MEAN FIELD EQUATIONS, HYPERELLIPTIC CURVES AND MODULAR FORMS: II

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ABSTRACT. A "pre-modular form" $Z_n(\sigma;\tau)$ of weight $\frac{1}{2}n(n+1)$ is introduced for each $n \in \mathbb{N}$, where $(\sigma,\tau) \in \mathbb{C} \times \mathbb{H}$. Let $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. Then the pre-modular form has the property that every non-trivial zero of $Z_n(\sigma;\tau)$, namely $\sigma \notin E_\tau[2]$, corresponds exactly to the solution to the non-linear mean field equation (MFE)

$$\triangle u + e^u = 8\pi n\,\delta_0$$

on the flat torus E_τ. This paper is a sequel to [3], where the hyperelliptic curve $\bar{X}_n(\tau) \subset \mathrm{Sym}^n E_\tau$ associated to the MFE is constructed. Our construction of $Z_n(\sigma;\tau)$ relies on a detailed study of the correspondence $\mathbb{P}^1 \leftarrow \bar{X}_n(\tau) \rightarrow E_\tau$ where the former map is the hyperelliptic projection and the latter map is induced from the addition law.

CONTENTS

0. INTRODUCTION

Consider the flat torus $E = E_\tau = \mathbb{C}/\Lambda_\tau$, $\tau = a + bi$, $b > 0$ and $\Lambda = \Lambda_\tau = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ with $\omega_1 = 1$ and $\omega_2 = \tau$. Let G be the Green function on E:

$$\begin{cases} -\triangle G = \delta_0 - \dfrac{1}{|E|} \quad \text{on } E, \\ \int_E G = 0, \end{cases} \tag{0.1}$$

With Appendix A written by You-Cheng Chou.

where δ_0 is the Dirac measure at the lattice point $0 \in E$. In this paper, we continue our study initiated in Part I [3] to study the equations

$$(0.2)\qquad n\nabla G(a_i) = \sum_{j=1,\neq i}^{n} \nabla G(a_i - a_j), \quad i = 1, \ldots, n,$$

with unknown $a = (a_1, \cdots, a_n)$, $a_i \in E^\times = E\backslash\{0\}$ and $a_i \neq a_j$ if $i \neq j$. The main theme is the connection between (0.2) and the following mean field equation:

$$(0.3)\qquad \triangle u + e^u = \rho\, \delta_0 \quad \text{on } E, \quad \rho \in \mathbb{R}^+.$$

Equation (0.3) is originated from the prescribed curvature problem in conformal geometry. It is also related to the higher dimensional complex Monge–Ampère equation:

$$(0.4)\qquad \det\Big(\frac{\partial^2 w}{\partial z_i \partial \bar{z}_j}\Big)_{i,j=1}^{d} = e^{-w} \quad \text{on } (E\backslash\{0\})^d.$$

For any solutions $u_i(z)$ to (0.3) corresponding to $\rho = \rho_i$, $i = 1, \ldots, d$, the function

$$w(z_1, \cdots, z_d) = -\sum_{i=1}^{d} u_i(z_i) + (\log 4)d$$

satisfies (0.4) with a logarithmic singularity along the normal crossing divisor $D = E^d\backslash(E\backslash\{0\})^d$. In particular, bubbling solutions to (0.3) give examples of bubbling solutions to (0.4).

The above discussions indicate the fundamental importance to study the concentration phenomenon of bubbling solutions to equation (0.3) in details. Indeed this is the heart inside the connection between (0.2) and (0.3) which we would like to explore. Suppose that $\{u_k\}$ is a sequence of bubbling solutions to (0.3) corresponding to $\rho = \rho_k$. Then it was proved in [4] by PDE method that $\rho_k \to 8\pi n$ with $n \in \mathbb{N}$, and the blow-up set $\{a_1, \cdots, a_n\}$ of $\{u_k\}$ must satisfy (0.2). The PDE method actually applies to a more general class of non-linear equations, but it is in general subtle to treat the reverse direction to construct solutions from $\{a_1, \cdots, a_n\}$. For (0.3) with $\rho = 8\pi n$, it turn out that a sequence of bubbling solutions could be straightforwardly constructed from a non-trivial solution $\{a_1, \cdots, a_n\}$ to (0.2) through the developing map f and the Liouville formula:

$$(0.5)\qquad u^\lambda(z) = \log \frac{8e^{2\lambda}|f'(z)|^2}{(1 + e^{2\lambda}|f(z)|^2)^2}, \quad \lambda \in \mathbb{R},$$

where f is a meromorphic function on $\mathbb{C}$ which has zeros precisely on $z \equiv a_i \pmod{\Lambda}$, $i = 1, \ldots, n$, and satisfies the type II constraints:

$$(0.6)\qquad f(z + \omega_i) = e^{2i\theta_j} f(z), \quad \theta_j \in \mathbb{R}, \quad j = 1, 2.$$

The function f could be explicitly written down in terms of $\{a_1, \cdots, a_n\}$, see e.g. (1.2) in Proposition 1.1.

The aim of this paper is to develop a theory towards understanding the structure of solutions to (0.2) and how it changes under deformations of E_τ in the moduli space. Our starting point is the hyperelliptic geometry developed in Part I [3] which encodes the essential algebraic constraints among the n equations in (0.2). Based on it, the complete information turns out depends on certain *pre-modular forms*, which is the main theme of this paper. To be more precise, the main task of this paper is to prove that if (0.2) has a non-trivial solution for $E = E_\tau$, then this τ must be a zero of a pre-modular form $Z_n(\sigma;\tau)$ of weight $\frac{1}{2}n(n+1)$, with $\sigma \in E\backslash E_\tau[2]$ and $\tau \in \mathbb{H} = \{\,\tau \in \mathbb{C} \mid \operatorname{Im}\tau > 0\,\}$. That is, $Z_n(\sigma;\tau)$ is a modular form of weight $\frac{1}{2}n(n+1)$ with respect to $\Gamma(N)$ whenever $\sigma \in E_\tau[N]$ (the N-torsion points). Our function $Z_n(\sigma;\tau)$ is holomorphic in $\tau \in \mathbb{H}$ and real analytic in $\sigma \in E_\tau$, therefore it also provides a tool to study the corresponding modular form at any N torsion points via deformations in σ. Recently, the idea of deformations was successfully applied in [2] in the case $n = 1$ ($\rho = 8\pi$) to provide a complete solution to (0.3). It is our hope that results of this paper and Part I [3] will lay the foundation to extend the study to all $n \in \mathbb{N}$.

Now we give the detailed descriptions. We start by explaining the notion "non-trivial solutions" to (0.2). In [3] we proved the following result:

Theorem A. *Let $a = (a_1, \cdots, a_n) \in E^n$ be a solution to (0.2), then either $\{a_1, \cdots, a_n\} \cap \{-a_1, \cdots, -a_n\} = \emptyset$ or $\{a_1, \cdots, a_n\} = \{-a_1, \cdots, -a_n\}$. Moreover, (0.2) is equivalent to*

$$\sum_{j=1}^{n} \nabla G(a_j) = 0, \tag{0.7}$$

and the following holomorphic system

$$\sum_{j=1,\neq i}^{n} \left(\zeta(a_i - a_j) + \zeta(a_j) - \zeta(a_i)\right) = 0, \qquad i = 1, \cdots, n. \tag{0.8}$$

We remark that the Weierstrass ζ function has singularities only at the lattice points and thus (0.8) is meaningful only if $a_i \neq 0$ for all i and $a_i \neq a_j$ for $i \neq j$ (as points in E). From the system of equations (0.8) we introduce a hyperelliptic curve $\bar{Y}_n$ as follows. Let

$$\begin{aligned} Y_n = \{\, (a_1, \cdots, a_n) \mid a_i \in E^\times \text{ for all } i,\ a_i \neq a_j \text{ for all } i \neq j, \\ \text{and } (a_1, \cdots, a_n) \text{ satisfies (0.8)}\,\}/S_n, \end{aligned}$$

(unordered n-tuples), and $\bar{Y}_n$ be the closure of Y_n in $\operatorname{Sym}^n E = E^n/S_n$, the n-th symmetric product of E. Then

$$\bar{Y}_n = Y_n \cup \{(0, \cdots, 0)\}$$

Define the map $B : Y_n \to \mathbb{C}$ by

$$a \mapsto B_a = (2n-1)\sum_{i=1}^{n} \wp(a_i),$$

and let $X_n = \{\, \{a_1, \cdots, a_n\} \in Y_n \mid \{a_1, \cdots, a_n\} \cap \{-a_1, \cdots, -a_n\} = \emptyset \,\}$. Then a classical result says that

(i) $Y_n \backslash X_n$ consists of $2n+1$ points (counting with multiplicities),
(ii) $B|_{X_n} : X_n \to \mathbb{C}$ is a two to one map (onto its image),
(iii) $\bar{X}_n = \bar{Y}_n$ is a hyperelliptic curve which can be parametrized by

$$Y_n \cong \{(B, C) \mid C^2 = \ell_n(B)\}$$

where $\ell_n(B)$ is a polynomial in B of degree $2n+1$. The branch points of the hyperelliptic curve is precisely $\bar{X}_n \backslash X_n$. In particular, the algebraic curve $\bar{X}_n$ is smooth if $\ell_n(B)$ has no multiple roots.

A proof of the above statements with rigorous details can be found in [3], which is based on a detailed study of the (integral) *Lamé equation* on E_τ

$$w'' = (n(n+1)\wp + B)w \tag{0.9}$$

from both the analytic and algebraic point of views. Indeed, let

$$w_a(z) := e^{z \sum \zeta(a_i)} \prod_{i=1}^{n} \frac{\sigma(z - a_i)}{\sigma(z)} \tag{0.10}$$

be the classical Hermite–Halphen ansatz. Then w_a and w_{-a} are independent solutions to (0.9) with $B = B_a$ if and only if $a \in X_n$. Historically $\bar{X}_n$ is also known as the *Lamé curve*.

By Theorem A, it is natural to separate the study of (0.2) into two stages. We first study the algebraic geometry associated to the hyperelliptic curve $\bar{X}_n$ and then study the remaining single equation on Green functions (0.7) for $a \in \bar{X}_n$. Notice that, by the anti-symmetry of ∇G, (0.7) holds automatically if a is a branch point of $\bar{X}_n$. Hence the system (0.2) contains all branch points of $\bar{X}_n$ as its solutions. We say that a solution a to (0.2) is *non-trivial* if a is not a branch point of $\bar{X}_n$.

We start by reviewing of the simplest case $n = 1$. Then $\bar{X}_1 \cong E$ and there is no new hyperelliptic geometry. Also a is a solution to (0.2) if and only if $\nabla G(a) = 0$, namely $a \in E$ is a critical point of G. The branch points of E consist of all half periods $\frac{1}{2}\omega_j$, $j = 1, 2, 3$. Hence a non-trivial solution p is simply a non half-period critical point of G. How many such points might G have? This had been answered in [11]:

Theorem B. *For any $\tau \in \mathbb{H}$, the Green function $G(z; \tau)$ on the flat torus E_τ has at most five critical points.*

Since $G(z)$ is an even function, the extra critical points appear in pair $\pm p$. The following result was announced in [11] and recently proved in [12]:

Theorem C. *Suppose that the pair of non half-period critical points $\{\pm p\}$ of G exists, the $\pm p$ are the minimal points of G. In fact any solution to (0.3) must be a minimizer of the non-linear functional*

$$J_{8\pi}(u) = \frac{1}{2}\int_E |\nabla u|^2 - 8\pi \log \int_E e^{-8\pi G + u}$$

on $u \in H^1(E) \cap \{u \mid \int_E u = 0\}$.

To proceed, we need the detailed analytic description of Green functions in [11]. Recall the theta function $\vartheta := \vartheta_1$, where

$$\vartheta_1(z;\tau) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+\frac{1}{2})^2} e^{(2n+1)\pi i z}, \quad q = e^{2\pi i \tau}.$$

Then the Green function is given by

$$G(z;\tau) = -\frac{1}{2\pi} \log|\vartheta(z;\tau)| + \frac{y^2}{2b} + C(\tau), \tag{0.11}$$

where $z = x + iy = t\omega_1 + s\omega_2$, $\eta_i = 2\zeta(\frac{1}{2}\omega_i)$, $i = 1, 2$, are quasi-periods and $C(\tau)$ is a constant so that $\int_{E_\tau} G = 0$. Using $(\log\vartheta)_z = \zeta(z) - \eta_1 z$ and the Legendre relation $\eta_1\omega_2 - \eta_2\omega_1 = 2\pi i$, we have

$$G_z(z;\tau) = -\frac{1}{4\pi}\big(\zeta(z;\tau) - t\eta_1(\tau) - s\eta_2(\tau)\big). \tag{0.12}$$

For any $z = t\omega_1 + s\omega_2$, $t, s \in \mathbb{R}$, we define the *Hecke function* $Z(z;\tau) = Z_{t,s}(\tau)$ by

$$Z_{t,s}(\tau) = Z(t\omega_1 + s\omega_2;\tau) = \zeta(t\omega_1 + s\omega_2;\tau) - t\eta_1(\tau) - s\eta_2(\tau).$$

Therefore z is a critical point of $G(z;\tau)$ if and only if $Z_{t,s}(\tau) = 0$. Since $Z_{t+1,s} = Z_{t,s} = Z_{t,s+1}$, we may assume that $0 \le t, s < 1$.

We found $Z_{t,s}(\tau)$ in [11]. But it first appeared in [9] where Hecke showed that if $z = t\omega_1 + s\omega_2$ is an N torsion point then $Z_{t,s}(\tau)$ is a modular form of weight one with respect to $\Gamma(N) = \{\, A \in \mathrm{SL}(2,\mathbb{Z}) \mid A \equiv I_2 \pmod N \,\}$. $Z(z;\tau)$ is holomorphic in τ, but not in z. $Z_{t,s}(\tau) \equiv 0$ for z a half-period, thus we are interested in equation $Z_{t,s}(\tau) = 0$ with $(t,s) \notin \frac{1}{2}\mathbb{Z}^2$.

Now we turn to (0.2) and (0.3) for general $n \ge 1$. A central issue is to study the following

Conjecture 0.1. *Under the deformations in τ, equation (0.2) has no bifurcations at any non-trivial solution.*

For $n = 1$, Conjecture 0.1 can be deduced from Theorem B (c.f. [12]). For $n \ge 2$, we do not expect an extension of Theorem B to work.

Our approach to Conjecture 0.1 for $n \ge 2$ is to construct a pre-modular form $Z_n(\sigma;\tau)$ with $\sigma \in E_\tau$ which generalizes the Hecke function ($Z_1 = Z$), and study this single analytic function Z_n instead. The pre-modular form $Z_n(\sigma;\tau)$ is naturally associated to the family of hyperelliptic curves $\bar{X}_n(\tau)$, $\tau \in \mathbb{H}$. The goal is to show that any non-trivial solution $a = (a_1, \cdots, a_n)$ to (0.2) comes from the zero of $Z_n(\sigma;\tau)$ with $\sigma = \sum_{i=1}^n a_i$, and vice versa.

Consider the Green function equation (0.7) in Theorem A. Let $\mathbf{z}_n(a) = \zeta(\sum_{i=1}^n a_i) - \sum_{i=1}^n \zeta(a_i)$. If $\sum_{i=1}^n a_i \ne 0$ then

$$\begin{aligned} -4\pi \sum \nabla G(a_i) &= \sum(\zeta(t_i\omega_1 + s_i\omega_2) - t_i\eta_1 - s_i\eta_2) \\ &= Z(\textstyle\sum a_i) - \mathbf{z}_n(a). \end{aligned}$$

In §1 we will show that in fact $\sum_{i=1}^n a_i \notin E_\tau[2]$ for any $a \in X_n(\tau)$. Hence (0.7) is equivalent to

$$\mathbf{z}_n(a) = Z\Big(\sum_{i=1}^n a_i\Big). \tag{0.13}$$

This motivates us to study the map

$$\sigma : \bar{X}_n \to E, \quad a \mapsto \sigma(a) := \sum_{i=1}^n a_i$$

induced from the addition map $E^n \to E$. The algebraic curve $\bar{X}_n(\tau)$ might be singular for some τ, but it must be irreducible (c.f. Theorem 1.5 (3)). In particular, σ is a finite morphism and $\deg \sigma$ is defined. The function field $K(\bar{X}_n)$ is also defined and is finite over $K(E) \cong \mathbb{C}[\wp, \wp']$. It is well known that $\deg \sigma = [K(\bar{X}_n) : K(E)] =$ number of points for a general fiber.

Theorem 0.2. *The map $\sigma : \bar{X}_n \to E$ has degree $\frac{1}{2}n(n+1)$.*

In §1, we will apply the *Theorem of the Cube* in the theory of abelian varieties (see e.g. [15]) to prove it. From Theorem 0.2, there is a polynomial $W_n(z) \in \mathbb{C}[g_2, g_3, \wp(\sigma), \wp'(\sigma)][z]$ of degree $\frac{1}{2}n(n+1)$ which defines the covering map σ between algebraic curves.

The major question is to find a natural primitive element of this covering map. Namely a rational function on $\bar{X}_n$ which has W_n as its minimal polynomial. This is achieved by the following fundamental theorem:

Theorem 0.3. *The rational function $\mathbf{z}_n \in K(\bar{X}_n)$ is a primitive generator for the field extension $K(\bar{X}_n)$ over $K(E)$ which is integral over the affine curve $E^\times$.*

Moreover, the induced map $\mathbf{z}_n : \bar{X}_n \to \mathbb{P}^1$ also has degree $\frac{1}{2}n(n+1)$, with its fibration structure over $\infty \in \mathbb{P}^1$ analytically isomorphic to $\sigma : \bar{X}_n \to E$ over 0.

This means that $W_n(\mathbf{z}_n) = 0$, and conversely for general τ and $\sigma = \sigma_0 \in E_\tau$, the roots of $W_n(z) = 0$ are precisely those $\frac{1}{2}n(n+1)$ values $z = \mathbf{z}_n(a)$ with $\sigma(a) = \sigma_0$.

The proof is given in §2, Theorem 2.1 and 2.9. A major tool used in the proof is the notion of *tensor product* of two Lamé equations $w'' = I_1 w$ and $w' = I_2 w$, where $I = n(n+1)\wp(z)$, $I_1 = I + B_a$ and $I_2 = I + B_b$. Indeed, for given τ with $\bar{X}_n$ not necessarily being smooth, and for a given general point $\sigma_0 \in E$, we need to show that the $\frac{1}{2}n(n+1)$ points on the fiber of $\bar{X}_n \to E$ above σ_0 has distinct $\mathbf{z}_n$ values. From the definition of $\mathbf{z}_n$, it is enough to show that for $\sigma(a) = \sigma(b) = \sigma_0$, the condition $\sum \zeta(a_i) = \sum \zeta(b_i)$ implies $B_a = B_b$ and then $a = b$ if $\sigma_0 \notin E[2]$.

If $w_1'' = I_1 w_1$ and $w_2'' = I_2 w_2$, then the product $q = w_1 w_2$ satisfies the fourth order ODE (tensor product) given by

$$q'''' - 2(I_1 + I_2)q'' - 6I'q' + ((B_a - B_b)^2 - 2I'')q = 0. \tag{0.14}$$

We remark that if $B_a = B_b$, then $I_1 = I_2$ and q actually satisfies a third order ODE as the second symmetric product of a Lamé equation. This is a useful tool in Part I [3] in the study of the Lamé curve.

If however $a \neq b$, by (0.10) and addition law we find that $q = w_a w_{-b} + w_{-a} w_b$ is an *even elliptic function* solution to (0.14), namely a *polynomial* in $x = \wp(z)$. This leads to strong constraints on (0.14) in variable x and eventually leads to a contradiction for generic choices of σ_0.

Now we set

$$Z_n(\sigma;\tau) = W_n(Z(\sigma;\tau)). \tag{0.15}$$

Then $Z_n(\sigma;\tau)$ is pre-modular of weight $\frac{1}{2}n(n+1)$, by which we mean that it is modular with respect to $\Gamma(N)$ whenever $\sigma \in E_\tau[N]$ being an N-torsion point. From the construction and (0.13) it is readily seen that $Z_n(\sigma;\tau)$ is the generalization of the Hecke function $Z = Z_1$ we are looking for. In fact, for general $n \geq 1$, we establish the following correspondences in §2:

(1) Solution u to the mean field equation (0.3) for $\rho = 8\pi n$.

(2) Periods integral $\displaystyle\int_L \frac{f'}{f} \in \sqrt{-1}\mathbb{R}$ for any one cycle L (c.f. (1.5)).

(3) Green equation $\displaystyle\sum_{i=1}^n \nabla G(a_i) = 0$ on the hyperelliptic curve X_n.

(4) Coincidence equation $\mathbf{z}_n(a) = Z(\sigma(a))$ for $a \in X_n$.

