Boundary feedback spectrum shift for distributed parameter systems on a planar domain

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Abstract: For a special class of linear distributed parameter systems defined on a planar domain and controlled by boundary inputs, a systematic procedure is established to reformulate them as systems defined over certain interpolation spaces. Based on this result, it is shown how certain spectrum shift can be achieved by using boundary state feedback control. An example is used to demonstrate the theoretical results.

Keywords: distributed parameter system, boundary control, spectrum shift

1 INTRODUCTION

Controlling distributed parameter systems is admittedly a difficult problem. In the literature (e.g. reference [**1**]), similar control problems are studied and parallel control theories are developed for both distributed and lumped parameter systems, especially when system models used to describe both kind of systems have a similar form. This is particularly true when the state-space approach is adopted. Nevertheless, the boundary control problem for distributed parameter systems has no counterpart in lumped systems, because when modelling lumped systems there are no boundary conditions. In this paper a class of distributed parameter systems with boundary inputs is considered. The state variables are represented as a function of time and spatial variables are defined over a planar domain. It is intended that state feedback control laws be found such that part of the spectrum of the closed-loop system may be shifted in a beneficial direction. In reference [**2**] it is shown that under some conditions the boundary inputs can be combined into system dynamic equations redefined over some interpolation space. Other methods, including those in references [**3**] to [**6**], give various ways of handling systems with boundary inputs. Using the method of combining boundary inputs into the system dynamic equations, it is possible to extend many control analysis and synthesis methods which were originally developed for distributed parameter systems with zero boundary conditions and inspired by methods for lumped systems. For example, this methodology is used in reference [**7**] to study the robustness stability problem of distributed parameter systems with boundary inputs and in reference [**8**] to study the spectrum shift problem via boundary state feedback for distributed parameter systems defined on a line.

Here, based on the idea of reference [**2**] and methods derived in references [**6**] and [**9**], the same problem of reference [**8**] for distributed parameter systems defined on a planar domain is discussed. Firstly, it is shown that this class of boundary control system can be reformulated as a bounded-input spectral system defined over a certain interpolation space. Secondly, it is shown how to adapt the results of references [**9**] and [**10**] to achieve the spectrum shift. Finally, an example and its simulation results are used to illustrate the theoretical results.

2 SYSTEM DESCRIPTION AND TRANSFORMATION

Let $\mathscr{N}$ be the set of non-negative integers, $\mathscr{R}$ be the field of real numbers and $\mathscr{R}^l$ be the vector space of l-tuple vectors with elements from $\mathscr{R}$. Furthermore, let $\boldsymbol{x} = [x_1, x_2]^{\mathrm{T}}$ represent a vector in $\mathscr{R}^2$. Consider the closed region $\Omega \subset \mathscr{R}^2$ with boundary $\partial\Omega = \cup_{i=1}^{4} \Gamma_i$, where

$$\begin{aligned}
\Gamma_1 &= \{[x_1, x_2]^{\mathrm{T}} \mid 0 \leqslant x_1 \leqslant l_1,\ x_2 = 0\} \\
\Gamma_2 &= \{[x_1, x_2]^{\mathrm{T}} \mid x_1 = 0,\ 0 \leqslant x_2 \leqslant l_2\} \\
\Gamma_3 &= \{[x_1, x_2]^{\mathrm{T}} \mid 0 \leqslant x_1 \leqslant l_1,\ x_2 = l_2\} \\
\Gamma_4 &= \{[x_1, x_2]^{\mathrm{T}} \mid x_1 = l_1,\ 0 \leqslant x_2 \leqslant l_2\}
\end{aligned} \tag{1}$$

At each boundary segment Γ_i, $\boldsymbol{n}_i$ is defined to be the unit

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outward normal on Γ_i and the second-order linear distributed parameter system is found to be

$$\frac{\partial z(t,\boldsymbol{x})}{\partial t} = a_{11}\frac{\partial^2 z(t,\boldsymbol{x})}{\partial x_1^2} + a_{22}\frac{\partial^2 z(t,\boldsymbol{x})}{\partial x_2^2}$$
$$\left.\frac{\partial z(t,\boldsymbol{x})}{\partial \boldsymbol{n}_i}\right|_{\Gamma_i} = u_i(t), \qquad i = 1, 2, 3, 4$$
$$z(0,\boldsymbol{x}) = z_0(\boldsymbol{x}), \qquad \forall\, \boldsymbol{x} \in \Omega \tag{2}$$

