Q}$ Hodge structures. But it shows that the original p-adic integral argument contains much more than just the equivalence of Betti numbers.

The third proof to be explained here is conceptually the simplest one. But it also contains some difficulities to be put in a rigorous proof for a while, until recently when proofs of the long-waiting "Weak Factorization Theorem" were announced.

Let $\mathcal{M}$ be the Grothendieck ring of algebraic varieties over a field k. For X a variety over k, we use $[X]$ to denote its class in $\mathcal{M}$. Now assume that X is smooth and let $\phi : Y \to X$ be a blow up of X along a smooth center $Z \subset X$ of codimension r, with exceptional divisor $E \subset Y$. Then we have the well-known motivic equation for projective bundles $E = \mathbf{P}_Z(N_{Z/X}) \to Z$:

$$[E] = [Z](1 + \mathbf{L} + \cdots \mathbf{L}^{r-1}) = [Z]\frac{\mathbf{L}^r - 1}{\mathbf{L} - 1}.$$

Sicne $[X] - [Z] = [Y] - [E]$, one gets

$$[X] = [Y] - [E] + [Z] = ([Y] - [E]) + [E]\frac{\mathbf{L} - 1}{\mathbf{L}^r - 1}.$$

We call this a "nice" change of variable formula since the "Jacobian factor" here depends only on the class $[E]$ instead of the precise structure of the normal bundle $N_{Z/X}$.

According to the Meta Theorem formulated in [Wang1], we will be able to deduce that two K-equivalent smooth varieties have isomorphic $\mathbf{Q}$ Hodge structures if we can prove such a change of variable formula for any birational morphism $\phi : Y \to X$. Because the Hodge structure realization functor commutes with the formation of Grothendieck rings — thanks to Deligne's Mixed Hodge Theory.

The above simple computation can be performed inductively to show that, for $\phi : Y \to X$ a composite of blowing-ups along smooth centers with

$$K_Y = \phi^* K_X + \sum\nolimits_{i=1}^{n} e_i E_i,$$

The change of variable from X to Y reads

$$[X] = \sum\nolimits_{I \subset \{1,\ldots,n\}} [E_I^\circ] \prod\nolimits_{i \in I} \frac{\mathbf{L} - 1}{\mathbf{L}^{e_i+1} - 1},$$

where $[E_I^\circ] := \bigcap_{i \in I} E_i \setminus \bigcup_{j \notin I} E_j$. This is exactly the formula occuring in the formula for motivic volume. The power for motivic integration allows one to have this formula in a suitable completed localized ring $\hat{\mathcal{M}}$ for arbitrary birational morphism ϕ. But if we are interested in the study of K-equivalence only, we may in fact conclude the more stronger equality that $[X] = [X']$ in $\mathcal{M}$ using the Weak Factorization Theorem whose proof was announced recently by Wlodrasczyk in [Wlod].

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REPORT ON NSC PROJECT 89-2115-M-002-012 A

CHIN-LUNG WANG
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This report is to summarize the work that was initiated during my visiting Professor Ching-Li Chai at University of Pennselvania during September 1999. Some of it was carried out later after my coming back Taiwan in a joint work with B. Hassett and H.-W. Lin [HLW] when we were visiting National Center for Theoretic Sciences in the summer of year 2000. I am very grateful to NSC for supporting me this research.

1. Introduction

Let X be an $(n+1)$-dimensional smooth complex projective variety and let D be a (n-dimensional) smooth ample divisor of X with inclusion map $i : D \to X$. The well known Weak Lefschetz Theorem (see [GrHa]) asserts that the restriction map $i^* : H^k(X; \mathbf{Z}) \to H^k(D; \mathbf{Z})$ is an isomorphism for $k \leq n-1$ and is compatible with the Hodge decomposition. For $n \geq 3$ one deduces from these results that $i^* : \mathrm{Pic}(X) \to \mathrm{Pic}(D)$ is also an isomorphism (Grothendieck has shown that this statement is true over any algebraically closed field [Hart]). While an ample line bundle on X always restricts to an ample line bundle on D, it is not at all clear whether $i^*\mathrm{Amp}(X) = \mathrm{Amp}(D)$.

During the visiting, Chai and me worked out several positive examples — eg. product of projective spaces of dimension at least two. However, further investigation shows that this "Weak Lefschetz Principle for the ample cone" fails in general. We found two types of counterexamples. One is of blow-up type and another is of product type with $\mathbf{P}^1$ factors. Also during this research we provide some partial positive results generalizing those obtained with Chai earlier, notably the one about Mori cones. Though we not yet have a complete picture of how the ample cone behaves under the Weak Lefschetz isomorphism, we pose the question that whether it holds if X admits no birational extremal contractions nor smooth curve fibration structures.

2. A Product Example With $\mathbf{P}^1$ Factors

The simpler examples was found for X being of product type. Take $X = \mathbf{P}^1 \times \mathbf{P}^d$ (for $d \geq 3$) and let D be a divisor of type (d, b) with $b \in \mathbf{N}$. It is very ample and generic D will be smooth. It has defining equation

$$x^d f_0 + x^{d-1} y f_1 + \ldots + f_d y^d = 0,$$

where x, y are coordinates of $\mathbf{P}^1$ and the f_i's are polynomials of degree b in $\mathbf{P}^d$. The projection $p : D \to \mathbf{P}^d$ has positive dimensional fibers exactly when $f_0 = f_1 = \ldots = f_d = 0$. This has no nontrivial solutions for general f_i's since there are more equations than variables. However, if p is a finite morphism then each ample divisor L on $\mathbf{P}^d$ pulls back to an ample divisor on D (this follows from either Nakai-Moishezon's criterion or Kleiman's criterion), yet this pull back divisor is the restriction of the divisor $\mathbf{P}^1 \times L$ on $X = \mathbf{P}^1 \times \mathbf{P}^d$, which is evidently not ample. This gives a simple counterexample to the Weak Lefschetz Principle.