(5) Zero of pre-modular form $Z_n(\sigma;\tau) := W_n(Z)$ with $\sigma \notin E_\tau[2]$.

(For more precise statements on (5) see Theorem 2.13.)

For σ being an N-torsion point, the modular form $Z_2(\sigma;\tau)$ and $Z_3(\sigma;\tau)$ were first constructed by Dahmen [6] in his study on integral Lamé equations (0.9) with algebraic solutions (i.e. with finite monodromy group). Indeed, for $\sigma \in E_\tau[N]$,

$$\begin{aligned} Z_2(\sigma;\tau) &= Z^3(\sigma) - 3\wp(\sigma;\tau)Z(\sigma) - \wp'(\sigma;\tau), \\ Z_3(\sigma;\tau) &= Z^6 - 15\wp Z^4 - 20\wp' Z^3 + (\tfrac{27}{4}g_2 - 45\wp^2)Z^2 - 12\wp'\wp Z - \tfrac{5}{4}\wp'^2. \end{aligned}$$

The construction of Z_2 is based on the addition formula, and the construction of Z_3 is based on the $n = 2$ case and a technical intermediate step: for $\mathbf{z}_3 = \zeta(\sigma) - \sum_{i=1}^3 \zeta(a_i)$ with $\sigma = \sum_{i=1}^3 a_i$, a classical cubic formula

$$\mathbf{z}_3^3 = 3(\wp(\sigma) + \textstyle\sum \wp(a_i))\mathbf{z}_3 + (\wp'(\sigma) - \sum \wp'(a_i)) \tag{0.16}$$

holds (c.f. Lemma 4.3). By eliminating terms involving a_i's, a degree six polynomial equation $W_3(\mathbf{z}_3) = 0$ is then achieved. The existence of a modular form Z_n of weight $\frac{1}{2}n(n+1)$ is also conjectured in [6], though it is unclear how his method would proceed for $n \geq 4$.

For all σ and n, the existence of the (pre)-modular forms $Z_n(\sigma;\tau)$ now follows from Theorem 0.3 and (0.15). However the explicit construction is still a challenging problem.

In a different direction, the Lamé curve had also been studied extensively in the *finite band integration theory*. In the complex case, this theory concerns about the eigenvalue problem on a second order ODE $Lw := w'' - Iw = Bw$ with eigenvalue B. The potential $I = I(z)$ is called a *finite-gap (band) potential* if the ODE has only logarithmic free solutions except for a finite number of $B \in \mathbb{C}$. The integral Lamé equations (with $I(z) = n(n+1)\wp(z)$) provide good (indeed earliest) examples of them. Using this theory, Maier [14] had recently written down *an explicit map* $\pi : \bar{X}_n \to E$ in terms of coordinates (B, C) on $\bar{X}_n$ (in our notations). It turns out we can prove

Theorem 0.4 (c.f. Theorem 3.5)**.** *The map π agrees with $\sigma : \bar{X}_n \to E$.*

In principle this allows us to compute the polynomial $W_n(z)$ for any $n \in \mathbb{N}$ by eliminating variables (B, C), though in practice the needed calculations are very demanding and time consuming. §3 is devoted to this explicit construction. In particular the weight 10 pre-modular form $Z_4(\sigma; \tau)$ is explicitly written down by a couple hours *Mathematica* calculations (c.f. Example 3.10):

(0.17)

$$\begin{aligned} Z_4(\sigma;\tau) = Z^{10} &- 45\wp Z^8 - 120\wp' Z^7 + (\tfrac{399}{4}g_2 - 630\wp^2)Z^6 - 504\wp\wp' Z^5 \\ &- \tfrac{15}{4}(280\wp^3 - 49g_2\wp - 115g_3)Z^4 + 15(11g_2 - 24\wp^2)\wp' Z^3 \\ &- \tfrac{9}{4}(140\wp^4 - 245g_2\wp^2 + 190g_3\wp + 21g_2^2)Z^2 \\ &- (40\wp^3 - 163g_2\wp + 125g_3)\wp' Z + \tfrac{3}{4}(25g_2 - 3\wp^2)(\wp')^2. \end{aligned}$$

One might be curious on the possibility to extend the more classical construction of $Z_n(\sigma; \tau)$ for $n \le 3$ to $n \ge 4$. In §4.2 we explain why no classical formula like (0.16) with degree n can exist for $n \ge 4$, and thus it is not possible to derive the explicit expression of $Z_n(\sigma; \tau)$ via a naive induction for $n \ge 4$. For this purpose, and for the sake of completeness, we present in §4.1 an alternative treatment of Dahmen's construction for $n = 2, 3$ in a format that is needed for our discussions in §4.2. At this point, it might be insightful to point out that $K(\bar{X}_n)$ is in general *not Galois* over $K(E)$ (under the morphism $\sigma : \bar{X}_n \to E$, c.f. Example 4.4).

The existence and effective construction of $Z_n(\sigma; \tau)$ opens the door to extend our results on equations (0.2) and (0.3) for $n = 1$ (established in [11, 12, 2]) to general $n \in \mathbb{N}$. As a related example, the explicit expression of $Z_4(\sigma; \tau)$ can be used to solve Dahmen's conjecture on a counting formula for integral Lamé equations with finite monodromy for $n = 4$ (c.f. §5.2 and Appendix A). Further applications of our pre-modular forms will be discussed in subsequent works. Some of them, especially those related to equations (0.2) and (0.3), are briefly described in §5.

1. Geometry on X_n for $\rho = 8n\pi$

We would like to extend some of the results proved in [11] for $\rho = 8\pi$ to the case $\rho = 8n\pi$ for all $n \in \mathbb{N}$. Especially we will generalize Hecke's function Z to Z_n whose non-trivial zeros again encode the structure of solutions to the corresponding mean field equation

$$\triangle u + e^u = 8\pi n\, \delta_0 \qquad \text{in } E = E_\tau. \tag{1.1}$$

We first recall the structural results from Part I [3].

Proposition 1.1 (Periods integrals and type II evenness [3])**.**

(1) *If solutions exist for $\rho = 8n\pi$, then there is a unique even solution within each type II scaling family.*
(2) *The solution u is determined by the zeros $a_1, \cdots, a_n$ of its developing map f (through expression (0.5) with $\lambda = 0$). In fact for $g = f'/f$,*

$$\begin{aligned} g(z) &= \sum_{i=1}^{n} \frac{\wp'(a_i)}{\wp(z) - \wp(a_i)} = \sum_{i=1}^{n} \Omega(z, a_i), \\ f(z) &= f(0) \exp \int^{z} g(\xi)\, d\xi \\ &= f(0) \prod_{i=1}^{n} \exp \int^{z} \Omega(\xi, a_i)\, d\xi, \end{aligned} \tag{1.2}$$

subject to the non-degenerate conditions $a_i \notin E[2]$, $a_i \neq \pm a_j$ for $i \neq j$.

The condition $\operatorname{ord}_{z=0} g(z) = 2n$ leads to $n-1$ equations for $a_1, \ldots, a_n$: Under the notations $(w, x_j, y_j) = (\wp(z), \wp(p_j), \wp'(p_j))$,

$$\begin{aligned} g(z) &= \sum_{j=1}^{n} \frac{1}{w} \frac{y_j}{1 - x_j/w} \\ &= \sum_{j=1}^{n} \frac{y_j}{w} + \sum_{j=1}^{n} \frac{y_j x_j}{w^2} + \cdots + \sum_{j=1}^{n} \frac{y_j x_j^r}{w^{r+1}} + \cdots . \end{aligned}$$

Since $g(z)$ has a zero at $z = 0$ of order $2n$ and $1/w$ has a zero at $z = 0$ of order two, we get $x_i \neq x_j$ for $i \neq j$ and

$$\sum_{j=1}^{n} y_j x_j^r = \sum_{j=1}^{n} \wp'(a_j) \wp(a_j)^r = 0, \quad 0 \le r \le n-2.$$

This gives the polynomial system describing the developing maps. Notice that the condition $y_i \neq 0$ follows automatically.

Theorem 1.2 (Green/polynomial system [3])**.** *For $\rho = 8n\pi$, $n \in \mathbb{N}$, the n equations for $a_1, \cdots, a_n$ with $a_i \neq \pm a_j$ for $i \neq j$ are precisely*

$$\wp'(a_1)\wp^r(a_1) + \cdots + \wp'(a_n)\wp^r(a_n) = 0, \tag{1.3}$$

where $r = 0, \ldots, n-2$, and

$$\nabla G(a_1) + \cdots + \nabla G(a_n) = 0. \tag{1.4}$$

Indeed the Green function equation (1.4) is equivalent to the type II condition (0.6). From Proposition 1.1 (2), this means that the periods integral $\int_{L_i} \Omega \in \sqrt{-1}\mathbb{R}$ for L_1, L_2 being the two fundamental periods of E. Let f be given by (1.2). By evaluating the periods integrals (c.f. Part I [3, (5.4) and (5.7)]), we have

$$\begin{aligned} f(z+\omega_1) &= e^{-4\pi i \sum_i s_i} f(z), \\ f(z+\omega_2) &= e^{4\pi i \sum_i t_i} f(z), \end{aligned} \tag{1.5}$$

where $a_i = t_i\omega_1 + s_i\omega_2$ for $i = 1, \ldots, n$. Then by the Liouville theorem, we obtain solutions u^λ to (1.1) by (0.5).

Note that the algebraic conditions in (1.3) are equivalent to

$$\sum_{j \neq i} \frac{\wp'(a_j) + \wp'(a_i)}{\wp(a_j) - \wp(a_i)} = 0, \qquad i = 1, \ldots, n. \tag{1.6}$$

(See Part I [3, Theorem 0.6].) By the addition formula for $\wp$, (1.6) is identical with (0.8) provided that $a_i \neq \pm a_j$ for $i \neq j$, or equivalently $\wp(a_i) \neq \wp(a_j)$ for $i \neq j$. Thus, a scaling family of solutions to (1.1) corresponds exactly to a non-trivial solution to (0.2).

Proposition 1.3 (Hyperelliptic geometry, the Liouville curve [3])**.** *For $x_i := \wp(a_i)$ with $x_i \neq x_j$ for $i \neq j$ and $y_i := \wp'(a_i)$, the first $n-1$ algebraic equations*

$$\sum y_i x_i^r = 0, \quad r = 0, \ldots, n-2, \tag{1.7}$$

defines a non-singular open algebraic curve $X_n \subset \mathrm{Sym}^n E$, called the Liouville curve. It has a hyperelliptic structure under the 2 to 1 map $a \mapsto \sum \wp(a_i)$, or algebraically:

$$X_n \to \mathbb{P}^1, \qquad (x_i, y_i)_{i=1}^n \mapsto \sum_{i=1}^n x_i.$$

This is closely related to (integral) Lamé equations: Write

$$\begin{aligned} f &= \exp \int g\, dz = \exp \int \sum_{i=1}^n (2\zeta(a_i) - \zeta(a_i - z) - \zeta(a_i + z))\, dz \\ &= e^{2\sum_{i=1}^n \zeta(a_i) z} \prod_{i=1}^n \frac{\sigma(z - a_i)}{\sigma(z + a_i)} = \frac{w_a}{w_{-a}}, \end{aligned}$$

where

$$w(z) = w_a(z) := e^{z \sum \zeta(a_i)} \prod_{i=1}^n \frac{\sigma(z - a_i)}{\sigma(z)} \tag{1.8}$$

is the Hermite–Halphen ansatz for solutions to integral Lamé equations (see [16, 3]).

Proposition 1.4 (Explicit map $a \mapsto B_a$ [3])**.** *$a \in X_n$ if and only if w_a and w_{-a} are independent solutions of the Lamé equation*

$$\frac{d^2 w}{dz^2} - \big(n(n+1)\wp(z) + B_a\big) w = 0, \tag{1.9}$$

where $B_a = (2n-1)\sum_{i=1}^n \wp(a_i)$.

The full information on the compactified hyperelliptic curve $\bar{X}_n \to \mathbb{P}^1$, especially on the branch points added, is described by

Theorem 1.5 (Hyperelliptic geometry on $\bar{X}_n$, the Lamé curve [3])**.**

(1) *The natural compactification $\bar{X}_n \subset \operatorname{Sym}^n E$ coincides with the, possibly singular, projective model of the hyperelliptic curve defined by*

$$\begin{aligned} C^2 &= \ell_n(B, g_2, g_3) \\ &= 4Bs_n^2 + 4g_3 s_{n-2}s_n - g_2 s_{n-1}s_n - g_3 s_{n-1}^2, \end{aligned} \tag{1.10}$$

in (B, C), where $s_k = s_k(B, g_2, g_3) = r_k B^k + \cdots \in \mathbb{Q}[B, g_2, g_3]$, is a recursively defined polynomial of homogeneous degree k with $\deg g_2 = 2$, $\deg g_3 = 3$, and $B = (2n-1)s_1$.

(2) *$\deg \ell_n = 2n+1$ and $\bar{X}_n$ has arithmetic genus $g = n$.*

(3) *The curve $\bar{X}_n$ is smooth except for a finite number of τ, namely the discriminant loci of $\ell_n(B, g_2, g_3)$ so that $\ell_n(B)$ has multiple roots. $\bar{X}_n$ is an irreducible curve which is smooth at infinity.*

(4) *The $2n+2$ branch points $a \in \bar{X}_n \backslash X_n$ are characterized by $-a = a$. In fact $\{-a_i\} \cap \{a_i\} \neq \emptyset \Rightarrow -a = a$. Also $0 \in \{a_i\} \Rightarrow a = (0, 0, \cdots, 0)$.*

(5) *The limiting system of (1.7) at $a = 0^n$ is given by*

$$\sum_{i=1}^n t_i^{2r+1} = 0, \quad r = 1, \ldots, n-1 \tag{1.11}$$

under the constraints

$$t_i \neq 0, \quad \text{and} \quad t_i \neq -t_j. \tag{1.12}$$

Moreover, the system (1.11) and (1.12) has a unique solution in $\mathbb{P}^{n-1}$ up to permutations.

The point $a = 0^n \in \bar{X}_n$ is referred as the point at infinity. For the other $2n+1$ finite branch points with $a = -a$, $w_a = w_{-a}$ which is still a solution to the Lamé equation. In the literature, these $2n+1$ functions are known as the Lamé functions. Indeed, there are four species of them, depending on the number of half periods contained in $\{a_i\}$. We call them being of type O, I, II, and III respectively. For $n = 2k$ being even, a must be of type O or II. For $n = 2k+1$ being odd, a must be of type I or III. There are factorizations of the polynomial $\ell_n(B)$ according to the types:

Proposition 1.6. [8, 16] *In terms of $e_i = \wp(\frac{1}{2}\omega_i)$, we may write*

$$\ell_n(B) = c_n^2 l_0(B) l_1(B) l_2(B) l_3(B),$$

where $c_n > 0$ is a constant, $l_i(B)$'s are monic polynomials in B such that

(1) *For $n = 2k$, $l_0(B)$ consists of type O roots with $\deg l_0(B) = \frac{1}{2}n + 1 = k + 1$. For $i = 1, 2, 3$, $l_i(B)$ consists of type II roots a which does not contain $\frac{1}{2}\omega_i$. Moreover, $\deg l_i(B) = \frac{1}{2}n = k$.*

(2) *For $n = 2k + 1$, $l_0(B)$ consists of type III roots with $\deg l_0(B) = \frac{1}{2}(n - 1) = k$. For $i = 1, 2, 3$, $l_i(B)$ consists of type I roots a which contains $\frac{1}{2}\omega_i$. Moreover, $\deg l_i(B) = \frac{1}{2}(n + 1) = k + 1$.*

We remark that both Proposition 1.6 and Theorem 1.5 (5) will be used in the proof of Theorem 0.2 (= Theorem 1.10 later in this section). Here are some examples to illustrate Proposition 1.6:

Example 1.7. Decomposition $\ell_n(B) = c_n^2 l_0(B) l_1(B) l_2(B) l_3(B)$ for $1 \le n \le 5$.

(1) $n = 1$, $k = 0$, $\bar{X}_1 \cong E$,

$$C^2 = \ell_1(B) = 4B^3 - g_2 B - g_3 = 4\prod_{i=1}^{3}(B - e_i).$$

(2) $n = 2$, $k = 1$, (notice that $e_1 + e_2 + e_3 = 0$)

$$\begin{aligned} C^2 = \ell_2(B) &= \tfrac{4}{81}B^5 - \tfrac{7}{27}g_2 B^3 + \tfrac{1}{3}g_3 B^2 + \tfrac{1}{3}g_2^2 B - g_2 g_3 \\ &= \frac{2^2}{3^4}(B^2 - 3g_2)\prod_{i=1}^{3}(B + 3e_i). \end{aligned}$$

(3) $n = 3$, $k = 1$, $\deg l_i(B) = 2$ for $i = 1, 2, 3$,

$$\begin{aligned} C^2 = \ell_3(B) &= \frac{1}{2^2 3^4 5^4} B(16B^6 - 504 g_2 B^4 + 2376 g_3 B^3 \\ &\qquad + 4185 g_2^2 B^2 - 36450 g_2 g_3 B + 91125 g_3^2 - 3375 g_2^3) \\ &= \frac{2^2}{3^4 5^4} B \prod_{i=1}^{3}(B^2 - 6e_i B + 15(3e_i^2 - g_2)). \end{aligned}$$

(4) $n = 4$, $k = 2$, $\deg l_0(B) = 3$,

$$C^2 = \ell_4(B) = \frac{1}{3^8 5^4 7^4}(B^3 - 52 g_2 B + 560 g_3)\prod_{i=1}^{3}(B^2 + 10 e_i B - 7(5e_i^2 + g_2)).$$

(5) $n = 5$, $k = 2$, $\deg l_i(B) = 3$ for $i = 1, 2, 3$,

$$\begin{aligned} C^2 = \ell_5(B) &= \frac{1}{3^{12} 5^4 7^4 11^2}(B^2 - 27 g_2) \\ &\quad \times \prod_{i=1}^{3}(B^3 - 15 e_i B^2 + (315 e_i^2 - 132 g_2)B + e_i(2835 e_i^2 - 540 g_2)). \end{aligned}$$

Now we study the last equation (1.4) in Theorem 1.2, namely the Green function equation on $\bar{X}_n$:

$$\sum_{i=1}^{n} \nabla G(a_i) = 0. \tag{1.13}$$

Definition 1.8 (Fundamental rational function)**.** *Consider the function on E^n:*

$$\mathbf{z}_n(a_1, \ldots, a_n) := \zeta(a_1 + \cdots + a_n) - \sum_{i=1}^{n} \zeta(a_i).$$

$\mathbf{z}_n$ is a rational function on E^n since it is meromorphic and periodic in each variable a_i, and E^n is an abelian variety.

It satisfies the reduction property along half periods: If $a = a' \cup \{\frac{1}{2}\omega_i\}$ then $\mathbf{z}_n(a) = \mathbf{z}_{n-1}(a')$.

Consider the identity (where $a_i = t_i\omega_1 + s_i\omega_2$):

$$\begin{aligned}
-4\pi \sum \nabla G(a_i) &= \sum Z(a_i) \\
&= \sum(\zeta(t_i\omega_1 + s_i\omega_2) - t_i\eta_1 - s_i\eta_2) \\
&= \zeta(\sum a_i) - (\sum t_i)\eta_1 - (\sum s_i)\eta_2 - \mathbf{z}_n(a) \\
&= Z(\sum a_i) - \mathbf{z}_n(a).
\end{aligned}$$

Hence the Green equation (1.13) is equivalent to

$$\mathbf{z}_n(a) = Z(\sum a_i). \tag{1.14}$$

This motivates us to study the branched covering map

$$\sigma : \bar{X}_n \to E, \qquad a \mapsto \sigma(a) := \sum_{i=1}^{n} a_i \tag{1.15}$$

induced from the addition map $E^n \to E$.

Now we may move to the discussions concerning new modular functions. We start by determining $\deg \sigma$.

Remark 1.9. For the reader's convenience we recall some definitions and facts. The function field $K(C)$ is defined for any irreducible algebraic curve C. Two irreducible curves which are isomorphic outside a finite number of points (birational) have the same function field. For a finite morphism of irreducible curves $f : X \to Y$, $K(X)$ is a finite extension of $K(Y)$ and the degree of f is defined by $\deg f = [K(X) : K(Y)]$. Geometrically $\deg f$ is also the number of points for a general fiber $f^{-1}(p)$, $p \in Y$. Indeed, for any $L \in \operatorname{Pic} Y$ (line bundle), we have $\deg f^*L = \deg f \deg L$. Taking $L = \mathcal{O}_Y(p)$ with p a general smooth point of Y then gives the result.