where $z(t,\boldsymbol{x})$ is the state function of the system, $z_0(\boldsymbol{x})$ is a given initial condition, $u_i(\cdot)$ is the ith input function and a_{11}, a_{22} are positive coefficients. Subsequently, $\mathscr{Z} = \mathscr{L}_2(\Omega)$ will be denoted as the space of the Lebesgue square integrable functions defined on Ω and $\langle\cdot,\cdot\rangle_{\mathscr{Z}}$ as the inner product defined on $\mathscr{Z}$. The space of all bounded linear operators from the space $\mathscr{X}$ to the space $\mathscr{Y}$ will be denoted by $\mathscr{B}(\mathscr{X},\mathscr{Y})$. For any operator A, $\mathscr{D}(A)$ will represent its domain, $\mathscr{J}(A)$ its range and $\mathscr{K}(A)$ its kernel.

The system (2) may also be represented by a more compact yet more general state-space model:

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = \tilde{A}z(t)$$
$$Hz(t) = u(t)$$
$$z(0) = z_0 \tag{3}$$

where

$$\tilde{A}z = a_{11}\frac{\partial^2 z(t,\boldsymbol{x})}{\partial x_1^2} + a_{22}\frac{\partial^2 z(t,\boldsymbol{x})}{\partial x_2^2} \tag{4}$$

and

$$Hz = \begin{bmatrix} -\left.\dfrac{\partial z(t,\boldsymbol{x})}{\partial x_2}\right|_{x_2=0} \\ \left.\dfrac{\partial z(t,\boldsymbol{x})}{\partial x_1}\right|_{x_1=l_1} \\ \left.\dfrac{\partial z(t,\boldsymbol{x})}{\partial x_2}\right|_{x_2=l_2} \\ -\left.\dfrac{\partial z(t,\boldsymbol{x})}{\partial x_1}\right|_{x_1=0} \end{bmatrix}$$
$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{bmatrix} \tag{5}$$

Note that $\tilde{A}:\mathscr{D}(\tilde{A})\subset\mathscr{Z}\to\mathscr{Z}$ is an unbounded linear operator densely defined on $\mathscr{Z}$. The state-space model (3) is of interest because if it owns the three properties listed in the following definition, then it may lead to another model that facilitates the task of designing state feedback control laws for the system described by system (2).

Definition 1 [**9**]

Distributed parameter systems described by the state-space model (3) are called general boundary input systems if:

1. $A = \tilde{A}|_{\mathscr{D}(\tilde{A})\cap\mathscr{K}(H)} : \mathscr{D}(A)\subset\mathscr{Z}\to\mathscr{Z}$ generates an analytic semigroup [**1**] $T(t)$ on $\mathscr{Z}$.
2. There exist an $\omega > \omega_0$, the growth constant [**1**] of $T(t)$, and a $G(\omega)\in\mathscr{B}(\mathscr{R}^2,\mathscr{Z})$ such that $(\omega I-\tilde{A})G(\omega) = 0$ and $HG(\omega)u = u$ for all $u = [u_1\ u_2\ u_3\ u_4]^{\mathrm{T}}\in\mathscr{R}^4$.
3. There exists a real number $\alpha\in[0,1)$ such that $\mathscr{J}[G(\omega)]\subset\mathscr{D}[(\omega I-A)^{\alpha}]$.

In the literature, many methods have been proposed for transforming the boundary conditions into various forms so that different types of boundary problems may be solved. The early results by Fattorini [**5**] and Lion and Magenes [**2**] suggest a way of thinking about this type of problem. The results by Lasiecka [**11**] and Amann [**3**] are very complete in the representation of the solutions to boundary problems of the parabolic type. However, this type of solution usually focuses on the expression of solutions themselves. This is adequate for solving the boundary problems but not for solving related control problems, which involve feedback controller design. The ability to transform the boundary conditions and obtain a set of standard state-space equations is very important for studying control problems. In fact, there are two different approaches. One is to generalize the theory to allow an unbounded input operator in the state-space description [**1**, **12**, **13**]. The other is to consider the state-space model over a larger interpolation space so that the input operator becomes a bounded operator.