The presence of one dimensional factors has rather special feature, as will be seen in the following section.

3. A Positive Result

It is trivial that the Lefschetz Principle $i^* \mathrm{Amp}(X) = \mathrm{Amp}(D)$ holds if X has Picard number one and $\dim X \geq 4$. One may generalize this in a straightforward manner to obtain:

Theorem 3.1 *Let $i : D \to X = \prod X_j$ be a smooth ample divisor in a finite product of smooth projective varieties such that each X_j has Neron-Severi rank 1, $\dim X_i \geq 2$ and with $\mathrm{Pic}(X) = \bigoplus p_j^* \mathrm{Pic}(X_j)$, ($p_j : X \to X_j$ is the projection map). Then $i^* \mathrm{Amp}(X) = \mathrm{Amp}(D)$ if $\dim X \geq 4$.*

Proof. Let H_j be an ample class on X_j and let $h_j = p_j^* H_j$. Notice the following fact: If $n_j = \dim X_j$, then $\prod h_j^{m_j}$ is an effective cycle if and only if $m_j \leq n_j$ for all j and $p = \prod h_j^{n_j}$ is a positive integer. It is then easy to see that $\sum a_j h_j$ is an ample class on X if and only if that $a_j > 0$ for all j: simply intersect it with $h_j^{n_j - 1} \cdot \prod_{k \neq j} h_k^{n_k}$ to get $a_j p > 0$.

Now let D be given by the ample class $\sum d_j h_j$ with $d_j > 0$. Since $\dim X \geq 4$, a divisor on D takes the form $L|_D = \sum a_j h_j|_D$. We need to show that $L|_D$ is ample implies that $a_j > 0$ for all j. For this, since $n_j \geq 2$, we simply intersect $L|_D$ with the effective cycle $h_j|_D^{n_j-2} . \prod_{k \neq j} h_k|_D^{n_k}$ on D to get

$$0 < h_j^{n_j-2} . \prod_{k \neq j} h_k^{n_k} . \sum a_j h_j . \sum d_j h_j = a_j d_j p.$$

That is, $a_j > 0$. Q.E.D.

Remark 3.2 The condition $\mathrm{Pic}(X) = \bigoplus p_j^* \mathrm{Pic}(X_j)$ holds if all but one X_j satisfy $h^1(X_j, \mathcal{O}_X) = 0$ or if X_j's are all non-isomorphic and their Jacobians are all non-isogenous to each other.

This condition is clearly satisfied by projective spaces of dimensions at least two. In this case the result was obtained earlier in a joint discussion with C.-L. Chai.

4 A Blow-Up Example

We describe here our second counterexample (X, D), which is more involved than the first product one. The details and proofs will be referred to the forthcoming publication work [HLW].

Let $\phi : X \to \mathbf{P}^4$ be the blow-up of $\mathbf{P}^4$ at two distinct points p_1 and p_2. Let ℓ' be the line spanned by p_1 and p_2 and D' a general smooth cubic hypersurface (threefold) containing p_1 and p_2 but not the line ℓ' — in reality we need to make precise the conditions satisfied by D'. We take D to be the proper transform of D' in X. We then show that D is a very ample divisor in X but $i^*\mathrm{Amp}(X) \neq \mathrm{Amp}(D)$. More precisely, we know that $\mathrm{Amp}(D)$ is strictly larger than $i^*\mathrm{Amp}(X)$ if and only if the Mori cone (the closure of the real cone generated by numerical classes of effective one-cycles) $\overline{\mathrm{NE}}(X)$ is strictly larger than $i_*\overline{\mathrm{NE}}(D)$ (by Kleiman's criterion for ampleness [Hart]). Let ℓ be the proper transform of ℓ' in X, which is an effective one-cycle in X. By Weak Lefschetz, $\ell = i_*\lambda$ for some one-cycle λ on D. Our main task is to show that $\lambda \notin \overline{\mathrm{NE}}(D)$. This was done through a detailed study of Del Pezzo surfaces and making use of a strengthening of its classification theory.

In fact we have determined both the ample cones and the Mori cones of X and D.

5 A Partial Theorem on Mori Cones

The counterexample we found in last section suggests the following equality on the negative part of Mori cones of D and X. The proof relies on Mori's theory of extremal rays [Mori] and is essentially contained in [Wis] and [Kollar].

Theorem 5.1 *Let $i : D \to X$ be a smooth ample divisor in a smooth variety X with* $\dim X \geq 4$. *Then* $i_* \mathrm{NE}(D)_{K_D \leq 0} = \mathrm{NE}(X)_{K_D \leq 0}$.

Proof. Since $K_D = K_X|_D + D|_D$, which is numerically strictly more positive than K_X, we know by Mori's theory that $\mathrm{NE}(X)_{K_D \leq 0}$ is a finite polyhedral cone generated by extremal rays. Let $R = \mathbf{R}C$ with $C \cong \mathbf{P}^1$ be such a ray and $\phi : X \to Y$ the corresponding contraction. We want to show that the class of C is also an effective class in D. To see this, if ϕ has a fiber F of dimension at least two, then $D \cap F$ will be such an effective class in D and we are done. If all fibers are one dimensional, Wiśniewski's Theorem shows that Y is smooth and either ϕ is a blow up of Y along a smooth codimension two subvariety Z or ϕ is in fact a conic bundle. These cases are ruled out when $\dim X \geq 4$ by results of Kollár in [Kollar]. Q.E.D.

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