A standard reference is [10, II.6, Proposition 6.9], where nonsingular curves are treated. The irreducible case is reduced to the nonsingular case through normalizations $\tilde{X} \to X$ and $\tilde{Y} \to Y$, since it is clear that the induced finite morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ has the same degree as f. Furthermore, the definition also extends to the case $f : X \to Y$ where $X = \bigcup_{i=1}^{k} X_i$ has a finite number of irreducible components. We require that $f|_{X_i}$ is a finite morphism for each i and then $\deg f := \sum_{i=1}^{k} \deg f|_{X_i}$. Since all curves considered here are proper (projective), it is enough to require $f|_{X_i}$ to be non-constant to ensure that it is a finite morphism.

Theorem 1.10. *The map $\sigma : \bar{X}_n \to E$ has degree $\frac{1}{2}n(n+1)$.*

Proof. The idea is to apply *Theorem of the Cube* [15, p.58, Corollary 2] for morphisms from an arbitrary variety V (not necessarily smooth) into abelian varieties (here the torus E): For any three morphisms $f, g, h : V \to E$ and a line bundle $L \in \operatorname{Pic} E$, we have

$$\begin{aligned}(f+g+h)^*L &\cong (f+g)^*L \otimes (g+h)^*L \otimes (h+f)^*L \\ &\qquad \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.\end{aligned} \tag{1.16}$$

We will apply it to the algebraic curve $V = V_n \subset E^n$ which consists of the ordered n-tuples a's so that $V_n/S_n = \bar{X}_n$.

Take L be any line bundle with $\deg L \neq 0$. Using the fact that $\deg F^*L = \deg F \deg L$ for any finite morphism $F : V \to E$, (1.16) implies that

$$\begin{aligned}&\deg(f+g+h) = \\ &\quad \deg(f+g) + \deg(g+h) + \deg(h+f) - \deg f - \deg g - \deg h.\end{aligned} \tag{1.17}$$

We prove inductively that for $j = 1, \ldots, n$ the branched covering map $f_j : V_n \to E$ defined by

$$f_j(a) := a_1 + \cdots + a_j$$

has degree $\frac{1}{2}j(j+1)n!$. The case $j = n$ then gives the result since f_n descends to σ under the S_n action. (Notice that the map f_j can not descend to a map on $\bar{X}_n$ for all $j < n$.) Since the curve V_n is not necessarily irreducible, we need to make sure that f_j is non-constant on each irreducible component to guarantee that f_j is a finite morphism. This will nevertheless be clear during the following proof.

Assuming first that it has been proved for $j = 1, 2$. To go from j to $j+1$, we let $f(a) = f_{j-1}(a)$, $g(a) = a_j$, and $h(a) = a_{j+1}$. Then by (1.17), f_{j+1} has degree $n!$ times

$$\tfrac{1}{2}j(j+1) + 3 + \tfrac{1}{2}j(j+1) - \tfrac{1}{2}(j-1)j - 1 - 1 = \tfrac{1}{2}(j+1)(j+2)$$

as expected.

It remains to investigate the case $j = 1$ and $j = 2$.

For $j = 1$, by Theorem 1.5 (4), the inverse image of $0 \in E$ under $f_1 : V_n \to E$ consists of a single point $\vec{0}$. By Theorem 1.5 (5), the limiting system of equations (1.11) and (1.12) has a unique non-degenerate solution in $\mathbb{P}^{n-1}$ up to permutations. From this, we conclude that there are precisely $n!$ branches of $V_n \to E$ near $\vec{0}$. For a point $b \in E$ close to 0, each branch will contribute a point a with $a_1 = b$. Thus the degree of f_1 is $n!$.

For $j = 2$, we consider the inverse image of $0 \in E$ under $f_2 : V_n \to E$. Namely $V_n \ni a \mapsto a_1 + a_2 = 0$.

The point $a = 0$ again contributes degree $n!$ by a similar branch argument: Indeed, over each branch near $\vec{0}$ we may represent $a = (a_i(t))$ as an analytic curve in t. Then $t \mapsto a_1(t) + a_2(t) \in E$ is still an one to one map for t close to 0. It is also clear that the same reasoning applies to any $f_k : V_n \to E$

and the point $\vec{0}$ always contributes $n!$ to the degree counting. Notice that f_k is non-constant on each branch near $\vec{0}$, and every irreducible component of V_n will contain at least one such branch, hence f_k is non-constant along each irreducible component of V_n.

Otherwise, for $a \neq 0$ with $f_2(a) = 0$, then $a_1 = -a_2$ and thus $a = -a$ by Theorem 1.5 (4).

If $n = 2k$, by Proposition 1.6 (1) the degree contribution from type O points $a = \{\pm a_1, \cdots, \pm a_k\}$ is given by

$$(k+1) \times (k \times 2 \times (n-2)!),$$

while the degree from the type II points $\{\pm a_1, \cdots, \pm a_{k-1}, \frac{1}{2}\omega_i, \frac{1}{2}\omega_j\}$ is

$$3 \times k \times ((k-1) \times 2 \times (n-2)!).$$

The sum is $2(4k^2 - 2k)(n-2)! = 2n!$.

If $n = 2k+1$, by Proposition 1.6 (2), the degree contribution from type III points $\{\pm a_1, \cdots, \pm a_{k-1}, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_3\}$ is

$$k \times ((k-1) \times 2 \times (n-2)!),$$

while the type I points $\{\pm a_1, \cdots, \pm a_k, \frac{1}{2}\omega_i\}$ contribute

$$3 \times (k+1) \times (k \times 2 \times (n-2)!).$$

The sum is again $2(4k^2 + 2k)(n-2)! = 2n!$.

Thus in both cases we get the total degree $n! + 2n! = 3n!$ as expected. □

During the proof we have actually shown that

Proposition 1.11. *The map σ is unramified at the infinity point $0^n \in \bar{X}_n$.*

We want to emphasize again that both Theorem 1.10 and Proposition 1.11 hold for all E_τ, $\tau \in \mathbb{H}$, regardless the smoothness of $\bar{X}_n$.

2. Pre-modular forms $Z_n(\sigma;\tau)$

When no confusion should arise, we denote the restriction $\mathbf{z}_n|_{\bar{X}_n}$ also by $\mathbf{z}_n$. Then $\mathbf{z}_n$ is a rational function on $\bar{X}_n$ with poles along the fiber $\sigma^{-1}(0)$ under the branch covering (summation) map $\sigma : \bar{X}_n \to E$. To avoid trivial situation we assume that $n \geq 2$ (since $\mathbf{z}_1 = 0$ by definition.)

Theorem 1.10 strongly suggests the following statement:

Theorem 2.1 (New pre-modular forms)**.**

(1) *There is a (weighted homogeneous) polynomial*

$$W_n(z) \in \mathbb{C}[g_2, g_3, \wp(\sigma), \wp'(\sigma)][z]$$

of z-degree $\frac{1}{2}n(n+1)$ such that for $\sigma = \sum a_i$

$$W_n(\mathbf{z}_n(a)) = 0.$$

Indeed, no matter $\bar{X}_n$ is smooth or not, $\mathbf{z}_n(a)$ is a primitive generator of the finite extension of rational function fields $K(\bar{X}_n)$ over $K(E)$, which has degree $\frac{1}{2}n(n+1)$, and with $W_n(z)$ being its minimal polynomial.

(2) *The function $Z_n(\sigma;\tau) := W_n(Z)$ is "pre-modular" of weight $\frac{1}{2}n(n+1)$, with Z, $\wp(\sigma)$, $\wp'(\sigma)$, g_2, g_3 being of weight 1, 2, 3, 4, 6 respectively. Here pre-modular means that it is modular in τ for $\Gamma(N)$ whenever we fix $\sigma \in E_\tau[N]$ being an N-torsion point.*

Proof. Since $\mathbf{z} \in K(\bar{X}_n)$, which is algebraic over $K(E)$ with degree $\frac{1}{2}n(n+1)$ by Theorem 1.10, its minimal polynomial $W_n(z) \in K(E)[z]$ certainly exists with $d := \deg W_n \mid \frac{1}{2}n(n+1)$.

Furthermore, since $\mathbf{z}$ has no poles over $E^\times$, it is indeed integral over the affine Weierstrass model of $E^\times$ where

$$R(E) = \mathbb{C}[x,y]/(y^2 - 4x^3 - g_2x - g_3)$$

with $x = \wp(\sigma)$ and $y = \wp'(\sigma)$. Thus the major statement in Theorem 2.1 (1) is the claim that $\mathbf{z}_n$ is essentially a primitive generator.

Notice that for $\sigma_0 \in E$ being outside the branch loci of $\sigma : \bar{X}_n \to E$, there are precisely $\frac{1}{2}n(n+1)$ different points $a = \{a_1, \cdots, a_n\} \in \bar{X}_n$ with $\sigma(a) = \sum a_i = \sigma_0$. Thus for the rational function $\mathbf{z}_n = \zeta(\sum a_i) - \sum \zeta(a_i) \in K(\bar{X}_n)$ to be a primitive generator, it is sufficient to show that $\mathbf{z}_n$ has exactly $\frac{1}{2}n(n+1)$ branches over $K(E)$. That is, $\sum \zeta(a_i)$ gives different values for different choices of those a above σ_0. Indeed, for any given $\sigma = \sigma_0$, the polynomial $W_n(z) = 0$ has at most d roots. But now $\mathbf{z}_n(a)$ with $\sigma(a) = \sigma_0$ gives $\frac{1}{2}n(n+1)$ distinct roots of $W_n(z)$, hence we must conclude $d = \frac{1}{2}n(n+1)$ and $\mathbf{z}_n$ is a primitive generator.

Hence it is sufficient to show the following more precise result:

Theorem 2.2. *Let $a, b \in Y_n$ and $(a_1, \cdots, a_n)$, $(b_1, \cdots, b_n) \in \mathbb{C}^n$ be representatives of a, b such that*

$$\sum_{i=1}^n a_i = \sum_{i=1}^n b_i, \qquad \sum_{i=1}^n \zeta(a_i) = \sum_{i=1}^n \zeta(b_i). \tag{2.1}$$

Suppose that $\sum \wp(a_i) \neq \sum \wp(b_i)$. Then a, b are branch points of $Y_n \to \mathbb{P}^1$ corresponding to Lamé functions of the same type.

We emphasize that $\bar{X}_n$ is not required to be smooth.

Theorem 2.1 (1) follows immediately by choosing σ_0 outside the branch loci of $\bar{X}_n \to E$ and $\sigma_0 \notin E[2]$. Indeed, let $a, b \in Y_n$ with $\sigma(a) = \sigma(b) = \sigma_0$ and $\mathbf{z}_n(a) = \mathbf{z}_n(b)$, or more precisely with conditions in (2.1) satisfied. By Theorem 2.2 we are left with the case $\sum \wp(a_i) = \sum \wp(b_i)$ but $a \neq b$. Then $a = -b$ by Theorem 1.5 (1), and in particular $\sigma(a) = -\sigma(b)$. Together with $\sigma(a) = \sigma(b)$ we conclude that $\sigma_0 = \sigma(a) = \sigma(b) \in E[2]$. This contradicts to the assumption that $\sigma_0 \notin E[2]$. Hence we must have $a = b$.

(2) follows from (1) and Hecke's theorem that $Z(\sigma;\tau)$ is a modular function of weight one whenever $\sigma = (k_1\omega_1 + k_2\omega_2)/N$ being an N-torsion point [9]. $\square$

We will give two proofs of Theorem 2.2. The first proof is longer but contains more informations.

Recall that the Hermite–Halphen ansatz

$$w_{\pm a}(z) = e^{\pm z \sum \zeta(a_i)} \prod_{i=1}^{n} \frac{\sigma(z \mp a_i)}{\sigma(z)}$$

are solutions to $w'' = (n(n+1)\wp(z) + B_a)w =: I_1 w$, and

$$w_{\pm b}(z) = e^{\pm z \sum \zeta(b_i)} \prod_{i=1}^{n} \frac{\sigma(z \mp b_i)}{\sigma(z)}$$

are solutions to $w'' = (n(n+1)\wp(z) + B_b)w =: I_2 w$. Then $q_{a,-b} := w_a w_{-b}$ and $q_{-a,b} := w_{-a} w_b$ are solutions to the fourth order ODE formed by the tensor product of the two Lamé equations. By assumption.

$$q_{a,-b}(z) = \prod_{i=1}^{n} \frac{\sigma(z - a_i)\sigma(z + b_i)}{\sigma^2(z)} \tag{2.2}$$

is an elliptic function since $\sum a_i = \sum b_i$. Similarly $q_{-a,b}(z) = q_{a,-b}(-z)$ is elliptic. In particular there exists an even elliptic function solution

$$Q := \tfrac{1}{2}(q_{a,-b} + q_{-a,b}) = (-1)^n \frac{\prod_{i=1}^{n} \sigma(a_i)\sigma(b_i)}{z^{2n}} + \text{higher order terms.}$$

Lemma 2.3. *The fourth order ODE is given by*

$$q'''' - 2(I_1 + I_2)q'' - 6I'q' + ((B_a - B_b)^2 - 2I'')q = 0. \tag{2.3}$$

Here $I = n(n+1)\wp(z)$, $I_1 = I + B_a$ and $I_2 = I + B_b$.

Proof. This follows from a straightforward computation. Indeed,

$$\begin{aligned} q' &= w_1' w_2 + w_1 w_2', \\ q'' &= (I_1 + I_2)q + 2w_1' w_2', \\ q''' &= 2I'q + (I_1 + I_2)q' + 2(I_1 w_1 w_2' + I_2 w_1' w_2). \end{aligned}$$

Notice that if $a = b$ (or just $B_a = B_b$) then $I_1 = I_2$ and we stop here to get the third order ODE as the symmetric product of the Lamé equation.

In general, we take one more differentiation to get

$$\begin{aligned} q'''' &= 2I''q + 4I'q' + (I_1 + I_2)q'' + 2I'q' + 2(I_1 + I_2)w_1' w_2' + 4I_1 I_2 q \\ &= 2(I_1 + I_2)q'' + 6I'q' + (2I'' - (I_1 - I_2)^2)q. \end{aligned}$$

This proves the lemma. □

Now we investigate the equation in variable $x = \wp(z)$. To avoid confusion, we denote $\dot{f} = \partial f / \partial x$ and $f' = \partial f / \partial z$.

Let $y^2 = p(x) = 4x^3 - g_2 x - g_3$. Then $\wp' = y$, $\wp'' = 6\wp^2 - \frac{1}{2}g_2 = \frac{1}{2}\dot{p}(x)$. $\wp''' = 12\wp\wp' = 12xy$, $\wp'''' = 12\wp'^2 + 12\wp\wp'' = 12p(x) + 6x\dot{p}(x)$. Also

$$\begin{aligned}
q' &= \dot{q}\wp' = y\dot{q},\\
q'' &= \ddot{q}\wp'^2 + \dot{q}\wp'' = p(x)\ddot{q} + \tfrac{1}{2}\dot{p}(x)\dot{q},\\
q''' &= \dddot{q}\,\wp'^3 + 3\ddot{q}\wp'\wp'' + \dot{q}\wp''',\\
q'''' &= \ddddot{q}\,\wp'^4 + 6\dddot{q}\,\wp'^2\wp'' + 3\ddot{q}(\wp'')^2 + 4\ddot{q}\wp'\wp''' + \dot{q}\wp''''\\
&= p(x)^2\ddddot{q} + 3p(x)\dot{p}(x)\dddot{q} + \big(\tfrac{3}{4}\dot{p}(x)^2 + 48xp(x)\big)\ddot{q} + \big(12p(x) + 6x\dot{p}(x)\big)\dot{q}.
\end{aligned}$$

By substituting these into (2.3) and get the ODE in x:

$$\begin{aligned}
L_4\, q := p^2\ddddot{q} + 3p\dot{p}\dddot{q} &+ \big(\tfrac{3}{4}\dot{p}^2 - 2(2(n^2+n-12)x+\beta)p\big)\ddot{q}\\
&- \big((2(n^2+n-3)x+\beta)\dot{p} + 6(n^2+n-2)p\big)\dot{q}\\
&+ \big(\alpha^2 - n(n+1)\dot{p}\big)q = 0.
\end{aligned} \tag{2.4}$$

where

$$\alpha := B_a - B_b \quad \text{and} \quad \beta := B_a + B_b. \tag{2.5}$$

For the rest of the proof, we want to discuss when $L_4\, q = 0$ with $\alpha \neq 0$ has a polynomial solution. Here g_2 and g_3 could be arbitrary, not necessarily satisfy the non-degenerate condition $g_2^3 - 27g_3^2 \neq 0$.

Suppose that $q(x)$ is a polynomial in x of degree $m \geq 1$:

$$q(x) = x^m - s_1x^{m-1} + s_2x^{m-2} - \cdots + (-1)^m s_m, \tag{2.6}$$

which satisfies

$$\deg_x L_4\, q(x) \leq 1. \tag{2.7}$$

Then we can solve s_j recursively in terms of α^2, β and g_2, g_3.

Indeed, the top degree x^{m+2} in (2.4) has coefficient

$$\begin{aligned}
&16m(m-1)(m-2)(m-3) + 144m(m-1)(m-2) + 108m(m-1)\\
&\quad - 16(n^2+n-12)m(m-1) - 24(n^2+n-3)m\\
&\qquad - 24(n^2+n-2)m - 12n(n+1)\\
&= (m-n)\Big(4m^3 + (4n+68)m^2 + (8n-101)m + 3(n+1)\Big),
\end{aligned}$$

which vanishes precisely when $m = n$. This we may assume that $m = n$.

The next order term x^{n+1} without the s_1 factor has coefficient

$$-8n(n-1)\beta - 12n\beta = -4n(2n+1)\beta,$$

and the coefficient of $-s_1 x^{n+1}$ is given by

$$\begin{aligned}
&16(n-1)(n-2)(n-3)(n-4)+144(n-1)(n-2)(n-3)\\
&\quad+108(n-1)(n-2)-16(n^2+n-12)(n-1)(n-2)\\
&\quad-24(n^2+n-3)(n-1)-24(n^2+n-2)(n-1)-12n(n+1)\\
&=-8n(2n-1)(2n+1).
\end{aligned}$$

Hence

$$s_1 = \frac{\beta}{2(2n-1)}. \tag{2.8}$$

Inductively the x^{n+2-i} coefficient in (2.4) gives recursive relations to solve s_i in terms of β, α^2 and g_2, g_3 for $i = 1, \ldots, n$. It implies that

Lemma 2.4. *For $i = 1, \ldots, n$, there is a polynomial expression*

$$s_i = s_i(\alpha^2, \beta, g_2, g_3) = C_i \beta^i + \cdots$$

which is homogeneous of degree i with $\deg \alpha = \deg \beta = 1$ *and* $\deg g_2 = 2$, $\deg g_3 = 3$. *Moreover, C_i is a non-zero rational number.*

A much detailed description will be given in the proof of Lemma 2.6 and the precise value of C_i can be determined (c.f. from (2.11)).

There are still two remaining terms in (2.7), that is,

$$L_4\, q = F_1(\alpha, \beta, g_2, g_3)x + F_0(\alpha, \beta, g_2, g_3). \tag{2.9}$$

The basic structure of the consistency equations is described by the following two lemmas:

Lemma 2.5. *We have*

$$\begin{aligned}
F_1(\alpha, \beta) &= \alpha^2 G_1(\alpha, \beta) = \alpha^2((-1)^{n-1} s_{n-1}(\alpha^2, \beta, g_2, g_3) + \cdots),\\
F_0(\alpha, \beta) &= \alpha^2 G_0(\alpha, \beta) = \alpha^2((-1)^{n} s_{n}(\alpha^2, \beta, g_2, g_3) + \cdots).
\end{aligned}$$

The remaining terms have either g_2 or g_3 as a factor, hence with lower α, β degree.

Proof. Equation (2.9) gives

$$\begin{aligned}
F_1(\alpha, \beta) &= (-1)^{n-1}\alpha^2 s_{n-1} + \text{terms in } s_1, \cdots, s_{n-2},\\
F_0(\alpha, \beta) &= (-1)^{n}\alpha^2 s_{n} + \text{terms in } s_1, \cdots, s_{n-1}.
\end{aligned}$$

We note that if $\alpha = 0$, then for any β there is a solution $q(x)$ to $L_4(q) = 0$ which is a polynomial in x of degree n.