The concept of general boundary input systems was first proposed by Inaba and Hinata [**6**, **14**]. It describes the systems with boundary inputs that can be transformed into distributed inputs. The most important condition in Definition 1 is the second one, which sometimes is called the translation condition. It is related to the actual finding of the equivalent input operators. In most cases the value of $G(\omega)$ needs to be found in order to show that the system is a general boundary input system.

In the following theorem, it is shown that model (3) is indeed a general boundary input system with (4) and (5).

Theorem 1

For $\tilde{A}$ and H from (4) and (5) respectively, suppose $A = \tilde{A}|_{\mathscr{D}(\tilde{A})\cap\mathscr{K}(H)}$ as in Definition 1.

1. The operator A has real eigenvalues λ_{n_1,n_2},

$(n_1, n_2) \in \mathcal{N} \times \mathcal{N}$, of finite magnitudes. Moreover, the corresponding eigenvectors ϕ_{n_1,n_2}, $(n_1, n_2) \in \mathcal{N} \times \mathcal{N}$, form a complete basis in $\mathcal{Z}$.

2. The system described by (3), (4) and (5) is a general boundary input system.

Proof

1. With (4) and (5), it is easy to verify that the operator $A = \tilde{A}|_{\mathcal{D}(\tilde{A}) \cap \mathcal{K}(H)}$ has only a point spectrum. Suppose $\{\lambda, \phi\}$ is a pair of eigenvalue and eigenvector in the point spectrum of A. Thus $(A - \lambda I)\phi = 0$, which in its detailed form is

$$
\begin{aligned}
&a_{11}\frac{\partial^2 \phi}{\partial x_1^2} + a_{22}\frac{\partial^2 \phi}{\partial x_2^2} - \lambda\phi = 0 \\
&-\left.\frac{\partial \phi}{\partial x_2}\right|_{x_2=0} = 0 \\
&\left.\frac{\partial \phi}{\partial x_1}\right|_{x_1=l_1} = 0 \\
&\left.\frac{\partial \phi}{\partial x_2}\right|_{x_2=l_2} = 0 \\
&-\left.\frac{\partial \phi}{\partial x_1}\right|_{x_1=0} = 0
\end{aligned}
\tag{6}
$$

Direct computation easily shows that

$$
\{\lambda, \phi_{n_1,n_2}\} = \begin{cases}
\{\lambda, \phi_{0,0}(x_1, x_2)\} = \{0, 1\}, & n_1 = 0,\ n_2 = 0 \\
\{\lambda, \phi_{0,n_2}(x_1, x_2)\} = \left\{\left(\dfrac{n_2\pi}{l_2}\right)^2, \sqrt{2}\cos\left(\dfrac{n_2\pi}{l_2}x_2\right)\right\}, & n_1 = 0,\ n_2 \geqslant 1 \\
\{\lambda, \phi_{n_1,0}(x_1, x_2)\} = \left\{\left(\dfrac{n_1\pi}{l_1}\right)^2, \sqrt{2}\cos\left(\dfrac{n_1\pi}{l_1}x_1\right)\right\}, & n_1 \geqslant 1,\ n_2 = 0 \\
\{\lambda, \phi_{n_1,n_2}(x_1, x_2)\} = \left\{\left(\dfrac{n_2\pi}{l_2}\right)^2 + \left(\dfrac{n_1\pi}{l_1}\right)^2, 2\cos\left(\dfrac{n_1\pi}{l_1}x_1\right)\cos\left(\dfrac{n_2\pi}{l_2}x_2\right)\right\}, & \\
 & n_1 \geqslant 1,\ n_2 \geqslant 1
\end{cases}
\tag{7}
$$

are the eigenvalue and eigenvector pairs of A and that $\{\phi_{n_1,n_2}\}$ forms a complete basis in $\mathcal{Z}$.

2. First of all, for every $z \in \mathcal{Z}$, the operator $T(t)$ defined by

$$
T(t)z = \sum_{(n_1,n_2) \in \mathcal{N} \times \mathcal{N}} \langle z, \phi_{n_1,n_2} \rangle_{\mathcal{Z}} e^{\lambda_{n_1,n_2} t} \phi_{n_1,n_2} \tag{8}
$$

is the analytic semigroup that A generates on $\mathcal{Z}$. Secondly, the appropriate ω and $G(\omega)$ values need to be found. For any $\omega > \omega_0$, the growth constant of $T(t)$, define the convenient symbol $f_u(\boldsymbol{x}) = [G(\omega)u](\boldsymbol{x})$. Then the equation $\tilde{A}G(\omega)u = \omega G(\omega)u$ can be rewritten as $\tilde{A}f_u(\boldsymbol{x}) = \omega f_u(\boldsymbol{x})$, which suggests examination of

$$
\begin{aligned}
f_u(\boldsymbol{x}) &= [G(\omega)u](\boldsymbol{x}) \\
&= c_1(t)\, e^{\sqrt{\omega/a_{11}}\, x_1} + c_2(t)\, e^{-\sqrt{\omega/a_{11}}\, x_1} \\
&\quad + c_3(t)\, e^{\sqrt{\omega/a_{22}}\, x_2} + c_4(t)\, e^{-\sqrt{\omega/a_{22}}\, x_2}
\end{aligned}
$$

for some $c_i(t)$, $i = 1, 2, 3, 4$, independent of $\boldsymbol{x}$. Substituting the above expression into the other requirement that $G(\omega)$ must satisfy, i.e. $HG(\omega)u = u$, yields the conditions for the functions $c_i(t)$:

$$
\begin{bmatrix} 1 & -1 \\ e^{\sqrt{\omega/a_{22}}\, l_2} & -e^{-\sqrt{\omega/a_{22}}\, l_2} \end{bmatrix}
\begin{bmatrix} c_3(t) \\ c_4(t) \end{bmatrix}
= \begin{bmatrix} \dfrac{1}{-\sqrt{\omega/a_{22}}} u_1(t) \\ \dfrac{1}{\sqrt{\omega/a_{22}}} u_3(t) \end{bmatrix}
\tag{9}
$$

$$
\begin{bmatrix} e^{\sqrt{\omega/a_{11}}\, l_1} & -e^{-\sqrt{\omega/a_{11}}\, l_1} \\ 1 & -1 \end{bmatrix}
\begin{bmatrix} c_1(t) \\ c_2(t) \end{bmatrix}
= \begin{bmatrix} \dfrac{1}{\sqrt{\omega/a_{11}}} u_2(t) \\ \dfrac{1}{-\sqrt{\omega/a_{11}}} u_4(t) \end{bmatrix}
\tag{10}
$$

These two sets of linear equations in $c_i(t)$ allow $c_i(t)$ to be expressed in terms of $u_i(t)$. For the sake of notational brevity, these can be written as

$$
\begin{aligned}
c_1(t) &= \alpha_{12}u_2(t) + \alpha_{14}u_4(t) \\
c_2(t) &= \alpha_{22}u_2(t) + \alpha_{24}u_4(t) \\
c_3(t) &= \alpha_{31}u_1(t) + \alpha_{33}u_3(t) \\
c_4(t) &= \alpha_{41}u_1(t) + \alpha_{43}u_3(t)
\end{aligned}
$$

where the α_{ij} values are constants depending only on a_{11}, a_{22} and ω. Thus

$$
\begin{aligned}
[G(\omega)u](\boldsymbol{x}) &= (\alpha_{31}\, e^{\sqrt{\omega/a_{22}}\, x_2} + \alpha_{41}\, e^{-\sqrt{\omega/a_{22}}\, x_2})u_1(t) \\
&\quad + (\alpha_{12}\, e^{\sqrt{\omega/a_{11}}\, x_1} + \alpha_{22}\, e^{-\sqrt{\omega/a_{11}}\, x_1})u_2(t) \\
&\quad + (\alpha_{33}\, e^{\sqrt{\omega/a_{22}}\, x_2} + \alpha_{43}\, e^{-\sqrt{\omega/a_{22}}\, x_2})u_3(t) \\
&\quad + (\alpha_{14}\, e^{\sqrt{\omega/a_{11}}\, x_1} + \alpha_{24}\, e^{-\sqrt{\omega/a_{11}}\, x_1})u_4(t)
\end{aligned}
$$