Indeed $q(x) = \prod_{i=1}^n (x - x_i)$, with $\beta = 2(2n-1)\sum_{i=1}^n x_i$, which comes from the Lamé equation (see [3, 16]). Thus $F_1(0, \beta) = 0 = F_0(0, \beta)$. Since F_i depends on α^2, we have $F_i(\alpha, \beta) = \alpha^2 G_i(\alpha, \beta)$, $i = 0, 1$, for some homogeneous polynomials G_0, G_1 in α^2, β, g_2, g_3 of degree n and $n-1$ respectively, and G_i's can be written as

$$\begin{aligned}
G_1(\alpha, \beta) &= (-1)^{n-1} s_{n-1} + \cdots,\\
G_0(\alpha, \beta) &= (-1)^{n} s_{n} + \cdots.
\end{aligned}$$

To see the dependence of the remaining terms on g_2 and g_3, we let $g_2 = 0 = g_3$, and then $L_4(q) \equiv \alpha^2((-1)^{n-1}s_{n-1}x + (-1)^n s_n) \pmod{x^2}$ because both $p(x) = 4x^3$ and $\dot{p}(x) = 12x^2$ vanish modulo x^2. Thus we have $F_1(\alpha, \beta) = (-1)^{n-1}\alpha^2 s_{n-1}$ and $F_0(\alpha, \beta) = (-1)^n \alpha^2 s_n$ whenever $g_2 = 0 = g_3$. This proves the lemma. □

Lemma 2.6. *The polynomials G_1 and G_0 have no common factors for any g_2, g_3.*

Proof. We consider first the special case $g_2 = g_3 = 0$. Then (2.7) becomes

$$\begin{aligned} &16x^6\dddot{q} + 144x^5\ddot{q} + \big(108x^4 - 8x^3(2(n^2+n-12)x + \beta)\big)\ddot{q} \\ &\quad - \big(12x^2(2(n^2+n-3)x + \beta) + 24x^3(n^2+n-2)\big)\dot{q} \\ &\qquad + \big(\alpha^2 - 12n(n+1)x^2\big)q \equiv 0 \pmod{\mathbb{C} \oplus \mathbb{C}x}. \end{aligned} \tag{2.10}$$

The coefficient of x^{n-k}, $k = 0, \cdots, n-2$, gives recursive equation

$$(-1)^k(m_k\, s_{k+2} + n_k\beta\, s_{k+1} + \alpha^2 s_k) = 0, \tag{2.11}$$

where the constants m_k and n_k are given by

$$\begin{aligned} m_k &= 16(n-(k+2))(n-(k+3))(n-(k+4))(n-(k+5)) \\ &\quad + 144(n-(k+2))(n-(k+3))(n-(k+4)) \\ &\quad\quad + (108 - 16(n^2+n-12))(n-(k+2))(n-(k+3)) \\ &\quad\quad\quad - 24(2n^2+2n-5)(n-(k+2)) - 12n(n+1) \\ &= -4(k+2)(2n-(k+1))(2n-(2k+1))(2n-(2k+3)), \\ n_k &= (8(n-(k+1))(n-(k+2)) + 12(n-(k+1))) \\ &= 4(n-(k-1))(n-(k+1)). \end{aligned}$$

Since $k \le n-2$, we have $m_k \neq 0$ and $n_k \neq 0$.

Let $\gamma(\alpha, \beta)$ be a non-trivial common factor of both G_1 and G_0.

In the case $g_2 = g_3 = 0$ we have $G_1 = (-1)^{n-1}s_{n-1}$ and $G_0 = (-1)^n s_n$. Then γ and α are co-prime, because if $\alpha = 0$ then $s_{n-1}(0, \beta) = c_{n-1}\beta^{n-1}$ and $s_n(0, \beta) = c_n\beta^n$ for some non-zero constants c_{n-1} and c_n. By (2.11) for $k = n-2$, we have $\gamma \mid s_{n-2}(\alpha^2, \beta, 0, 0)$ too. By induction on k for $k = n-3, \cdots, 0$ in decreasing order we conclude that $\gamma \mid s_0 = 1$, which leads to a contradiction.

For $g_2, g_3 \in \mathbb{C}$, we see by Lemma 2.5 that the leading terms of G_1, G_0, as polynomials of α and β, are $(-1)^{n-1}s_{n-1}(\alpha^2, \beta, 0, 0)$ and $(-1)^n s_n(\alpha^2, \beta, 0, 0)$ respectively. Since $s_{n-1}(\alpha^2, \beta, 0, 0)$ and $s_n(\alpha^2, \beta, 0, 0)$ are co-prime, we conclude that $G_1(\alpha, \beta, g_2, g_3)$ and $G_0(\alpha, \beta, g_2, g_3)$ are also co-prime. The proof is complete. □

Proposition 2.7. *The common zeros of $G_1 = 0$ and $G_0 = 0$ are precisely given by the pair of branch points (a, b) corresponding to Lame functions of the same type. If $\bar{X}_n$ is non-singular, there are exactly $n(n-1)$ such ordered pairs (a, b)'s.*

Proof. It suffices to prove the (generic) case that $\bar{X}_n$ is non-singular, namely the case that all the Lamé functions are distinct. The general case follows from the non-singular case by a limiting argument.

For any two Lamé functions w_a, w_b of the same type, it is easy to see that we may arrange the representatives of a and b so that (2.1) holds. It follows that $q := q_{a,-b} = q_{-a,b}$ (see (2.2)) is an even elliptic function solution to (2.3), or equivalently $q(x)$ is a polynomial solution to $L_4\, q(x) = 0$.

From the above discussion, (α, β) must be a common root of G_1 and G_0 (where $\alpha = B_a - B_b$, $\beta = B_a + B_b$). By Lemma 2.4 and 2.5, we have $\deg G_1 = n - 1$ and $\deg G_0 = n$ and G_1, G_0 are co-prime to each other by Lemma 2.6. Hence by Bezout theorem there are at most $n(n-1)$ common roots.

On the other hand, the number of such ordered pairs can be determined by Proposition 1.6. Indeed, if $n = 2k$ is even, then we have

$$(k+1)k + 3k(k-1) = 4k^2 - 2k = n(n-1)$$

such pairs. If $n = 2k+1$ is odd, the number of pairs is given by

$$k(k-1) + 3(k+1)k = 4k^2 + 2k = n(n-1).$$

Hence in all cases the number of ordered pairs coming from the Lamé functions of the same type agrees with the Bezout degree of the polynomial system defined by $G_1 = 0 = G_0$. Thus these $n(n-1)$ pairs form the zero locus as expected (and there is no infinity contribution). □

The above discussions from Lemma 2.3 to Proposition 2.7 constitute a complete proof of Theorem 2.2. Here is a summary: We already know that Q is an even elliptic function with singularity only at $0 \in E$. Thus

$$Q(x) = C \prod_{i=1}^{n} (\wp(z) - \wp(c_i)) =: C \prod_{i=1}^{n} (x - x_i)$$

is a polynomial solution to the ODE (2.4) with $\alpha = B_a - B_b$, $\beta = B_a + B_b$.

Since $\alpha = B_a - B_b \neq 0$, by Lemma 2.5 (α, β) must be a common root of $G_1(\alpha, \beta) = 0 = G_0(\alpha, \beta)$. Then Proposition 2.7 says that (α, β) is pair of Lamé functions of the same type. This proves Theorem 2.2.

For future reference, we combine Theorem 2.2 and Proposition 2.7 into the following statement on a fourth order ODE which arises from the *tensor product of two different (integral) Lamé equations* with the same parameter n.

Due to its importance, we will give a second (shorter and more direct) proof of the part corresponding to Theorem 2.2.

Theorem 2.8. *Let $I(z) = n(n+1)\wp(z)$. The fourth order ODE*

$$q''''(z) - 2(I + \beta)q''(z) - 6I'q'(z) + (\alpha^2 - 2I'')q(z) = 0 \tag{2.12}$$

with $\alpha \neq 0$ has an elliptic function solution if and only if (α, β) is a pair of common root to $G_0(\alpha, \beta) = 0$ and $G_1(\alpha, \beta) = 0$. Moreover, this solution must be even.

Second Proof to Theorem 2.2. Following the definition of $q_{a,-b}(z)$ in (2.2), we now consider the odd elliptic solution to (2.12) (= (2.3)) instead:

$$q(z) = \tfrac{1}{2}(q_{a,-b}(z) - q_{-a,b}(z)),$$

which has a pole of order $3+2l$ at $0 \in E$ with $l \le n-2$. Thus $q(z)/\wp'(z)$ is an even elliptic function with the only pole at 0 since $q(\frac{1}{2}\omega_i) = 0$ for $1 \le i \le 3$. If $q(z)$ does not vanish completely, then

$$q(z) = C\wp'(z)\prod_{i=1}^{l}(\wp(z) - \wp(c_i)) =: C\wp'(z)f(\wp(z)),$$

where $f(x) = \prod_{i=1}^{l}(x - \wp(c_i)) = x^l - s_1x^{l-1} + \cdots + (-1)^l s_l$.

By Lemma 2.3, $q(z)$ satisfies

$$\begin{aligned} &q''''(z) - 2(\beta + 2n(n+1)\wp(z))q''(z) \\ &\quad - 6n(n+1)\wp'(z)q'(z) + (\alpha^2 - 2n(n+1)\wp''(z))q(z) = 0. \end{aligned} \tag{2.13}$$

By straightforward calculations, we can compute all derivatives of q in terms of derivatives of $\wp(z)$ and $f'(x)$. For example,

$$\begin{aligned} q'(z) &= \wp''(z)f(x) + \wp'(z)^2 f'(x), \\ q''(z) &= \wp'''(z)f(x) + 3\wp''(z)\wp'(z)f'(x) + \wp'(z)^3 f''(x), \quad \text{etc.} \end{aligned}$$

Then (2.13) is equivalent to

$$\begin{aligned} &f(x)\Big((360 - 96n(n+1))x^2 - 24\beta x + (4n(n+1) - 18)g_2 + \alpha^2\Big) \\ &+f'(x)\Big((1320 - 96n(n+1))x^3 - 36\beta x^2 \\ &\qquad + (12n(n+1) - 150)g_2x + (6n(n+1) - 60)g_3 + 3\beta g_2\Big) \\ &+f''(x)\Big((1020 - 16n(n+1))x^4 - 8\beta x^3 + (4n(n+1) - 210)g_2x^2 \\ &\qquad + (2\beta g_2 + (4n(n+1) - 120)g_3)x + 2\beta g_3 + \tfrac{15}{4}g_2^2\Big) \\ &+f'''(x)(60x^2 - 30g_2)(4x^3 - g_2x - g_3) \\ &+f''''(x)(4x^3 - g_2x - g_3)^2 = 0. \end{aligned}$$

By comparing the coefficients of x^{l+2}, we obtain

$$\begin{aligned} &(360 - 96n(n+1)) + l(1320 - 96n(n+1)) + l(l-1)(1020 - 16n(n+1)) \\ &\quad + 240l(l-1)(l-2) + 16l(l-1)(l-2)(l-3) = 0. \end{aligned}$$

After simplification, this is reduced to

$$4n(n+1) = (2l+3)(2l+5),$$

which obviously leads to a contradiction since the RHS is odd. Therefore we must have $q \equiv 0$ from the beginning. That is, $\{a_i, -b_i\} = \{-a_i, b_i\}$.

If one of a, b does not correspond to a Lamé function, say $a \in X_n$, then $\{a_1, \cdots, a_n\} \cap \{-a_1, \cdots, -a_n\} = \emptyset$ and we conclude that $\{a_i\} = \{b_i\}$. Otherwise a and b correspond to Lamé functions of the same type. $\square$

Theorem 2.9. *The map* $\mathbf{z}_n : \bar{X}_n \to \mathbb{P}^1$ *has degree* $\frac{1}{2}n(n+1)$*, with its fibration structure over* $\infty \in \mathbb{P}^1$ *analytically isomorphic to* $\sigma : \bar{X}_n \to E$ *over* 0*.*

Proof. By definition, $\mathbf{z}_n^{-1}(\infty) = \sigma^{-1}(0)$ as sets. So the crucial point is to compare the fibration (ramification) structures of $\bar{X}_n \to E$ at $0 \in E$ and $\bar{X}_n \to \mathbb{P}^1$ at $\infty \in \mathbb{P}^1$. Let $a \in \bar{X}_n$ with $\sigma(a) = 0$. Then for $b = \{b_i\}_{i=1}^n \in \bar{X}_n$ in a small analytic neighborhood of a we have $b_i \neq 0$ all i. Moreover,

$$\sigma(b) = b_1 + \cdots b_n,$$

and

$$\mathbf{z}_n(b) = \zeta(\sigma(b)) - \sum_{i=1}^n \zeta(b_i) = \frac{1}{b_1 + \cdots + b_n}(1 + O(\sigma(b))).$$

In the local coordinate of $\mathbb{P}^1$ at ∞, the map $\mathbf{z}_n$ is then represented by

$$(\mathbf{z}_n(b))^{-1} = (b_1 + \cdots + b_n)(1 + O(\sigma(b))),$$

which is clearly analytically equivalent to $\sigma(b)$.

In particular, $\deg \mathbf{z}_n = \deg \sigma = \frac{1}{2}n(n+1)$ by Theorem 2.1. $\square$

Example 2.10. For $n = 2$, $\beta = B_a + B_b$, $\alpha = B_a - B_b$, we have

$$s_1 = \tfrac{1}{6}\beta, \qquad s_2 = \tfrac{1}{36}\beta^2 + \tfrac{1}{72}\alpha^2 - \tfrac{1}{4}g_2.$$

The first compatibility equation from x^1 is

$$s_1(\alpha^2 + 36 g_2) - 6\beta g_2 = 0.$$

After substituting s_1 we get

$$\tfrac{1}{6}\alpha^2\beta = 0. \tag{2.14}$$

The second compatibility equation from x^0 is

$$s_2(\alpha^2 + 6g_2) - s_1(\beta g_2 + 24 g_3) + 4\beta g_3 + \tfrac{3}{2}g_2^2 = 0.$$

By substituting s_1, s_2 and noticing the (expected) cancellations we get

$$\alpha^2\left(\tfrac{1}{36}\beta^2 + \tfrac{1}{72}\alpha^2 - \tfrac{1}{6}g_2\right) = 0. \tag{2.15}$$

If $B_a \neq B_b$ then (2.14) implies that $B_b = -B_a$ and then (2.15) leads to

$$B_a^2 = 3g_2 \Longrightarrow \wp(a_1) + \wp(a_2) = \pm\sqrt{g_2/3}.$$

By Example 1.7 (2), such $a \in \bar{X}_2$ lies in the branch loci of the hyperelliptic (Lamé) curve. In particular, $a, b \in \sigma^{-1}(0)$ and they are excluded by the assumption in Proposition 2.2. Denote by $\wp(\pm q_\pm) = \pm\sqrt{g_2/12}$. Then $a := \{q_+, -q_+\} \neq b := \{q_-, -q_-\}$ unless $g_2 = 0$. When $g_2 \neq 0$, $\mathbf{z}_2$ fails to distinguish the two points a and b. When $g_2 = 0$ (equivalently $\tau = e^{\pi i/3}$), $a = b$ becomes a (singular) branch point for $\sigma : \bar{X}_2 \to E_\tau$.

Example 2.11. For $n = 3$, $\beta = B_a + B_b$, $\alpha = B_a - B_b$. Then

$$\begin{aligned} s_1 &= \tfrac{1}{10}\beta, \\ s_2 &= \tfrac{1}{600}(4\beta^2 + \alpha^2 - 150g_2), \\ s_3 &= \tfrac{1}{3600}(2\beta^3 + 3\alpha^2\beta - 120\beta g_2 + 900g_3). \end{aligned}$$

The two compatibility equations from x^1 and x^0 are

$$\begin{aligned} 0 &= \tfrac{1}{600}\alpha^2(4\beta^2 + \alpha^2 + 60g_2), \\ 0 &= \tfrac{1}{3600}\alpha^2(2\beta^3 + 3\alpha^2\beta - 90\beta g_2 + 540g_3). \end{aligned}$$

If $\alpha \neq 0$ then $\alpha^2 = -4\beta^2 - 60g_2$ and the second equation becomes

$$\beta^3 + 27g_2\beta - 54g_3 = 0.$$

It is clear that there are only finite solutions (B_a, B_b)'s to this, though it may not be so straightforward to see that these 6 solution pairs (for generic tori) come from the branch loci as proved in Proposition 2.7.

We summarize the key points of our discussions:

We start with $n = 1$, $Z_1 \equiv Z = -4\pi\nabla G$ which is essentially the Hecke modular function. The non-trivial solutions, i.e. non half periods, to the critical point equation ($\Longleftrightarrow$ solutions to the mean field equation (1.1) for $\rho = 8\pi$) is transformed into zeros of pre-modular form Z (which is modular for $\sigma \in E$ being an N torsion point).

For general $n \geq 1$, we consider the following statements:

(1) Solution u to the mean field equation (1.1) for $\rho = 8\pi n$.
(2) Periods integral $\displaystyle\int_{L_j} g \in \sqrt{-1}\mathbb{R}$ (c.f. (1.5)).
(3) Green equation $\displaystyle\sum_{i=1}^n \nabla G(a_i) = 0$ on the hyperelliptic curve X_n.
(4) Coincidence equation $\mathbf{z}_n(a) = Z(\sigma(a))$ for $a \in X_n$.
(5) Zero of pre-modular form $Z_n(\sigma;\tau) := W_n(Z)$ with $\sigma \notin E_\tau[2]$.

We have proved the equivalence of (1), (2), (3) and (4).

As in the case $n = 1$, we call the $2n + 1$ finite branch points $a \in \bar{X}_n \backslash X_n$ the *trivial critical points* (since $a = -a$ and the Green equation (3) holds trivially). They satisfy a nice compatibility condition with the case $n = 1$ under the addition map:

Lemma 2.12. *Let $a = (a_1, \cdots, a_n) \in Y_n$ be a solution to the Green equation $\sum_{i=1}^n \nabla G(a_i) = 0$. Then a is trivial, i.e. $a = -a$, if and only if $\sigma(a) \in E[2]$.*

Proof. If a is trivial, then $\sigma(a) \in E[2]$ clearly. If a is non-trivial, i.e. $a \in X_n$, by Proposition 1.1 and (1.5), it gives rise to a type II developing map f with

$$f(z + \omega_1) = e^{-4\pi i \sum_i s_i} f(z), \qquad f(z + \omega_2) = e^{4\pi i \sum_i t_i} f(z).$$

Here $a_i = t_i\omega_1 + s_i\omega_2$ for $i = 1, \ldots, n$.

If $\sigma(a) \in E[2]$, then both exponential factors reduce to one and we conclude that $f(z)$ is an elliptic function on E. Notice that the only zero of $f'(z)$ is at $z = 0$ which has order $2n$, and the only poles of $f'(z)$ are at $-a_i$ of order 2, $i = 1, \ldots, n$. This forces that $\sigma(a) \equiv 0 \pmod{\Lambda}$ and

$$f'(z) = \sum_{j=1}^{n} E_j \wp(z + a_j) + C_1$$

for some constants $E_1, \ldots, E_n$ and C_1, since f' is residue free. Then

$$f(z) = -\sum_{j=1}^{n} E_j \zeta(z + a_i) + C_1 z + C_2$$

for some constant C_2. But $f(z)$ is elliptic, which implies that $C_1 = 0$ and $\sum_{j=1}^{n} E_j = 0$. Now $f^{2k-1}(0) = 0$ for $k = 1, \ldots, n$ leads to a system of linear equations in E_j's (c.f. [3, Lemma 2.5]):

$$\sum_{j=1}^{n} \wp^k(a_j) E_j = 0, \qquad k = 1, \ldots, n.$$

But then $\wp(a_i) \neq \wp(a_j)$ for $i \neq j$ forces that $E_j = 0$ for all j. This is a contradiction and so we must have $\sigma(a) \notin E[2]$. □

Now the following theorem completes the analogy with the case $n = 1$.

Theorem 2.13 (Extra critical points vs zeros of pre-modular forms)**.**

(i) *Given* $\sigma_0 \in E_\tau \backslash E_\tau[2]$ *with* $Z_n(\sigma_0; \tau) = 0$*, there is a unique* $a \in X_n$ *such that* $\sigma(a) = \sigma_0$ *and* $\mathbf{z}_n(a) = Z(\sigma_0)$.

(ii) *Conversely, if* $a \in X_n$ *and* $\mathbf{z}_n(a) = Z(\sigma(a))$*, then* $Z_n(\sigma(a); \tau) = 0$ *and* $\sigma(a) \notin E_\tau[2]$.

Proof. (i) For any given σ_0, by substituting σ by σ_0 in $W_n(z)$, we get a polynomial $W_{n,\sigma_0}(z)$ of degree $\frac{1}{2}n(n+1)$. Since $W_n(z)$ is the minimal polynomial of the rational function $\mathbf{z}_n \in K(\bar{X}_n)$ over $K(E)$, those $\mathbf{z}_n(a)$ with $a \in \bar{X}_n$ and $\sigma(a) = \sigma_0$ give precisely all the roots of $W_{n,\sigma_0}(a)$, counted with multiplicities.

Now $Z(\sigma_0)$ is a root of $W_{n,\sigma_0}(z)$ with $\sigma_0 \notin E[2]$, hence there is a point $a \in X_n$ corresponds to it, i.e. $Z(\sigma_0) = \mathbf{z}_n(a)$ with $\sigma(a) = \sigma_0$, which is unique by Theorem 2.2. Notice that if $a \in \bar{X}_n \backslash X_n$ then $a = -a$ and then $\sigma(a) \in E[2]$. So in fact we must have $a \in X_n$.