This explicit expression of $[G(\omega)u](\boldsymbol{x})$ can be expanded with respect to the basis $\{\phi_{n_1,n_2}\}$. The result is written as

$$
\begin{aligned}
[G(\omega)u](\boldsymbol{x}) = & \left[\sum_{n_2\in\mathcal{N}} g^1_{0,n_2}\phi_{0,n_2}(x_2)\right]u_1(t) \\
& + \left[\sum_{n_1\in\mathcal{N}} g^2_{n_1,0}\phi_{n_1,0}(x_1)\right]u_2(t) \\
& + \left[\sum_{n_2\in\mathcal{N}} g^3_{0,n_2}\phi_{0,n_2}(x_2)\right]u_3(t) \\
& + \left[\sum_{n_1\in\mathcal{N}} g^4_{n_1,0}\phi_{n_1,0}(x_1)\right]u_4(t)
\end{aligned}
$$

This is a full characterization of the operator $G(\omega)$, with which the boundedness property can be easily derived. Finally, since $\mathcal{R}^4$ is a finite dimensional space and $G(\omega)\in\mathcal{B}(\mathcal{R}^4,\mathcal{Z})$, $\mathcal{J}[G(\omega)]$ is also finite dimensional, which means that at least $\alpha=0$ is such that $\mathcal{J}[G(\omega)]\subset\mathcal{D}[(\omega I-A)^\alpha]=\mathcal{Z}$. This completes the proof.

After establishing that under the assumptions, model (3) with (4) and (5) is a general boundary input system, the following theorem can be quoted to obtain a model with a more familiar appearance. The symbols are as defined before.

Theorem 2 [**9**]

For a general boundary input system (3), if $\mathcal{Z}_0=\mathcal{Z}$, $\mathcal{Z}_1=\mathcal{D}(A)$, the interpolation space $\mathcal{Z}_\alpha=\mathcal{D}[(\omega I-A)^\alpha]$ and the operators $A_\alpha:\mathcal{D}(A_\alpha)=\mathcal{Z}_\alpha\subset\mathcal{Z}_{\alpha-1}\to\mathcal{Z}_{\alpha-1}$ and $B_\alpha\in\mathcal{B}(\mathcal{R}^4,\mathcal{Z}_{\alpha-1})$ respectively can be defined as

$$A_\alpha z=[(\omega I-A)^{\alpha-1}]^{-1}A(\omega I-A)^{\alpha-1}z \tag{11}$$

$$B_\alpha u=(\omega I-A_\alpha)G(\omega)u \tag{12}$$

then the system model (3) is equivalent to

$$
\begin{aligned}
&\frac{\mathrm{d}z(t)}{\mathrm{d}t}=A_\alpha z(t)+B_\alpha u(t)\\
&z(0)=z_0
\end{aligned}
\tag{13}
$$

The model (13) provides a description of the systems to be studied with no explicit formulation of the boundary conditions, and facilitates the application of existing state feedback controller design methods. In fact, if the analysis results presented in Theorem 1 continue to be utilized, series expansions of the operators A_α and B_α can be found.

Corollary 1

Let $\mu_{n_1,n_2}=\omega-\lambda_{n_1,n_2}$. Then the following expansion is obtained:

$$\mathcal{Z}_\alpha=\left\{z=\sum_{(n_1,n_2)\in\mathcal{N}\times\mathcal{N}}\zeta_{n_1,n_2}\phi_{n_1,n_2}\,\middle|\,\sum_{(n_1,n_2)\in\mathcal{N}\times\mathcal{N}}(\mu^\alpha_{n_1,n_2}\zeta_{n_1,n_2})^2<\infty\right\}$$

$$A_\alpha z=\sum_{(n_1,n_2)\in\mathcal{N}\times\mathcal{N}}\lambda_{n_1,n_2}\zeta_{n_1,n_2}\phi_{n_1,n_2},\qquad \forall z=\sum_{(n_1,n_2)\in\mathcal{N}\times\mathcal{N}}\zeta_{n_1,n_2}\phi_{n_1,n_2}\in\mathcal{Z}_\alpha$$

$$
\begin{aligned}
B_\alpha u = & \left(\sum_{n_2\in\mathcal{N}}\mu_{0,n_2}g^1_{0,n_2}\phi_{0,n_2}\right)u_1(t)+\left(\sum_{n_1\in\mathcal{N}}\mu_{n_1,0}g^2_{n_1,0}\phi_{n_1,0}\right)u_2(t)+\left(\sum_{n_2\in\mathcal{N}}\mu_{0,n_2}g^3_{0,n_2}\phi_{0,n_2}\right)u_3(t)\\
& +\left(\sum_{n_1\in\mathcal{N}}\mu_{n_1,0}g^4_{n_1,0}\phi_{n_1,0}\right)u_4(t),\qquad \forall u\in\mathcal{R}^4
\end{aligned}
$$

Proof. The results are derived from straightforward computations.

To simplify the notation of $B_\alpha u$, ψ^i_n, $i=1,2,3,4$, can be defined directly from Corollary 1 as

$$
\begin{aligned}
B_\alpha u = & \left(\sum_{n_2\in\mathcal{N}}\psi^1_{n_2}\phi_{0,n_2}\right)u_1(t)+\left(\sum_{n_2\in\mathcal{N}}\psi^3_{n_2}\phi_{0,n_2}\right)u_3(t)\\
& +\left(\sum_{n_1\in\mathcal{N}}\psi^2_{n_1}\phi_{n_1,0}\right)u_2(t)+\left(\sum_{n_1\in\mathcal{N}}\psi^4_{n_1}\phi_{n_1,0}\right)u_4(t)
\end{aligned}
$$

With regard to the extended space $\mathcal{Z}_\alpha$ in this corollary, it is noted that $\mathcal{Z}_0=\mathcal{Z}$ and $\mathcal{Z}_1=\mathcal{D}(A)$. Moreover, $\mathcal{Z}_\alpha\subset\mathcal{Z}_\beta$ for $\alpha>\beta$. Thus, for $\alpha<0$, $\mathcal{Z}_\alpha$ is an extension of $\mathcal{Z}$ and a series of spaces $\{\mathcal{Z}_\alpha\}$ is constructed from the operator A. With the extended space $\mathcal{Z}_\alpha$, the solution with boundary conditions can be properly formulated, and the boundary input can be described as a distributed input over the extended interpolation space. In section 4, there is an example showing how to handle the boundary problem in interpolation space.

At the end of this section it is mentioned that due to part 1 of Theorem 1, the systems studied here are called the spectral systems [**1**] when they are described by model (13).

3 SPECTRUM SHIFT

The boundary state feedback spectrum shift problem to be considered here is not very different in objective from the problem considered for finite dimensional systems. For a distributed parameter system with spectra $\{\lambda_{n_1,n_2}\}$, it is possible to find $k_i(\boldsymbol{x})$, $i=1, 2, 3, 4$, such that with the state feedback $u_i(t)=\langle k_i(\boldsymbol{x}), z(t)\rangle_{\mathscr{L}}$ the closed-loop system has a new set of spectra $\{\beta_{n_1,n_2}\}$. However, compared with the well-known results of the finite dimensional pole assignment problem, the results of spectrum assignment and assignability problems for distributed parameter systems are much more restrictive. For the spectral systems defined in section 2, it turns out that the results in references [**9**] and [**10**] can be adapted to solve the spectrum shift problem under certain conditions.

In reference [**8**], the spectrum shift problem is discussed for one-dimensional distributed parameter systems and it is shown that, with the help of reference [**9**], it is possible to achieve uniform spectrum shift. More specifically, for certain one-dimensional distributed parameter systems with the set of spectra $\{\lambda_j\}$ and a given $\eta<0$, a proper state feedback controller can be found to shift the spectrum of the closed-loop systems to $\{\beta_j=\lambda_j+\eta\}$. However, for distributed parameter systems on a two-dimensional domain, the situation is usually more difficult. Hence for the systems described in this paper, the problem of how to shift all the spectra $\{\lambda_{n_1,n_2}\}$ to the new set of spectra $\{\beta_{n_1,n_2}=\lambda_{n_1,n_2}+\eta\}$ is not discussed. Instead, the possibility is demonstrated of shifting the subsets $\{\lambda_{n_1,0}\}$ and $\{\lambda_{0,n_2}\}$ of the overall spectrum to $\{\beta_{n_1,0}=\lambda_{n_1,0}+\eta_1\}$ and $\{\beta_{0,n_2}=\lambda_{0,n_2}+\eta_2\}$ respectively, for given constants $\eta_1<0$ and $\eta_2<0$. Two weighting parameters $\delta_i\in[0, 1]$, $i=1, 2$, are also introduced, which 'divide the work' of the spectrum shift of the two subsets of spectra into four parts. More specifically, an attempt is made to find a state feedback law $k_1(\boldsymbol{x})$ that moves $\{\lambda_{0,n_2}\}$ to $\{\lambda_{0,n_2}+\delta_2\eta_2\}$ and a $k_3(\boldsymbol{x})$ that moves $\{\lambda_{0,n_2}+\delta_2\eta_2\}$ to $\{\beta_{0,n_2}=\lambda_{0,n_2}+\eta_2\}$ and, similarly, a $k_2(\boldsymbol{x})$ that moves $\{\lambda_{n_1,0}\}$ to $\{\lambda_{n_1,0}+\delta_1\eta_1\}$ and a $k_4(\boldsymbol{x})$ that moves $\{\lambda_{n_1,0}+\delta_1\eta_1\}$ to $\{\beta_{n_1,0}=\lambda_{n_1,0}+\eta_1\}$. As in the one-dimensional case [**8**], this approach increases the flexibility of utilizing multiple inputs in the system. Furthermore, by sometimes spreading the load it is possible to avoid saturation of individual inputs.