(ii) It is clear that $Z_n(\sigma(a)) \equiv W_n(Z(\sigma(a)) = W_n(\mathbf{z}_n(a)) = 0$. Since $a \in X_n$, we know that $\sum_{i=1}^{n} \nabla G(a_i) = 0$ by the above equivalence of (4) and (3). But since a is non-trivial ($a \in X_n$ by assumption), Lemma 2.12 implies that $\sigma(a) \notin E[2]$. □

Remark 2.14. For $\sigma = \frac{k_1}{N}\omega_1 + \frac{k_2}{N}\omega_2$, Z_n is $\Gamma(N)$ modular. This corresponds to the finite monodromy problem of integral Lamé equations (c.f. [1, 6], see also §5.2 and Appendix A).

3. Explicit determination of Z_n

We have studied extensively the Hermite–Halphen ansatz

$$w_a(z) := e^{z \sum \zeta(a_i)} \prod_{i=1}^{n} \frac{\sigma(z-a_i)}{\sigma(z)} \tag{3.1}$$

with $a \in Y_n$ to represent solutions to the integral Lamé equation

$$w'' = (n(n+1)\wp(z) + B)w. \tag{3.2}$$

There is another ansatz, the *Hermite–Krichever ansatz*, which can also be used to construct solutions to (3.2). It takes the form

$$\phi(z) := \Big(U(\wp(z)) + V(\wp(z)) \frac{\wp'(z) + \wp'(a_0)}{\wp(z) - \wp(a_0)}\Big) \frac{\sigma(z-a_0)}{\sigma(z)} e^{(\zeta(a_0)+\kappa)z}, \tag{3.3}$$

where $U(x)$ and $V(x)$ are polynomials in x, $a_0 \in E^\times$, and $\kappa \in \mathbb{C}$ is a constant. As usual, we set $(x,y) = (\wp(z), \wp'(z))$ and $(x_0, y_0) = (\wp(a_0), \wp'(a_0))$ to be the corresponding algebraic coordinates.

Notice that (3.3) makes sense since ϕ only has poles at $z = 0$ (the one at $z = a_0$ from $(\wp(z) - \wp(a_0))^{-1}$ cancels with the zero from $\sigma(z - a_0)$). Moreover, in order for $\operatorname{ord}_{z=0} \phi(z) = -n$, we must have

Lemma 3.1 (Degree constraints)**.**

(i) *If* $n = 2m$ *with* $m \in \mathbb{N}$ *then* $\deg U \le m-1$ *and* $\deg V = m - 1$.
(ii) *If* $n = 2m+1$ *with* $m \in \mathbb{N} \cup \{0\}$ *then* $\deg U = m$ *and* $\deg V \le m-1$.

By an obvious normalization, in case (i) we may assume that $U(x) = \sum_{i=0}^{m-1} u_i x^i$, $V(x) = \sum_{i=0}^{m-1} v_i x^i$ with $v_{m-1} = 1$, and in case (ii) $U(x) = \sum_{i=0}^{m} u_i x^i$ with $u_m = 1$ and $V(x) = \sum_{i=0}^{m-1} v_i x^i$. In both cases, the requirement that $\phi(z)$ satisfies (3.2) leads to recursive relations on u_i's and v_i's. In doing so, it is more convenient to work on the algebraic coordinates. This had been carried out by Maier in [14, §4]. The following is a summary:

In case (i) the recursion determines v_i ($v_{m-1} = 1$) and then u_i for $i = m-1, m-2, \cdots$ in decreasing order. In case (ii) it starts with $u_m = 1$ and determines v_i and then u_i for $i = m-1, m-2, \cdots$. There are two compatibility equations coming from $u_{-1}(B, \kappa, x_0, y_0) = 0$ and $v_{-1}(B, \kappa, x_0, y_0) = 0$. The two parameters x_0, y_0 satisfy $y_0^2 = 4x_0^3 - g_2 x_0 - g_3$. Hence there are four variables $(B, \kappa, x_0, y_0) \in \mathbb{C}^4$ which are subject to three polynomial equations. By taking in to account the limiting cases with $(x_0, y_0) = (\infty, \infty)$, this recovers the Lame curve $\bar{Y}_n$, which was denoted by Γ_ℓ in [14] with $\ell = n$.

There are four natural coordinate projections (rational functions) $\bar{Y}_n \to \mathbb{P}^1$, namely B, κ, x_0 and y_0 respectively. The first one $B : \bar{Y}_n \to \mathbb{P}^1$ is simply the hyperelliptic structure map. The main result in [14] is an explicit description of the other 3 maps in terms of the coordinates (B, C) on $\bar{Y}_n$:

Theorem 3.2 ([14, Theorem 4.1])**.** *For all $n \in \mathbb{N}$ and $i \in \{1,2,3\}$,*

$$\begin{aligned} x_0(B) &= e_i + \frac{4}{n^2(n+1)^2}\frac{l_i(B)lt_i(B)^2}{l_0(B)lt_0(B)^2}, \\ y_0(B,C) &= \frac{16}{n^3(n+1)^3}\frac{C}{c_n}\frac{lt_1(B)lt_2(B)lt_3(B)}{l_0(B)^2lt_0(B)^3}, \\ \kappa(B,C) &= -\frac{(n-1)(n+2)}{n(n+1)}\frac{C}{c_n}\frac{l_\theta(B)}{l_0(B)lt_0(B)}. \end{aligned} \tag{3.4}$$

The formula for $x_0(B)$ is independent of the choices of i.

In the above formulas, $lt_j(B)$, $j = 0,1,2,3$, are the *twisted Lamé polynomials* whose zeros correspond to solutions to the Lamé equation given by the Hermite–Krichever ansatz with $\kappa \neq 0$ and $a_0 = 0, \frac{1}{2}\omega_1, \frac{1}{2}\omega_2, \frac{1}{2}\omega_3$ respectively, i.e. $(x_0,y_0) = (\infty,\infty),(e_1,0),(e_2,0),(e_3,0)$ respectively. The polynomial $l_\theta(B)$ is the *theta-twisted polynomial* whose roots correspond to the case $\kappa = 0$ and $a_0 \notin E[2]$. (For $\kappa = 0$ and $a_0 \in E[2]$ they correspond to the *ordinary* Lamé polynomials $l_i(B)$'s.) All the polynomials are taken to be *monic* in B. They are indeed homogeneous by setting the weights of B, e_i, g_2, g_3 to be $1,1,2,3$ respectively.

Remark 3.3. In [14] $\nu = C/c_n$ is used instead. Also $l_0(B), l_i(B), lt_0(B), lt_i(B)$, and $l_\theta(B)$ ($i = 1,2,3$) are written there as $L^I_\ell(B;g_2,g_3)$, $L^{II}_\ell(B;e_i,g_2,g_3)$, $Lt^I_\ell(B;g_2,g_3)$, $Lt^{II}_\ell(B;e_i,g_2,g_3)$, and $L\theta_\ell(B;g_2,g_3)$ respectively, where $\ell = n$.

The compatibility equations from the recursive formulas for these special cases give rise to explicit formulas for $lt_j(B)$'s and $l_\theta(B)$'s. Tables for $lt_0(B)$, $l_\theta(B)$ up to $n = 8$, and for $lt_i(B)$ up to $n = 6$, are given in [14, Table 5, 6].

Example 3.4. For later usage, we recall Maier's formulas for $lt_j(B)$ and $l_\theta(B)$ for $n \leq 4$.

(1) First of all, $l_\theta(B) = 1$ for $n \leq 3$. For $n = 4$,

$$l_\theta(B) = B^2 - \tfrac{193}{3}g_2.$$

Also for $n = 1$, $lt_j(B) = 1$ for all j.

(2) $n = 2$: $lt_0(B) = 1$, $lt_i(B) = B - 6e_i$ for $i = 1,2,3$.

(3) $n = 3$: $lt_0(B) = B^2 - \frac{75}{4}g_2$, and for $i = 1,2,3$,

$$lt_i(B) = B^2 - 15e_iB + \tfrac{75}{4}g_2 - 225e_i^2.$$

(4) $n = 4$: $lt_0(B) = B^3 - \frac{343}{4}g_2B - \frac{1715}{2}g_3$. For $i = 1,2,3$,

$$\begin{aligned} lt_i(B) = B^4 &- 55e_iB^3 + (\tfrac{539}{4}g_2 - 945e_i^2)B^2 \\ &+ (1960e_ig_2 + 2450g_3)B + 61740e_i^2g_2 - 68600e_ig_3 - 9261g_2^2. \end{aligned}$$

To apply Theorem 3.4, we need to compare the projection map $\pi : \bar{Y}_n \to E$ defined by $a \mapsto a_0$, or equivalently $(\wp(a_0),\wp'(a_0)) = (x_0(B_a), y_0(B_a))$, with the addition map $\sigma : \bar{Y}_n \to E$. They turn out to be the same!

Theorem 3.5. *$\pi(a) = \sigma(a) = \sum_{i=1}^n a_i$. Moreover, $\kappa(a) = -\mathbf{z}_n(a)$.*

Proof. During the proof we view $a_i \in \mathbb{C}$ instead of its image $[a_i] \in E$.

Let $a \in Y_n$. The two expressions (3.1) and (3.3), which correspond to the same solution to the Lamé equation (3.2), must be proportional to each other by a constant. Hence we get

$$\kappa(a) = \sum_{i=1}^n \zeta(a_i) - \zeta(a_0).$$

Recall that $\mathbf{z}_n(a) = \zeta(\sigma(a)) - \sum_{i=1}^n \zeta(a_i)$. Then

$$\mathbf{z}_n(a) + \kappa(a) = \zeta(\sigma(a)) - \zeta(a_0). \tag{3.5}$$

As a well defined meromorphic function on $\bar{Y}_n$, we conclude that

$$a_0(a) = \sigma(a) + c$$

for some constant $c \in \mathbb{C}$. Consider a point $a \in Y_n \backslash X_n$ with $\sigma(a) = \frac{1}{2}\omega_1$, i.e. $l_1(B_a) = 0$. Such a exists by Proposition 1.6. Then $\mathbf{z}_n(a) = 0$ trivially. We also have $\kappa(a) = 0$ by Theorem 3.4 since

$$C_a^2 = c_n^2 l_0(B_a) l_1(B_a) l_2(B_a) l_3(B_a) = 0$$

(again by Proposition 1.6). So (3.5) implies $0 = \frac{1}{2}\eta_1 - \zeta(\frac{1}{2}\omega_1 + c)$, and hence $c = 0$. This proves $\sigma(a) = a_0$, which represents $\pi(a)$ in E, and also $\kappa(a) = -\mathbf{z}_n(a)$. The proof is complete. $\square$

Now we may describe the explicit construction of the polynomial $W_n(z)$ in Theorem 2.1 based on Theorem 3.2. It is indeed merely an application of the elimination theory using resultant.

By Theorem 3.2 and 3.5, we may eliminate C to get

$$\frac{y_0}{\mathbf{z}_n} = \frac{16}{n^2(n+1)^2(n-1)(n+2)} \frac{lt_1(B) lt_2(B) lt_3(B)}{l_0(B) lt_0(B)^2 l_\theta(B)}, \tag{3.6}$$

which leads to a polynomial equation $g = 0$ for

$$g := \mathbf{z}_n \prod_{i=1}^3 lt_i(B) - y_0 \frac{n^2(n+1)^2(n-1)(n+2)}{16} l_0(B) lt_0(B)^2 l_\theta(B). \tag{3.7}$$

On the other hand, the 3 rational expressions of x_0 lead to $f = 0$ for

$$\begin{aligned} f &:= l_i(B) lt_i(B)^2 - (x_0 - e_i) \frac{n^2(n+1)^2}{4} l_0(B) lt_0(B)^2 \\ &= \frac{1}{3} \sum_{i=1}^3 l_i(B) lt_i(B)^2 - x_0 \frac{n^2(n+1)^2}{4} l_0(B) lt_0(B)^2. \end{aligned} \tag{3.8}$$

Notice that f, g are polynomials in g_2, g_3 (and B, x_0, y_0) instead of e_i's.

Let $R(f, g; B)$ be the resultant of the two polynomials f and g arising from the elimination of the variable B.

Proposition 3.6. *$R(f, g; B)(z) = \lambda_n W_n(z) \in \mathbb{C}[g_2, g_3, x_0, y_0][z]$, where $\lambda_n = \lambda_n(g_2, g_3, x_0, y_0)$ is independent of z and with $y_0^2 \mid \lambda_n$.*

Proof. From the explicit rational expressions (3.4) in Theorem 3.2, the resultant of f, g, defined in (3.8), (3.7) gives rise to the equation of the branched covering map $\sigma : \bar{Y}_n \to E$ over the loci outside $E[2]\backslash\{0\}$.

More precisely, if $C \neq 0$ then the formulas for y_0 and $\kappa = -\mathbf{z}_n$ are equivalent to $g = 0$. However, if $C = 0$ we have $l_i(B) = 0$ for some $0 \leq i \leq 3$ by Proposition 1.6. From (3.8), we get $1 \leq i \leq 3$ in order to have $f = 0$. Hence $l_0(B)lt_0(B) \neq 0$ and then $\mathbf{z}_n = 0 = y_0$ in (3.4). But the equation $g = 0$ in (3.7) gives extra solutions. Indeed by (3.7) we conclude only that $y_0 = 0$ and $\mathbf{z}_n$ could be arbitrary. In conclusion, the additional solutions consist of $(x_0, y_0) = (e_i, 0)$, $i = 1, 2, 3$, with $z = \mathbf{z}_n$ being arbitrary. Thus the additional solutions contribute a factor to $R(f, g; B)(z)$ which divides $\prod_{i=1}^{3}(x_0 - e_i) = y_0^2/4$ and is independent of z.

In particular $R(f, g; B)(z)$ has degree $\frac{1}{2}n(n+1)$ in z. Also by construction $R(f, g; B)(\mathbf{z}_n) = 0$, hence it must be a multiple of the minimal polynomial $W_n(z)$ of $\mathbf{z}_n$. Since $\deg W_n = \frac{1}{2}n(n+1)$, the full multiplier λ_n is independent of z, and with $y_0^2 \mid \lambda_n$. $\square$

Proposition 3.7. *The pre-modular form $Z_n(\sigma; \tau) = W_n(Z)$ can be explicitly computed for any $n \in \mathbb{N}$.*

In practice, such a computation is very time consuming even using computer. In the following, we apply it to the initial cases up to $n = 4$. As before we denote $x_0 = \wp(\sigma) =: \wp$ and $y_0 = \wp'(\sigma) =: \wp'$.

Example 3.8. For $n = 2$, it is easy to see that

$$f = B^3 - 9\wp B^2 + 27(g_2\wp + g_3),$$
$$g = zB^3 - 9\wp' B^2 - 9zg_2 B + 27(g_2\wp' - 2zg_3).$$

The resultant $R(f, g; B)$ is calculated by the 6×6 Sylvester determinant:

$$\begin{vmatrix} 1 & -9\wp & 0 & 27(g_2\wp + g_3) & 0 & 0 \\ 0 & 1 & -9\wp & 0 & 27(g_2\wp + g_3) & 0 \\ 0 & 0 & 1 & -9\wp & 0 & 27(g_2\wp + g_3) \\ z & -9\wp' & -9zg_2 & 27(g_2\wp' - 2zg_3) & 0 & 0 \\ 0 & z & -9\wp' & -9zg_2 & 27(g_2\wp' - 2zg_3) & 0 \\ 0 & 0 & z & -9\wp' & -9zg_2 & 27(g_2\wp' - 2zg_3) \end{vmatrix}.$$

A direct evaluation gives

$$R(f, g; B) = -3^9\Delta(\wp')^2(z^3 - 3\wp z - \wp').$$

Here $\Delta = g_2^3 - 27g_3^2$ is the discriminant. This gives $W_2(z) = z^3 - 3\wp z - \wp'$ and $Z_2(\sigma; \tau) = W_2(Z) = Z^3 - 3\wp Z - \wp'$.

Example 3.9. For $n = 3$, we have

$$\begin{aligned} f &= 16B^6 - 576B^5\wp + 360B^4 g_2 + 5400B^3(5g_3 + 4g_2\wp) \\ &\quad - 3375B^2 g_2^2 - 84375\Delta - 101250Bg_2(3g_3 + 2g_2\wp), \\ g &= 16B^6 z - 1440B^5\wp' - 1800B^4 g_2 z + 54000B^3(g_2\wp' - g_3 z) \\ &\quad - 16875B^2 g_2^2 z - 506250Bg_2^2\wp' + 421875\Delta z. \end{aligned}$$

It takes a couple seconds to evaluate the corresponding 12×12 Sylvester determinant (e.g. using *Mathematica*) to get

$$R(f, g; B) = 2^{36}3^{27}5^{30}\Delta^5(\wp')^4 W_3(z),$$

where $W_3(z)$ is given by

$$W_3(z) = z^6 - 15\wp z^4 - 20\wp' z^3 + (\tfrac{27}{4}g_2 - 45\wp^2)z^2 - 12\wp\wp' z - \tfrac{5}{4}\wp'^2.$$

It seems impractical to evaluate this resultant by hand. .

In §4 we will discuss a classical approach, namely a "two-steps" procedure, to construct $Z_n(\sigma; \tau)$. It is essentially the method used in [6] which works well for $n = 2, 3$ and can allow computations by hand. The above two examples agree with Example 4.1 and Example 4.2 in §4 respectively, which appeared in [6] already. Unfortunately this classical approach fails for all $n \geq 4$ (see Proposition 4.5). Hence Proposition 3.7 is the only general method to construct $Z_n(\sigma; \tau)$.

Here is a new example:

Example 3.10. For $n = 4$, the expansions of the polynomials f and g, as given in (3.8) and (3.7) by a direct substitution, are already too complicate to put here. Nevertheless, a couple hours *Mathematica* calculation gives

$$R(f, g; B) = -2^{80}3^{63}5^{60}7^{63}\Delta^{18}(\wp')^8 W_4(z),$$

where $W_4(z)$ is the degree 10 polynomial:

$$\begin{aligned} W_4(z) = z^{10} &- 45\wp z^8 - 120\wp' z^7 + (\tfrac{399}{4}g_2 - 630\wp^2)z^6 - 504\wp\wp' z^5 \\ &- \tfrac{15}{4}(280\wp^3 - 49g_2\wp - 115g_3)z^4 + 15(11g_2 - 24\wp^2)\wp' z^3 \\ &- \tfrac{9}{4}(140\wp^4 - 245g_2\wp^2 + 190g_3\wp + 21g_2^2)z^2 \\ &- (40\wp^3 - 163g_2\wp + 125g_3)\wp' z + \tfrac{3}{4}(25g_2 - 3\wp^2)(\wp')^2. \end{aligned} \tag{3.9}$$

The weight 10 pre-modular form $Z_4(\sigma; \tau)$ is then obtained.

Applications of these explicit pre-modular forms will be discussed in a subsequent work.

We end this section with a brief discussion on the *rationality property*. We have constructed two affine curves from $\bar{X}_n$. One is the hyperelliptic model $Y_n = \{(B, C) \mid C^2 = \ell_n(B)\}$, another one is $Y'_n := \{(x_0, y_0, \mathbf{z}_n) \mid y_0^2 = 4x_0^2 - g_2x_0 - g_3,\ W_n(x_0, y_0; \mathbf{z}_n) = 0\}$ which is understood as a degree $\frac{1}{2}n(n+1)$ branched cover of the original curve $E = \{(x_0, y_0) \mid y_0^2 = 4x_0^3 - g_2x_0 - g_3\}$ under the projection $\sigma' : Y'_n \to E$ with defining equation $W_n(\mathbf{z}_n) = 0$.

Y_n is birational to Y'_n over E, namely the addition map $\sigma : Y_n \to E$ is compatible with $\sigma' : Y'_n \to E$. Notice that both ℓ_n and W_n have coefficients in $\mathbb{Q}[g_2, g_3]$. The explicit birational map $\phi : (B, C) \dashrightarrow (x_0, y_0, \mathbf{z}_n)$ (given in Theorem 3.2 and 3.5 via $\mathbf{z}_n = -\kappa$) also has coefficients in $\mathbb{Q}[g_2, g_3]$. This implies that ϕ is defined over $\mathbb{Q}$. Moreover ϕ extends to a birational morphism

$$\begin{tikzcd} \bar{Y}_n \cong \bar{X}_n \arrow[rr, "\phi"] \arrow[dr, "\sigma"'] & & \bar{Y}'_n \arrow[dl, "\sigma'"] \\ & E & \end{tikzcd}$$

by identifying $\sigma^{-1}(0_E)$ with $\mathbf{z}_n^{-1}(\infty)$ via Theorem 2.9. From (the proof of) Proposition 3.6, the morphism ϕ is an isomorphism outside those branch points for $Y_n \to \mathbb{P}^1$ lying over $E[2]\backslash\{0\}$. In particular, the non-isomorphic loci lie in $\mathbf{z}_n = 0$ by (3.4) and Theorem 3.5.

Remark 3.11. In contrast to the smoothness of $Y_n(\tau)$ for general τ, for all $n \geq 3$ the model $Y'_n(\tau)$ is singular at points $\mathbf{z}_n = 0 = y_0$ (and hence $x_0 = e_i$ for some i). Indeed from (3.4) this is equivalent to $C = 0$ and $l_i(B)lt_i(B)^2 = 0$ for some $1 \leq i \leq 3$. For $n = 2$, there is only one solution B for each fixed i (c.f. Example 3.4). However, for $n \geq 3$ there are more than one solutions B. These points $(B, 0) \in Y_n$ are collapsed to the same point $(x_0, y_0, \mathbf{z}_n) = (e_i, 0, 0) \in Y'_n$ under ϕ, thus $(e_i, 0, 0)$ is a singular point of Y'_n.

For $n = 3, 4$ this is easily seen from the equation $W_n(z) = 0$ given above since it contains a quadratic polynomial in $(z, \wp')$ (i.e. in $(\mathbf{z}_n, y_0)$) as its lowest degree terms.

In particular, the birational map ϕ^{-1} is also represented by rational functions $B = B(x_0, y_0, \mathbf{z}_n)$ and $C = C(x_0, y_0, \mathbf{z}_n)$ with coefficients in $\mathbb{Q}[g_2, g_3]$ and with at most poles along $\mathbf{z}_n = 0$. In principle such an explicit inverse can be obtained by a Groebner basis calculation associated to the ideal of the graph Γ_ϕ.

Example 3.12. For small n, these formulas can be obtained from Example 3.4. For $n = 2, 3$ they can also be obtained from the intermediate step of the classical construction of $W_n(z)$ in §4.1. For $n = 2$, from (4.1) we have

$$B = 3(\mathbf{z}_2^2 - x_0), \tag{3.10}$$

and then C is obtained from $\mathbf{z}_2(B, C)$ by substituting B. We get

$$C = \mathbf{z}_2(\tfrac{2}{3}B^2 - \tfrac{9}{2}g_2) = 6\mathbf{z}_2(\mathbf{z}_2^4 - 2x_0\mathbf{z}_2^2 + x_0^2 - \tfrac{3}{4}g_2). \tag{3.11}$$

For $n = 3$, (4.3) easily gives

$$B = \tfrac{5}{3}\left(\mathbf{z}_3^2 - \frac{y_0}{\mathbf{z}_3}\right) - 5x_0, \tag{3.12}$$

and then $C = \frac{4}{375}B(B^2 - \frac{75}{4}g_2)\mathbf{z}_3$ accordingly.

The following statement is clear from the above description:

Proposition 3.13. *Let E be defined over $\mathbb{Q}$, i.e. $g_2, g_3 \in \mathbb{Q}$. Then the Lamé curve $\bar{Y}_n$ is also defined over $\mathbb{Q}$ for all $n \in \mathbb{N}$. Moreover, $\bar{Y}'_n$ and all the morphisms σ, σ', ϕ are also defined over $\mathbb{Q}$.*

A rational point $(B, C) \in \bar{Y}_n$ is mapped to a rational point $(x_0, y_0, \mathbf{z}_n) \in \bar{Y}'_n$ by ϕ. For the converse, given $(x_0, y_0) \in E(\mathbb{Q})$, a point $(x_0, y_0, \mathbf{z}_n)$ in the σ' fiber gives a unique $(B, C) \in \bar{Y}_n(\mathbb{Q})$ if $\mathbf{z}_n \in \mathbb{Q}$ and $(x_0, y_0, \mathbf{z}_n) \neq (e_i, 0, 0)$ for any i.

Remark 3.14. It is well known that there are only few (i.e. at most finite) rational points on a non-elliptic hyperelliptic curve. This phenomenon is consistent with the irreducibility of the polynomial $W_n(z)$ over $K(E)$ in light of Hilbert's irreducibility theorem that there is a infinite (Zariski dense) set of $(g_2, g_3, x_0, y_0) \in \mathbb{Q}^4$ so that the specialization of $W_n(z)$ is still irreducible. Nevertheless, it might be interesting to see if $\mathbf{z}_n$ plays any role in the study of rational points.

4. A remark on the classical approach to Z_n

A "two stpes" approach to the determination of Z_2 and Z_3 based on the addition law (4.1) and a classical cubic identity (4.3) of elliptic functions was developed in [6]. One might hope that a more sophisticated application of the Frobenius–Stickelberger type identities (e.g. [16, p.458]) may lead to a construction of Z_n for $n \geq 4$. The purpose of this appendix is to show that this classical approach fails for all $n \geq 4$ (see Proposition 4.5).

4.1. **Explicit constructions for $n = 2, 3$.** We describe the first two cases $n = 2, 3$ in this subsection using a "two steps" procedure.

Example 4.1 ($n = 2$). For $\mathbf{z} = \mathbf{z}_2(a_1, a_2)$, we have on X_2:

$$W_2(\mathbf{z}) = \mathbf{z}^3 - 3\wp(a_1 + a_2)\mathbf{z} - \wp'(a_1 + a_2) = 0.$$

This is essentially equivalent to the addition law:

$$\mathbf{z}^2 = \wp(a_1 + a_2) + \wp(a_1) + \wp(a_2). \tag{4.1}$$

In particular, the weight 3 pre-modular form is

$$Z_2(\sigma; \tau) = W_2(Z) = Z^3(\sigma) - 3\wp(\sigma)Z(\sigma) - \wp'(\sigma).$$

We start by applying the *symmetrized operator* $\delta := \frac{1}{2}(\partial_{a_1} + \partial_{a_2})$ to $\mathbf{z} := \zeta(\sigma) - \zeta(a_1) - \zeta(a_2)$ sucessively to get

$$\delta \mathbf{z} = \tfrac{1}{2}(\wp(a_1) + \wp(a_2)) - \wp(\sigma),$$
$$\delta^2 \mathbf{z} = \tfrac{1}{4}(\wp'(a_1) + \wp'(a_2)) - \wp'(\sigma).$$

We rewrite (4.1) in the following *admissible* form:

$$\mathbf{z}^2 = A_2(\delta \mathbf{z}) := 3\wp(\sigma) + 2\delta \mathbf{z}. \tag{4.2}$$

Then $2\mathbf{z}\delta\mathbf{z} = 3\wp'(\sigma) + 2\delta^2\mathbf{z} = \wp'(\sigma) + \frac{1}{2}(\wp'(a_1) + \wp'(a_2))$. Hence

$$\mathbf{z}^3 = 3\wp(\sigma)\mathbf{z} + 2\mathbf{z}\delta\mathbf{z} = 3\wp(\sigma)\mathbf{z} + \wp'(\sigma) + \tfrac{1}{2}(\wp'(a_1) + \wp'(a_2)).$$

On X_2 this reduces to $W_2(\mathbf{z}) = 0$. We emphasize that only (4.2) is used.

Example 4.2 ($n = 3$). For $\mathbf{z} = \mathbf{z}_3(a)$, we have on X_3:

$$W_3(\mathbf{z}) = \mathbf{z}^6 - 15\wp\mathbf{z}^4 - 20\wp'\mathbf{z}^3 + (\tfrac{27}{4}g_2 - 45\wp^2)\mathbf{z}^2 - 12\wp'\wp\mathbf{z} - \tfrac{5}{4}\wp'^2 = 0.$$

Here $\wp$ and $\wp'$ are evaluated at $\sigma = \sum_{i=1}^3 a_i$.

Hence, $Z_3(\sigma;\tau) = W_3(Z)$ is of weight 6.

The derivation of $W_3(z)$ is more involved. It consists of two steps. The first step is the following classical identity. We supply a detailed proof of it since a variant of the proof will be used for our later discussions on the general cases $n \geq 4$.

Lemma 4.3 (c.f. [16, p.459]). *For $\mathbf{z} = \zeta(\sigma) - \sum_{i=1}^3 \zeta(a_i)$ with $\sigma = \sum_{i=1}^3 a_i$,*

$$\mathbf{z}^3 = 3(\wp(\sigma) + \sum \wp(a_i))\mathbf{z} + (\wp'(\sigma) - \sum \wp'(a_i)). \tag{4.3}$$

Proof. We will prove it by viewing both sides as functions of $s = a_3$ and by comparing the principal parts on both sides. In doing so we emphasize that the case $n = 2$ is used in an essential way.

For a better presentation on signs we work on $\mathbf{f} = -\mathbf{z}_3$. Let $\sigma_2 = a_1 + a_2$ and $\mathbf{f}_2 = -\mathbf{z}_2 = \zeta(a_1) + \zeta(a_2) - \zeta(\sigma_2)$. In the following, all elliptic functions are evaluated at $s = \sigma_2$ if not written explicitly. Then

$$\mathbf{f} = \zeta(a) + \zeta(b) + \zeta(s) - \zeta(\sigma_2 + s) = \frac{1}{s} + \mathbf{f}_2 + \wp s + \tfrac{1}{2}\wp' s^2 + O(s^3),$$

by noting $\zeta(s) = 1/s + O(s^3)$.

We want to compare

$$\mathbf{f}^3 = \frac{1}{s^3} + \frac{3\mathbf{f}_2}{s^2} + \frac{3\mathbf{f}_2^2 + 3\wp}{s} + \mathbf{f}_2^3 + 6\mathbf{f}_2\wp + \tfrac{3}{2}\wp' + O(s)$$

with (all summations are for $i = 1, 2$)

$$\begin{aligned}
&3\Big(\sum \wp(a_i) + \wp(s) + \wp(\sigma_2 + s)\Big)\mathbf{f} + \Big(\sum \wp'(a_i) + \wp'(s) - \wp'(\sigma_2 + s)\Big)\\
&= 3\Big(\frac{1}{s^2} + (\sum \wp(a_i) + \wp) + \wp' s\Big)\Big(\frac{1}{s} + \mathbf{f}_2 + \wp s + \tfrac{1}{2}\wp' s^2\Big)\\
&\qquad + \frac{-2}{s^3} + (\sum \wp'(a_i) - \wp') + O(s)\\
&= \frac{1}{s^3} + \frac{3\mathbf{f}_2}{s^2} + \frac{3(\sum \wp(a_i) + \wp) + 3\wp}{s}\\
&\qquad + 3(\sum \wp(a_i) + \wp)\mathbf{f}_2 + \tfrac{9}{2}\wp' + (\sum \wp'(a_i) - \wp') + O(s).
\end{aligned}$$

Now the case for $n = 2$ in (4.1) says that $\mathbf{f}_2^2 = \sum \wp(a_i) + \wp$. Thus the s^{-1} terms match and the equality for the constant terms is equivalent to

$$\mathbf{f}_2^3 = 3\wp\,\mathbf{f}_2 - (\wp' + \tfrac{1}{2}\sum \wp'(a_i)). \tag{4.4}$$

By symmetric differentiation, the $n = 2$ case implies also

$$\wp' + \tfrac{1}{2}\sum \wp'(a_i) = \delta(\mathbf{f}_2^2) = 2\mathbf{f}_2\delta\mathbf{f}_2 = 2\mathbf{f}_2(\wp - \tfrac{1}{2}\sum \wp(a_i)). \tag{4.5}$$

Thus the right hand side of (4.4) equals

$$\mathbf{f}_2(\wp + \sum \wp(a_i)) = \mathbf{f}_2^3,$$

which is precisely the left hand side.

This identifies the principal parts at the pole $s = 0$. For the other pole $s = -(a_1 + a_2)$ the comparison follows from the case $s = 0$ under the symmetry $s \mapsto -(a_1 + a_2) - s$. This proves the lemma. $\square$

The important thing we learn from the above proof is in the last step: The formula (4.5) is a non-monic polynomial relation for $\mathbf{f}_2$ which has lower degree than the monic one (whose minimal degree is 2). Though (4.5) is merely a symmetric analogue of the addition law for ζ:

$$\mathbf{z}_2 = \zeta(a_1 + a_2) - \zeta(a_1) - \zeta(a_2) = \frac{1}{2}\frac{\wp'(a_1) - \wp'(a_2)}{\wp(a_1) - \wp(a_2)},$$

it allows us to deduce higher monic polynomial relations like (4.4).

In the same way, by applying $\delta := \frac{1}{3}\sum_{i=1}^3 \partial_{a_i}$ to (4.3), we may compute $\delta\mathbf{z}^3 = 3\mathbf{z}^2\delta\mathbf{z}$ in both ways to get a non-monic degree 2 relation

$$(\wp - \tfrac{1}{3}\sum \wp_i)\mathbf{z}^2 = -(\wp' - \tfrac{1}{3}\sum \wp_i')\mathbf{z} - \tfrac{1}{3}(\wp'' - \tfrac{1}{3}\sum \wp_i'') + (\wp + \sum \wp_i)(\wp - \tfrac{1}{3}\sum \wp_i).$$

Here $\wp_i := \wp(a_i)$ and the sum is from 1 to 3.

By multiplying $\mathbf{z}$ to (4.3) and replace $\mathbf{z}^2$ terms which involve $\sum \wp_i$ by the above relation, we get a degree 4 monic relation

$$\mathbf{z}^4 = 12\wp\mathbf{z}^2 + (10\wp' - 2\sum \wp_i')\mathbf{z} + 3(\wp'' - \tfrac{1}{3}\sum \wp_i'') - 9(\wp + \sum \wp_i)(\wp - \tfrac{1}{3}\sum \wp_i).$$

It is unclear if we may proceed in this way to eventually get an expression involving σ only. Our second step towards the proof of Example 4.2 is a more systematic usage of the symmetrized derivatives.

Proof of Example 4.2. Before we apply $\delta := \frac{1}{3}\sum_{i=1}^3 \partial_{a_i}$ to (4.3) successively, we need to first have a good sense about the expression $\delta^r\mathbf{z}$ for $r \in \mathbb{N}$. We compute

$$\begin{aligned} \delta\mathbf{z} &= \tfrac{1}{3}\sum \wp(a_i) - \wp(\sigma), \\ \delta^2\mathbf{z} &= \tfrac{1}{9}\sum \wp'(a_i) - \wp'(\sigma), \\ \delta^3\mathbf{z} &= \tfrac{2}{9}\sum \wp^2(a_i) - \tfrac{1}{18}g_2 - \wp''(\sigma), \\ \delta^4\mathbf{z} &= \tfrac{4}{27}\sum \wp(a_i)\wp'(a_i) - \wp'''(\sigma). \end{aligned} \tag{4.6}$$

The key observation is that, $\delta^2\mathbf{z}$ and $\delta^4\mathbf{z}$ are good terms (which depend on σ only) allowed in our calculations, while $\delta\mathbf{z}$ and $\delta^3\mathbf{z}$ are terms not allowed in our final formula which need to be replaced by known terms.

Now, in terms of $\delta^r\mathbf{z}$ in (4.6), we rewrite (4.3) as

$$\mathbf{z}^3 = A_3(\delta^*\mathbf{z}) := 12\wp\mathbf{z} + 9\mathbf{z}\delta\mathbf{z} - 8\wp' - 9\delta^2\mathbf{z}. \tag{4.7}$$

Here, and in the following, all $\wp$ and its derivatives are evaluated at σ. This implies that $\mathbf{z}\delta\mathbf{z}$ is indeed a sum of good terms.

By applying δ to (4.7) we get

$$3\mathbf{z}^2\delta\mathbf{z} = 12\wp'\mathbf{z} + 12\wp\delta\mathbf{z} + 9(\delta\mathbf{z})^2 + 9\mathbf{z}\delta^2\mathbf{z} - 8\wp'' - 9\delta^3\mathbf{z}. \tag{4.8}$$

By multiplying $\mathbf{z}^2$ to (4.8), it follows that $\mathbf{z}^2\delta^3\mathbf{z}$ is a sum of good terms.

By applying δ to (4.8) we get

$$\begin{aligned} 6\mathbf{z}(\delta\mathbf{z})^2 + 3\mathbf{z}^2\delta^2\mathbf{z} = 12\wp''\mathbf{z} + 24\wp'\delta\mathbf{z} + 12\wp\delta^2\mathbf{z} \\ + 27\delta\mathbf{z}\delta^2\mathbf{z} + 9\mathbf{z}\delta^3\mathbf{z} - 8\wp''' - 9\delta^4\mathbf{z}. \end{aligned} \tag{4.9}$$

By multiplying $\mathbf{z}$ to (4.9) we find that all the terms appeared are now good terms. Hence it give rise to the polynomial $W_3(z)$ by explicit substitution.

In fact in this last step we have $\delta^2\mathbf{z} = -\wp'$ and $\delta^4\mathbf{z} = -\wp'''$ on the curve X_3. Thus we have

$$\begin{aligned} 9\mathbf{z}\delta\mathbf{z} &= \mathbf{z}^3 - 12\wp\mathbf{z} - \wp', \\ 9^2\mathbf{z}^2\delta^3\mathbf{z} &= -2\mathbf{z}^6 + 24\wp\mathbf{z}^4 + 28\wp'\mathbf{z}^3 - 72\wp''\mathbf{z}^2 + 12\wp\wp'\mathbf{z} + \wp'^2. \end{aligned}$$

Then we get $W_3(z)$ by a straightforward manipulation with (4.9). □

Example 4.4. The branched cover $\bar{X}_n \to E$ is in general not a Galois cover. Namely, $W_n(z)$ does not split into product of linear factors in $K(\bar{X}_n)$.

Indeed, for $n = 2$ we may factorize $W_2(z)$ by division to get

$$W_2(z) = (z - \mathbf{z}_2)(z^2 + \mathbf{z}_2 z + (\mathbf{z}_2^2 - 3\wp(\sigma))).$$

Thus the other two roots of $W_2(z)$ are given by

$$\mathbf{w}_\pm := \frac{1}{2}\Big(-\mathbf{z}_2 \pm \sqrt{3(4\wp(\sigma) - \mathbf{z}_2^2)}\Big).$$

We will show that $\mathbf{w}_\pm$ are not single valued on $\bar{X}_2$ for general tori.

The rational function $h(a) := 4\wp(\sigma(a)) - \mathbf{z}_2^2(a)$ has poles of total order six by Theorem 2.9. By Example 2.10, if $g_2 \ne 0$, they consists of 3 poles with each of order 2, and for $g_2 = 0$, $0^2 \in \bar{X}_2$ is of order 2 and $(q, -q) \in \bar{X}_2$ with $\wp(\pm q) = 0$ is of order 4. In order for $\mathbf{w}_\pm$ being single valued, the zeros of h must also be of even order.

If $h(a) = 0$ but $\wp(\sigma(a)) \ne 0$ for some $a \in \bar{X}_2$, then it is easy to see that there must be some zeros of h with odd order. Thus we only need to consider the case that all zeros of h are also zeros of $\wp(\sigma(a))$. Since

$$h(a) = 3\wp(a_1 + a_2) - (\wp(a_1) + \wp(a_2)),$$

we have $\wp(a_1) + \wp(a_2) = 0$ too. The addition theorem then implies that $\wp'(a_1) = \wp'(a_2)$. But the equation for X_2 is given by $\wp'(a_1) + \wp'(a_2) = 0$, hence $\wp'(a_1) = 0 = \wp'(a_2)$. That is, $a_1 = \frac{1}{2}\omega_i$, $a_2 = \frac{1}{2}\omega_j$ and $a_1 + a_2 = \frac{1}{2}\omega_k$ is the third half period. Then we get the non-trivial equation $e_k = 0$. Thus for general $E = E_\tau$, $W_2(z)$ does not split into product of linear factors.

4.2. **A remark on $n \ge 4$.** A closer look at the proof of Example 4.2 shows that the overall important equation to start with is not the classical polynomial identity (4.3) in $\mathbf{z}$. Instead, the polynomial equation (4.7) on $\mathbf{z}$ and $\delta\mathbf{z}$ with *admissible coefficients* (depending only on σ) is what we really need for the proof. (For simplicity we use the notation $A(\delta^*\mathbf{z})$ for it.)

To the authors' knowledge, historically a reasonably clean polynomial identity of degree n with symmetric coefficients in a_i's like (4.1) and (4.3) was unknown for $n \ge 4$. For $n = 4$, naive generalization of the proof of Lemma 4.3 to get a degree 4 polynomial fails immediately. Thus we try to get an *admissible equation* in $\mathbf{z}$ and $\delta^i\mathbf{z}$'s instead. (These two type of expressions are equivalent by (4.10) below.) As a result, we will be able to prove that the degree 4 polynomial indeed does not exist.

Notice that while $\mathbf{z} = \zeta(\sum a_i) - \sum \zeta(a_i)$ is symmetric in the last variable $s = a_n$ under $s \mapsto -\sigma_n = -\sigma_{n-1} - s$,

$$\delta\mathbf{z} = \frac{1}{n}\sum\nolimits_{i=1}^{n} \wp(a_i) - \wp(\sigma_n)$$

breaks such a symmetry. It thus distinguishes the two poles $s = 0$ and $s = -\sigma_{n-1}$ which is a key property we shall use.

As usual, we have $\mathrm{wt}(\wp^{(j)}) = j + 2$, $\mathrm{wt}(\mathbf{z}) = 1$ and $\mathrm{wt}(\delta\mathbf{z}) = 2$. Since

$$\delta^{r+1}\mathbf{z} = \frac{1}{n^{r+1}}\sum\nolimits_{i=1}^{n} \wp^{(r)}(a_i) - \wp^{(r)}(\sigma_n), \tag{4.10}$$

we see that $\sum \wp^{(r)}(a_i) = n^{r+1}(\delta^{r+1}\mathbf{z} + \wp^{(r)}(\sigma_n))$ is admissible of weight $r+2$ for all $r \ge 0$. By the elementary properties of $\wp$ function, this is equivalent to that $\sum \wp^j(a_i)$ (of weight $2j$) and $\sum \wp^j(a_i)\wp'(a_i)$ (of weight $2j+1$), $j \ge 0$, are all admissible. (We notice that admissible terms of odd weights are *good terms* when we restrict to X_n.) As before, we denote by $\wp$ (without variable) the admissible term $\wp(\sigma_n)$, and similarly for other elliptic functions.

We have already shown in (4.2) and (4.7) that

$$\begin{aligned} \mathbf{z}_2^2 &= 2\delta_2\mathbf{z}_2 + 3\wp(\sigma_2), \\ \mathbf{z}_3^3 &= 9\mathbf{z}_3\delta_3\mathbf{z}_3 - 9\delta_3^2\mathbf{z}_3 + 12\wp(\sigma_3)\mathbf{z}_3 - 8\wp'(\sigma_3). \end{aligned}$$

However, a naive generalization of this is not true for $n \ge 4$:

Proposition 4.5. *For all $n \in \mathbb{N}_{\ge 4}$, there is no admissible polynomial equation*

$$\mathbf{z}^n = A_n(\delta^*\mathbf{z}).$$

Here A_n is homogeneous of weight n with coefficients in $\mathbb{Q}[\wp^{(j)}]_{j=0,\ldots,n-2}$.

Proof. Let $n \ge 2$, $\mathbf{z} = \mathbf{z}_n$, $\sigma = \sigma_n$, $\delta = \delta_n$, and $s = a_{n+1}$. We omit the argument in a function if it is evaluated at σ. Near the pole $s = 0$,

$$\begin{aligned} \mathbf{z}_{n+1} &= \mathbf{z} - \zeta(s) + \zeta(\sigma + s) - \zeta(\sigma) \\ &= -\frac{1}{s} + \mathbf{z} - \wp\, s - \tfrac{1}{2}\wp' s^2 - \left(\tfrac{1}{6}\wp'' - \tfrac{1}{60}g_2\right)s^3 + \cdots . \end{aligned}$$

Since $\delta_{n+1} = \frac{n}{n+1}\delta + \frac{1}{n+1}\partial_s$, we have for $k \geq 1$

$$\begin{aligned}
\delta_{n+1}^k \mathbf{z}_{n+1} &= \tfrac{n^k}{(n+1)^k}(\delta^k \mathbf{z} + \wp^{(k-1)}) + \tfrac{1}{(n+1)^k}\wp^{(k-1)}(s) - \wp^{(k-1)}(\sigma + s) \\
&= \tfrac{(-1)^{k-1}k!}{(n+1)^k}\frac{1}{s^{k+1}} + \left(\tfrac{n^k}{(n+1)^k}\delta^k \mathbf{z} + \left(\tfrac{n^k}{(n+1)^k} - 1\right)\wp^{(k-1)} + \tfrac{(k-1)!}{(n+1)^k}c_{k-1}\right) \\
&\quad - \sum_{j\geq 1}\left(\tfrac{1}{j!}\wp^{(k+j-1)} - \tfrac{(k+j-1)!}{(n+1)^k j!}c_{k+j-1}\right)s^j,
\end{aligned}$$

where $\wp(z) = z^{-2} + \sum_{j\geq 2} c_j z^j$, $c_2 = \frac{1}{20}g_2$, $c_4 = \frac{1}{28}g_3$, and $c_{2i+1} = 0$ for all i. Notice that the principal part has only one term.

Now let $n = 3$, and we will prove the case for $n + 1 = 4$. We have

$$\begin{aligned}
\mathbf{z}_4^2 &= \frac{1}{s^2} - \frac{2\mathbf{z}}{s} + (\mathbf{z}^2 + 2\wp) + (\wp' - 2\wp\mathbf{z})s \\
&\quad + (\wp^2 - \wp'\mathbf{z} + \tfrac{1}{3}\wp'' - \tfrac{1}{30}g_2)s^2 + O(s^3), \\
\delta_4 \mathbf{z}_4 &= \tfrac{1}{4}\frac{1}{s^2} + (\tfrac{3}{4}\delta\mathbf{z} - \tfrac{1}{4}\wp) - \wp' s - (\tfrac{1}{2}\wp'' - \tfrac{1}{80}g_2)s^2 + O(s^3), \\
\delta_4^2 \mathbf{z}_4 &= -\tfrac{1}{8}\frac{1}{s^3} + (\tfrac{9}{16}\delta^2\mathbf{z} - \tfrac{7}{16}\wp') - (\wp'' - \tfrac{1}{160}g_2)s + O(s^2) \\
\delta_4^3 \mathbf{z}_4 &= \tfrac{3}{32}\frac{1}{s^4} + (\tfrac{27}{64}\delta^3\mathbf{z} - \tfrac{37}{64}\wp'' + \tfrac{1}{640}g_2) + O(s).
\end{aligned}$$

Hence (we still denote δ_4 by δ for simplicity)

$$\begin{aligned}
\mathbf{z}_4^2\delta\mathbf{z}_4 &= \frac{1}{4s^4} - \frac{\mathbf{z}}{2s^3} + (\tfrac{3}{4}\delta\mathbf{z} + \tfrac{1}{4}\mathbf{z}^2 + \tfrac{1}{4}\wp)\frac{1}{s^2} - (\tfrac{3}{4}\wp' + \tfrac{3}{2}\mathbf{z}\delta\mathbf{z})\frac{1}{s} \\
&\quad + \tfrac{3}{4}\mathbf{z}^2\delta\mathbf{z} + \tfrac{3}{2}\wp\,\delta\mathbf{z} - \tfrac{1}{4}\wp\,\mathbf{z}^2 + \tfrac{7}{4}\wp'\mathbf{z} - (\tfrac{5}{12}\wp'' + \tfrac{1}{4}\wp^2 - \tfrac{1}{240}g_2) + O(s), \\
\mathbf{z}_4\delta^2\mathbf{z}_4 &= \frac{1}{8s^4} - \frac{\mathbf{z}}{8s^3} + \frac{\wp}{8s^2} + (\tfrac{1}{2}\wp' - \tfrac{9}{16}\delta^2\mathbf{z})\frac{1}{s} \\
&\quad + \tfrac{9}{16}\mathbf{z}\delta^2\mathbf{z} - \tfrac{7}{16}\wp'\mathbf{z} + (\tfrac{49}{48}\wp'' - \tfrac{1}{120}g_2) + O(s), \\
(\delta\mathbf{z}_4)^2 &= \frac{1}{16s^4} + \frac{3\delta\mathbf{z} - \wp}{8s^2} - \frac{\wp'}{2s} + (\tfrac{3}{4}\delta\mathbf{z} - \tfrac{1}{4}\wp)^2 - (\tfrac{1}{4}\wp'' - \tfrac{1}{160}g_2) + O(s).
\end{aligned}$$

We would like to represent

$$\begin{aligned}
\mathbf{z}_4^4 &= \frac{1}{s^4} - \frac{4\mathbf{z}}{s^3} + \frac{6\mathbf{z}^2 + 4\wp}{s^2} - \frac{4\mathbf{z}^3 + 12\wp\mathbf{z} - 2\wp'}{s} \\
&\quad + \mathbf{z}^4 + 12\wp\mathbf{z}^2 + 6(\wp^2 - \wp'\mathbf{z}) + (\tfrac{2}{3}\wp'' - \tfrac{1}{15}g_2) + O(s).
\end{aligned}$$

by $A_4(\delta^*\mathbf{z})$, whose general form is

$$a\mathbf{z}_4^2\delta\mathbf{z}_4 + b\mathbf{z}_4^2\delta^2\mathbf{z}_4 + c\delta^3\mathbf{z}_4 + d(\delta\mathbf{z}_4)^2 + e\wp\,\mathbf{z}_4^2 + f\wp\delta\mathbf{z}_4 + g\wp'\mathbf{z}_4 + (h\wp'' + ig_2).$$

The equations for s^{-4}, s^{-3}, s^{-2} in $\mathbf{z}_4^4 = A_4(\delta^*\mathbf{z})$ read as

$$\begin{aligned}
1 &= \tfrac{1}{4}a + \tfrac{1}{8}b + \tfrac{3}{32}c + \tfrac{1}{16}d \quad (\text{for } 1/s^4),\\
4 &= \tfrac{1}{2}a + \tfrac{1}{8}b \quad (\text{for } \mathbf{z}/s^3),\\
6 &= \tfrac{1}{4}a \quad (\text{for } \mathbf{z}^2/s^2),\\
0 &= \tfrac{3}{4}a + \tfrac{3}{8}d \quad (\text{for } \delta\mathbf{z}/s^2),\\
4 &= \tfrac{1}{4}a + \tfrac{1}{8}b - \tfrac{1}{8}d + e + \tfrac{1}{4}f \quad (\text{for } \wp/s^2).
\end{aligned}$$

The middle 3 equations give $a = 24$, $b = -64$, $d = -48$. The first equation then gives $c = 64$, an the last equation reduces to

$$e + \tfrac{1}{4}f = 0.$$

The equation for s^{-1} then reads as $-4\mathbf{z}^3 - 12\wp\mathbf{z} + 2\wp' = (-18\wp' - 36\mathbf{z}\delta\mathbf{z}) + (-32\wp' + 36\delta^2\mathbf{z}) + 24\wp' - 2e\wp\mathbf{z} - g\wp'$. By collecting terms we get

$$\mathbf{z}^3 = 9\mathbf{z}\delta\mathbf{z} - 9\delta^2\mathbf{z} + (\tfrac{1}{2}e - 3)\wp\mathbf{z} + (7 + \tfrac{1}{4}g)\wp'.$$

Now we plug in the result for $\mathbf{z}^3$. By comparing with (4.7) we get $e = 30$, $g = -60$. Hence $f = -120$.

The final equation for the constant term is given by

$$\begin{aligned}
&\mathbf{z}^4 + 12\wp\mathbf{z}^2 + 6(\wp^2 - \wp'\mathbf{z}) + (\tfrac{2}{3}\wp'' - \tfrac{1}{15}g_2)\\
&= 24\big(\tfrac{3}{4}\mathbf{z}^2\delta\mathbf{z} + \tfrac{3}{2}\wp\,\delta\mathbf{z} - \tfrac{1}{4}\wp\,\mathbf{z}^2 + \tfrac{7}{4}\wp'\mathbf{z} - (\tfrac{5}{12}\wp'' + \tfrac{1}{4}\wp^2 - \tfrac{1}{240}g_2)\big)\\
&\quad - 64\big(\tfrac{9}{16}\mathbf{z}\delta^2\mathbf{z} - \tfrac{7}{16}\wp'\mathbf{z} + (\tfrac{49}{48}\wp'' - \tfrac{1}{120}g_2)\big)\\
&\quad + 64(\tfrac{27}{64}\delta^3\mathbf{z} - \tfrac{37}{64}\wp'' + \tfrac{1}{640}g_2)\\
&\quad - 48\big((\tfrac{3}{4}\delta\mathbf{z} - \tfrac{1}{4}\wp)^2 - (\tfrac{1}{4}\wp'' - \tfrac{1}{160}g_2)\big)\\
&\quad + 30\wp(\mathbf{z}^2 + 2\wp) - 120\wp(\tfrac{3}{4}\delta\mathbf{z} - \tfrac{1}{4}\wp) - 60\wp'\,\mathbf{z} + h\wp'' + ig_2.
\end{aligned}$$

By collecting terms we get

$$\begin{aligned}
\mathbf{z}^4 &= 18\mathbf{z}^2\delta\mathbf{z} - 36\mathbf{z}\delta^2\mathbf{z} + 27\delta^3\mathbf{z} - 27(\delta\mathbf{z})^2\\
&\quad + 12\wp\mathbf{z}^2 - 36\wp\delta\mathbf{z} + 16\wp'\mathbf{z} + 75\wp^2 + (h - 101)\wp'' + (i + \tfrac{1}{2})g_2.
\end{aligned} \tag{4.11}$$

From (4.7), by symmetric differentiation we compute

$$3\mathbf{z}^2\delta\mathbf{z} = 9(\delta\mathbf{z})^2 + 9\mathbf{z}\delta^2\mathbf{z} - 9\delta^3\mathbf{z} + 12\wp'\mathbf{z} + 12\wp\delta\mathbf{z} - 8\wp''. \tag{4.12}$$

Multiplying (4.7) by $\mathbf{z}$ and substituting $\mathbf{z}^2\delta\mathbf{z}$ by (4.12), we get

$$\begin{aligned}
\mathbf{z}^4 &= 9\mathbf{z}^2\delta\mathbf{z} - 9\mathbf{z}\delta^2\mathbf{z} + 12\wp\mathbf{z}^2 - 8\wp'\mathbf{z}\\
&= 18\mathbf{z}\delta^2\mathbf{z} - 27\delta^3\mathbf{z} + 27(\delta\mathbf{z})^2 + 12\wp\mathbf{z}^2 + 36\wp\delta\mathbf{z} + 28\wp'\mathbf{z} - 24\wp''.
\end{aligned} \tag{4.13}$$

On the other hand, by substituting $\mathbf{z}^2\delta\mathbf{z}$ in (4.11) by (4.12), we get

$$\begin{aligned}
\mathbf{z}^4 &= 18\mathbf{z}\delta^2\mathbf{z} - 27\delta^3\mathbf{z} + 27(\delta\mathbf{z})^2 + 12\wp\mathbf{z}^2 + 36\wp\delta\mathbf{z} + 88\wp'\mathbf{z}\\
&\quad + 75\wp^2 + (h - 149)\wp'' + (i + \tfrac{1}{2})g_2.
\end{aligned} \tag{4.14}$$

Comparing (4.13) and (4.14) we must have

$$60\wp'(\sigma)\mathbf{z} = -99\wp^2(\sigma) - (h-149)\wp''(\sigma) - (i+\tfrac{1}{2})g_2.$$

The function $\mathbf{z} = \zeta(a_1+a_2+a_3) - \zeta(a_1) - \zeta(a_2) - \zeta(a_3)$, viewed as a function in $t = a_3$, has a pole at $t = 0$. But the right hand side is clearly regular at $t = 0$. Hence a contradiction.

This shows that for $n = 4$ no monic admissible polynomial equations of degree n for $\mathbf{z}_n$ may exist. The proof shows also that if $A_n(\delta^*\mathbf{z})$ exists for some $n \in \mathbb{N}$, then $A_{n-1}(\delta^*\mathbf{z})$ must also exist by looking at the residue term in the Laurent expansion of $\mathbf{z}_n^n = A_n(\delta^*\mathbf{z}_n)$. By induction this implies $A_n(\delta^*\mathbf{z})$ does not exists for all $n \geq 4$. □

Thus in order to make Theorem 2.1 effective we have to construct the degree $\frac{1}{2}n(n+1)$ polynomial $W_n(z)$ directly without the intermediate step. This issue is now resolved in §3 based on the method of resultant.

5. FUTURE PERSPECTIVES

As discussed in the introduction, one of our purposes in this series of papers is to study the structure of non-trivial solutions to (0.2), and to understand how non-trivial solutions change during the deformation of torus in the moduli space. Several important issues are listed below.

5.1. **Does Conjecture 0.1 hold?**

If we could show that

(5.1) $\qquad Z_n(\sigma;\tau)$ has only simple zeros in τ for any $\sigma \notin E_\tau[2]$,

then Conjecture 0.1 follows from Lemma 2.12 and Theorem 2.13 immediately. The reason is as follows: Let $a(\tau_0)$ be a non-trivial solution to (0.2) with $E = E_{\tau_0}$. By Lemma 2.12 and Theorem 2.13, we have $\sigma_0 := \sigma(a(\tau_0)) \notin E_{\tau_0}[2]$ and $Z_n(\sigma_0;\tau_0) = 0$. If (5.1) holds, then near (σ_0, τ_0) the equation $Z_n(\sigma;\tau) = 0$ for fixed σ has only one solution (σ,τ). This implies that $a(\tau_0)$ is not a bifurcation point of equation (0.2). In fact, it is easy to see that (5.1) is equivalent to Conjecture 0.1.

For $n = 1$, (5.1) is proved in [2, Theorem 4.1]. For $n = 1, 2, 3$ with σ being an N-torison point, the simple zero property of $Z_n(\sigma;\tau)$ had been shown by Dahmen in [6, 7] (see also [1]), by counting the number of integral Lamé equations with finite monodromy group by means of *dessin d'enfants* (see §5.2 for more explanations). However their results could not provide a complete answer to the question whether (5.1) holds (for $1 \leq n \leq 3$). The proof for $n = 1$ in [2, §5] used deformations and the fact that equation (1.1) has no solution for rectangular tori [11]. Hence we pose the following:

Conjecture 5.1. *There is no solutions to $\triangle u + e^u = 8\pi n\,\delta_0$ for $\tau = ib$, $b > 0$.*

By applying the theory developed in Part I [3] and this paper, we are able to prove it for $n = 2$ in the forthcoming Part III. Thus, together with the result of Dahmen, we could prove (5.1) in its fully generality for $n = 2$.

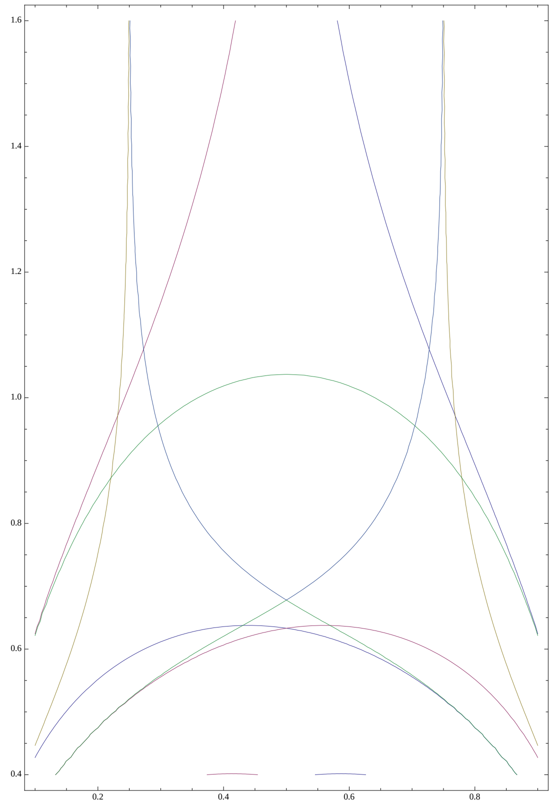

FIGURE 1. Five degenerate curves in F (marked in 5 different colors) for $n = 2$.

The affirmative answers to both (5.1) and Conjecture 5.1 are the first important steps towards the complete picture of (0.2), or equivalently (1.1) (= (0.3) with $\rho = 8\pi n$), over the moduli space of flat tori. As in the case $n = 1$ treated in [2, §5], we work on the fundamental domain F. When $n = 2$, there are 5 *trivial critical points* coming from the branch points of the hyperelliptic curve Y_2. We could deform the pre-modular form $Z_2(\sigma;\tau)$ in σ and study the geometry of the degeneracy curves in F corresponding to these 5 *trivial solutions*. Our numerical study of these curves (using *Mathematica*) is shown in Figure 1.

In Part III, we will show that the number of non-trivial solutions to (0.2) is constant in τ when τ moves in a connected region in F separated by these degenerate curves. Moreover we shall be able to determine the number of solutions explicitly in each connected region.

5.2. **Dahmen's conjecture.**

Although we have shown the existence of $Z_n(\sigma;\tau)$ for all n in Theorem 2.1 and provided the constructions of them in Proposition 3.7, in practice we still need to find the explicit expressions of them in order to get useful informations, e.g. to attack (5.1).

Indeed, in [6, 7] Dahmen developed a theory to count the number of equivalence classes of the algebraic form of integral Lamé equations

$$p(x)\frac{d^2w}{dx^2} + \frac{1}{2}p'(x)\frac{dw}{dx} - (n(n+1)\wp(z) + B)w = 0, \tag{5.2}$$

where $x = \wp(z)$, $p(x) = 4x^3 - g_2x - g_3$, $n \in \mathbb{N}$, with finite monodromy group M. In this case $M \cong D_N$ (dihedral) for some $N \in \mathbb{N}$. Denote this number by $L_n(N)$ for each $n, N \in \mathbb{N}$, then Dahmen conjectured that

$$L_n(N) = \frac{1}{2}\left(\frac{n(n+1)}{24}\Psi(N) - a_n\phi(N) - b_n\phi\Big(\frac{N}{2}\Big)\right) + \frac{2}{3}\epsilon_n(N), \tag{5.3}$$

where $a_{2m} = a_{2m+1} := m(m+1)/2$ and $b_{2m} = b_{2m-1} := m^2$. Also $\phi(N) = \#\{\, k \in \mathbb{Z} \mid \gcd(k, N) = 1,\, 0 \le k < N \,\}$ is the Euler function (we set $\phi(N) = 0$ if $N \notin \mathbb{N}$), $\Psi(N) := \#\{\, (k_1, k_2) \mid \gcd(k_1, k_2, N) = 1,\, 0 \le k_i < N \,\}$, and $\epsilon_n(N)$ is 0 or 1 to make the expression in (5.3) an integer.

Under the assumption of the existence of suitable modular forms (which is now provided by our $Z_n(\sigma;\tau)$ with $\sigma \in E_\tau[N]$), Dahmen proved that if (5.3) holds for some n, N then

$$Z_n\Big(\frac{k_1}{N}\omega_1 + \frac{k_2}{N}\omega_2;\tau\Big) \tag{5.4}$$

has only simple zeros. Furthermore, by the method of dessin d'enfants, he established (5.3) for any N and $n = 1, 2, 3$. In general, we may reduced his conjecture for all n, N to the counting of vanishing order of (5.4) at ∞.

With the explicit expression of $W_4(z)$ in (3.9) and hence the weight 10 pre-modular form $Z_4(\sigma;\tau)$, we are able to determine the precise vanishing order at ∞. It turns out agrees with the expected one, hence confirms Dahmen's conjecture for $n = 4$ (and for all N). The detailed verification is written up by Y.-C. Chou, as part of his undergraduate thesis supervised by the third author, and is included as Appendix A.

5.3. The geometry of $\bar{X}_n$ at its branch points over $\mathbb{P}^1$.

Equation (0.3) gives us the first motivation to study the geometry of $\bar{X}_n$ as follows. It was proved in Part I [3, Theorem 0.8] that if $\{u_k\}_{k=1}^\infty$ is a sequence of blowup solutions to (0.3) with $\rho_k \to 8\pi n$ and $\rho_k \ne 8\pi n$ for large k, then the blowup set $p = \{p_1, \cdots, p_n\}$ of $\{u_k\}$ is a trivial solution to (0.2). Here $\{u_k\}$ is said to blowup if $\max_E u_k \to +\infty$ as $k \to \infty$. Given such a blowup sequence, it was proved in [13] that

$$\rho_k - 8\pi n = (D(p) + O(1)) \exp\Big(-\max_E u_k\Big). \tag{5.5}$$

To define the invariant $D(p)$, we first recall the regular part of the Green function $\tilde{G}(z,q) := G(z-q) + \frac{1}{2\pi}\log|z-q|$, and define

$$\begin{aligned} f_{p_i}(z) &:= 8\pi\Big(\tilde{G}(z,p_i) - \tilde{G}(p_i,p_i) + \sum_{j\neq i}(G(z,p_j) - G(p_i,p_j)) \\ &\qquad - n(G(z) - G(p_i))\Big), \\ \mu_i &:= \exp\Big(8\pi(\tilde{G}(p_i,p_i) + \sum_{j\neq i} G(p_i,p_j) - nG(p_i))\Big). \end{aligned}$$

Then

$$D(p) := \lim_{r\to 0}\sum_{i=1}^{n}\mu_i\left(\int_{\Omega_i\setminus B_r(p_i)}\frac{e^{f_{p_i}(z)}-1}{|z-p_i|^4} - \int_{\mathbb{R}^2\setminus\Omega_i}\frac{1}{|z-p_i|^4}\right), \tag{5.6}$$

where Ω_i is any open neighborhood of p_i in E such that $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$, and $\bigcup_{i=1}^n \overline{\Omega}_i = E$.

We note that in a neighborhood of p_i, $e^{f_{p_i}(z)-1}$ is a harmonic function with quadratic terms in (x,y) plus $O(|z-p_i|^3)$, where $z = x+iy$, hence the limit in (5.6) is finite and well-defined.

It is clear that when $D(p) \neq 0$ its sign provides important information when we study bubbling solutions (blow-up sequence) with $\rho \neq 8\pi n$ (e.g. if $D(p) > 0$ then the bubbling may only occur from the right hand side). Also in the case $\rho_k = 8\pi n$ for all k, if the blow-up family u_λ exists then $D(p) = 0$ trivially, provided that p is a non-trivial solution to (0.2).

From the analytic point of view, the sign of $D(p)$ plays many important roles in the concentration of bubbling solutions of a class of non-linear PDE in two dimensions. For example, in the mean field equations, Chern–Simons-Higgs equations, and even a system of equations like $SU(3)$ Chern–Simons system etc.. Recently, it is also related to the so-called *local uniqueness* of bubbling solutions, which states that for any two sequence of bubbling solutions under some non-degenerate conditions (with $D(p) \neq 0$ being one of them) and with the same ρ_k, if their blowup sets are identical then the two sequence of solutions are equal. For recent applications of these quantities, we refer the readers to [13].

However, $D(p)$ or even its sign is in general very difficult to compute, except for the case $n = 1$ [5, 12, 2]. Thus the question how the analytic invariant $D(p)$ would be related to the geometry of $\bar{X}_n$ is one of the main issues we are going to pursue. Recall the conjecture we posed in Part I:

Conjecture 5.2 ([3]). *For rectangular tori, there are n branch points on $\bar{X}_n$ with $D < 0$ and $n+1$ branch points with $D > 0$.*

Conjecture 5.2 was proved in [5] for $n = 1$ and in [13] for $n = 2$. We are speculating that it is closely related to Conjecture 5.1 on non-existence of solutions to the mean field equations on rectangular tori.

APPENDIX A. A COUNTING FORMULA FOR LAMÉ EQUATIONS

By You-Cheng Chou [1]

Using the pre-modular forms constructed in §2 and §3, we verify the $n = 4$ case of Dahmen's conjectural counting formula (5.3) (= Conjecture 73 in [6]) for integral Lamé equations with finite monodromy.

A.1. **Dahmen's conjecture.** Let $L_n(N)$ be the number of Lamé equations (5.2) up to linear equivalence which has monodromy group isomorphic to the dihedral group D_N. Using the Hermite–Halphen ansatz (1.8) and the theory in §2, the problem is reduced to the zero counting of the $\mathrm{SL}(2,\mathbb{Z})$ modular form

$$M_n(N) := \prod_{\substack{0 \le k_1, k_2 < N \\ \gcd(k_1,k_2,N)=1}} Z_n\Big(\frac{k_1 + k_2\tau}{N};\tau\Big).$$

Using this, by repeating Dahmen's argument in [6], Lemma 65, 74, we get

Proposition A.1. *Suppose that for all $N \in \mathbb{Z}_{\ge 3}$ and $n \in \mathbb{N}$ we have that*

$$\nu_\infty(M_n(N)) = a_n\phi(N) + b_n\phi\Big(\frac{N}{2}\Big),$$

where $a_{2m} = a_{2m+1} = m(m+1)/2$, $b_{2m} = b_{2m-1} = m^2$. Then

$$L_n(N) = \frac{1}{2}\left(\frac{n(n+1)\Psi(N)}{24} - \left(a_n\phi(N) + b_n\phi\Big(\frac{N}{2}\Big)\right)\right) + \frac{2}{3}\epsilon_n(N),$$

where $\epsilon_n(N) = 1$ if $N = 3$ and $n \equiv 1 \pmod 3$, and $\epsilon_n(N) = 0$ otherwise.

Furthermore, $Z_n(\sigma;\tau)$ with σ a torsion point has only simple zeros in $\tau \in \mathbb{H}$.

Proof. Recall the formula for $\mathrm{SL}(2,\mathbb{Z})$ modular forms of weight k:

$$\sum_{P \ne \infty, i, \rho} \nu_P(f) + \nu_\infty(f) + \frac{\nu_i(f)}{2} + \frac{\nu_\rho(f)}{3} = \frac{k}{12}.$$

For $f = M_n(N)$, the weight $k = \frac{1}{2}n(n+1)\Psi(N)$. Notice that the counting is always doubled under the symmetry $(k_1,k_2) \to (N-k_1, N-k_2)$, thus by [6], Lemma 65, an upper bound for $L_n(N)$ is given by

$$U_n(N) := \frac{1}{2}\left(\frac{n(n+1)\Psi(N)}{24} - \left(a_n\phi(N) + b_n\phi\Big(\frac{N}{2}\Big)\right)\right) + \frac{2}{3}\epsilon_n(N).$$

That is, $L_n(N) \le U_n(N)$. Moreover, the equality holds if and only if each factor $Z_n((k_1+k_2\tau)/N;\tau)$ has only simple zeros.

We will show the equality holds by comparing it with the counting formula for the projective monodormy group $PL_n(N)$ (c.f. [6], Lemma 74).

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We recall the relation between $L_n(N)$ and $PL_n(N)$:

$$PL_n(N) = \begin{cases} L_n(N) + L_n(2N) & \text{if } N \text{ is odd,} \\ L_n(2N) & \text{if } N \text{ is even.} \end{cases}$$

If n is even and N is odd, we have

$$\begin{aligned} PL_n(N) &= L_n(N) + L_n(2N) \\ &\le \frac{1}{2}\left(\frac{n(n+1)\Psi(N)}{24} - \left(\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}\phi(N) + \frac{n^2}{4}\phi\Big(\frac{N}{2}\Big)\right)\right) + \frac{2}{3}\epsilon_n(N) \\ &\quad + \frac{1}{2}\left(\frac{n(n+1)\Psi(2N)}{24} - \left(\frac{\frac{n}{2}(\frac{n}{2}+1)}{2}\phi(2N) + \frac{n^2}{4}\phi(N)\right)\right) + \frac{2}{3}\epsilon_n(2N) \\ &= \frac{n(n+1)}{12}\left(\Psi(N) - 3\phi(N)\right) + \frac{2}{3}\epsilon_n(N) \end{aligned}$$

For the last equality, we use $\epsilon_n(2N) = 0$, $\Psi(2N) = 3\Psi(N)$ and $\phi(2N) = \phi(N)$. (If N is even, the relations are $\epsilon_n(N) = 0$, $\Psi(2N) = 4\Psi(N)$ and $\phi(2N) = \phi(N)$.) For the other three cases with (n, N) being (even, even), (odd, odd) or (odd, even), the computations are similar, and all lead to

$$PL_n(N) \le \frac{n(n+1)}{12}\left(\Psi(N) - 3\phi(N)\right) + \frac{2}{3}\epsilon_n(N).$$

On the other hand, using the method of *dessin d'enfants*, Dahmen showed directly that the equality holds [7]. Thus all the intermediate inequalities are indeed equalities, and in particular $L_n(N) = U_n(N)$ holds. $\square$

A.2. **q-expansions for some modular forms.** Recall that

$$\begin{aligned} \sum_{m\in\mathbb{Z}} \frac{1}{(m+z)^k} &= \frac{1}{(k-1)!}(-2\pi i)^k \sum_{n=1}^{\infty} n^{k-1} e^{2\pi i n z}, \\ \sum_{n\in\mathbb{Z}} \frac{1}{(x+n)^2} &= \pi^2 \cot^2(\pi x) + \pi^2, \\ \sum_{n\in\mathbb{Z}} \frac{1}{(x+n)^3} &= \pi^3 \cot^3(\pi x) + \pi^3 \cot(\pi x). \end{aligned}$$

We compute the q-expansions for $g_2, g_3, \wp, \wp', Z$, where $q = e^{2\pi i \tau}$:

$$g_2 = 60 \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^4} = 60\Big(2\zeta(4) + 2\frac{(-2\pi i)^4}{3!}\sum_{n=1}^{\infty}\sigma_3(n)q^n\Big),$$

where $\sigma_k(n) := \sum_{d|n} d^k$. Similarly,

$$g_3 = 140 \sum_{(n,m)\neq(0,0)} \frac{1}{(n+m\tau)^6} = 140\Big(2\zeta(6) + 2\frac{(-2\pi i)^6}{5!}\sum_{n=1}^{\infty}\sigma_5(n)q^n\Big).$$

Let $z = t + s\tau$. For $s = 0$, we have

$$\begin{aligned}
\wp'(\tau; t, 0) &= -2 \sum_{n,m\in\mathbb{Z}} \frac{1}{(t+n+m\tau)^3} \\
&= -2 \sum_{n\in\mathbb{Z}} \frac{1}{(t+n)^3} - 2\sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \left(\frac{1}{(m\tau+n+t)^3} - \frac{1}{(m\tau+n-t)^3}\right) \\
&= -2 \sum_{n\in\mathbb{Z}} \frac{1}{(t+n)^3} - 2\sum_{m=1}^{\infty} \frac{(-2\pi i)^3}{2!} \sum_{n=1}^{\infty} n^2 \left(e^{2\pi i n(m\tau+t)} - e^{2\pi i n(m\tau-t)}\right) \\
&= -2\pi^3 \cot(\pi t) - 2\pi^3 \cot^3(\pi t) + 16\pi^3 \sum_{n,m=1}^{\infty} n^2 \sin(2\pi n t)\, q^{nm}.
\end{aligned}$$

$$\begin{aligned}
\wp(\tau; t, 0) &= \frac{1}{t^2} + \sum_{(n,m)\neq(0,0)} \left(\frac{1}{(t+n+m\tau)^2} - \frac{1}{(n+m\tau)^2}\right) \\
&= \sum_{n\in\mathbb{Z}} \frac{1}{(t+n)^2} - \sum_{n=1}^{\infty}\frac{2}{n^2} + \sum_{m=1}^{\infty}\sum_{n\in\mathbb{Z}} \left(\frac{1}{(m\tau+t+n)^2} + \frac{1}{(m\tau-t+n)^2} - \frac{2}{(m\tau+n)^2}\right) \\
&= \pi^2\cot^2(\pi t) + \frac{2}{3}\pi^2 + \sum_{m=1}^{\infty}(-2\pi i)^2 \sum_{n=1}^{\infty} \left(e^{2\pi i n(m\tau+t)} + e^{2\pi i n(m\tau-t)} - 2e^{2\pi i n m\tau}\right) \\
&= \pi^2\cot^2(\pi t) + \frac{2}{3}\pi^2 + 8\pi^2 \sum_{n,m=1}^{\infty} (1-\cos 2n\pi t) q^{nm}.
\end{aligned}$$

Also, the Hecke function

$$Z(\tau; t, 0) = \pi\cot(\pi t) + 4\pi \sum_{n.m=1}^{\infty} (\sin 2n\pi t) q^{nm}.$$

For $s = 1/2$, we have

$$\begin{aligned}
\wp'(\tau; t, 1/2) &= -2 \sum_{(n,m)\neq(0,0)} \frac{1}{(t+n+(\frac{1}{2}+m)\tau)^3} \\
&= -2\sum_{m=1}^{\infty}\left(\sum_{m\in\mathbb{Z}} \frac{1}{(n+t+(m-\frac{1}{2})\tau)^3} - \sum_{n\in\mathbb{Z}} \frac{1}{(n-t+(m-\frac{1}{2})\tau)^3}\right) \\
&= -2\frac{(-2\pi i)^3}{2!} \sum_{n,m=1}^{\infty} n^2 \left(e^{2\pi i n(t+(m-\frac{1}{2})\tau)} - e^{2\pi i n(-t)+(m-\frac{1}{2})\tau}\right) \\
&= 16\pi^3 \sum_{n,m=1}^{\infty} n^2 (\sin 2\pi n t) q^{n(m-\frac{1}{2})}.
\end{aligned}$$

Similarly,

$$\wp(\tau; t, 1/2) = -\frac{1}{3}\pi^2 + 8\pi^2 \sum_{n,m=1}^{\infty} n q^{nm} - 8\pi^2 \sum_{n,m=1}^{\infty} n(\cos 2\pi n t) q^{n(m-\frac{1}{2})},$$

and

$$Z(\tau; t, 1/2) = 4\pi \sum_{n,m=1}^{\infty} (\sin 2\pi n t) q^{n(m-\frac{1}{2})}.$$

A.3. **The counting formula for $n = 4$.** Now we give the computations for $n = 4$ and prove the formula $L_4(N) = U_4(N)$ from Proposition A.1.

Theorem A.2. *For $n = 4$ and $N \in \mathbb{Z}_{\geq 3}$, we have*

$$L_4(N) = \frac{1}{2}\left(\frac{5\Psi(N)}{6} - \left(3\phi(N) + 4\phi\Big(\frac{N}{2}\Big)\right)\right).$$

Moreover, $Z_4(\sigma;\tau)$ with $\sigma \in E_\tau[N]$ has only simple zeros in $\tau \in \mathbb{H}$.

Proof. For $n = 4$, the pre-modular form $Z_4 = W_4(Z)$ is given in (3.9):

$$\begin{aligned} W_4(Z) = {} & Z^{10} - 45\wp Z^8 - 120\wp' Z^7 + (\frac{399}{4}g_2 - 630\wp^2)Z^6 - (504\wp\wp')Z^5 \\ & - \frac{15}{4}(280\wp^3 - 49g_2\wp - 115g_3)Z^4 + 15(11g_2 - 24\wp^2)\wp' Z^3 \\ & - \frac{9}{4}(140\wp^4 - 245g_2\wp^2 + 190g_3\wp + 21g_2^2)Z^2 \\ & - (40\wp^3 - 163g_2\wp + 125g_3)\wp Z + \frac{3}{4}(25g_2 - 3\wp^2)\wp'^2, \end{aligned}$$

where Z is the Hecke function.

We compute the asymptotic behavior of $W_4(Z)$ when $\tau \to \infty$. Let $z = t + s\tau$. We divide the problem into two cases

(1) $s \equiv 0 \pmod 1$. In this case, according to the q-expansion given in §A.2, we have

$$g_2 \to \frac{3}{4}\pi^4, \qquad g_3 \to \frac{8}{27}\pi^6, \qquad Z(z) \to \pi\cot(\pi t),$$
$$\wp'(z) \to -2\pi^3\cot(\pi t) - 2\pi^3\cot^3(\pi t),$$
$$\wp(z) \to \pi^2\cot^2(\pi t) + \frac{2}{3}\pi^2.$$

A direct computation shows that $W_4(Z)$ has zeros at ∞ when $s = 0$.

By replacing all the modular forms $g_2, g_3, \wp, \wp'$ and Z in $W_4(Z)$ with their q-expansions, we have (e.g. using *Mathematica*)

$$W_4(Z) = 2^{14}3^3 5^2 7\,\pi^{10}\cos^2(\pi t)\sin^2(\pi t)q^3 + O(q^4)$$

(2) $s \not\equiv 0 \pmod 1$. In this case we have

$$Z \to 2\pi i\left(s - \frac{1}{2}\right), \qquad \wp(z) \to -\frac{1}{3}\pi^2,$$
$$\wp'(z) \to 0, \qquad g_2 \to \frac{4}{3}\pi^4, \qquad g_3 \to \frac{8}{27}\pi^6.$$

Hence the constant term of $W_4(Z)$ is given by

$$\begin{aligned} W_4(z) = {} & -64\pi^{10}(-2+s)(-1+s)^2 s^2(1+s) \\ & \times(-3+2s)(-1+2s)^2(1+2s) + O(q). \end{aligned}$$

Notice that if $s \not\equiv 0 \pmod 1$ then $W_4(Z)$ has zero at $\tau = \infty$ if and only if $s \equiv 1/2 \pmod 1$.

Now we fix $s = 1/2$ and replace the modular forms $g_2, g_3, \wp, \wp'$ and Z in $W_4(Z)$ with their q-expansions. We get

$$W_4(Z) = 2^{10}3^3 5^2 7\, \pi^{10} \cos(\pi t)^2 \sin(\pi t)^2 q^2 + O(q^3).$$

These computations for the q-expansions imply that

$$\begin{aligned} \nu_\infty(M_4(N)) &= 3\,\#\{\, 1 \le k_1 \le N \mid \gcd(N, k_1) = 1 \,\} \\ &\qquad + 2\,\#\{\, 0 \le k_1 \le N \mid \gcd(N/2, k_1) = 1 \,\} \\ &= 3\phi(N) + 4\phi(N/2). \end{aligned}$$

Since the value of $\nu_\infty(M_4(N))$ coincides with the assumption in Proposition A.1 for $n = 4$, the theorem follows from it accordingly. □

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