Based on the results of references [**8**] and [**9**], the following theorem is proposed for the systems described by system (2). Again, the notation used is as defined before.

Theorem 3

For the spectral systems defined above, suppose $\psi_n^i\neq 0$ for $i=1, 2, 3, 4$,

$$\sum_{n_1\in\mathscr{N}}\frac{1}{|\lambda_{n_1,0}|}<\infty$$

$$\sum_{n_2\in\mathscr{N}^{n_2\neq 0}}\frac{1}{|\lambda_{0,n_2}|}<\infty$$

and

$$\sum_{m=0,m\neq n_1}^{\infty}\frac{1}{|\lambda_{m,0}-\lambda_{n_1,0}|}=v_{n_1}^1\leqslant v^1<\infty \qquad \text{for } n_1\in\mathscr{N}$$

$$\sum_{m=1,m\neq n_2}^{\infty}\frac{1}{|\lambda_{0,m}-\lambda_{0,n_2}|}=v_{n_2}^2\leqslant v^2<\infty \qquad \text{for } n_2\in\mathscr{N},\ n_2\neq 0$$

Let $\eta_1, \eta_2\in\mathscr{R}$ and $0\leqslant\delta_1, \delta_2\leqslant 1$ be chosen such that

$$\{\lambda_{n_1,0}\}\cap\{\lambda_{n_1,0}+\delta_1\eta_1\}\cap\{\lambda_{n_1,0}+\eta_1\}=\varnothing$$

and

$$\{\lambda_{0,n_2}\}\cap\{\lambda_{0,n_2}+\delta_2\eta_2\}\cap\{\lambda_{0,n_2}+\eta_2\}=\varnothing$$

Then the state feedback laws $u_i(t)=\langle k_i(\boldsymbol{x}), z(t)\rangle_{\mathscr{L}}$ $i=1, 2, 3, 4$, with

$$k_1(\boldsymbol{x})=\delta_2\eta_2\sum_{n_2\in\mathscr{N}^{n_2\neq 0}}\frac{1}{\psi^1_{n_2}}\prod_{m\in\mathscr{N}^{m\neq n_2,m\neq 0}}\times\left(1-\frac{\delta_2\eta_2}{\lambda_{0,n_2}-\lambda_{0,m}}\right)\phi_{0,n_2}(\boldsymbol{x}) \tag{14}$$

$$k_2(\boldsymbol{x})=0$$

$$k_3(\boldsymbol{x})=0$$

$$k_4(\boldsymbol{x})=\delta_1\eta_1\sum_{n_1\in\mathscr{N}}\frac{1}{\psi^2_{n_1}}\prod_{m\in\mathscr{N}^{m\neq n_1}}\times\left(1-\frac{\delta_1\eta_1}{\lambda_{n_1,0}-\lambda_{m,0}}\right)\phi_{n_1,0}(\boldsymbol{x}) \tag{15}$$

will shift part of the spectrum of the closed-loop system from $\{\lambda_{0,n_2}\}$ and $\{\lambda_{n_1,0}\}$ to $\{\lambda_{0,n_2}+\delta_2\eta_2\}$ and $\{\lambda_{n_1,0}+\delta_1\eta_1\}$ respectively. Moreover, with $k_1(\boldsymbol{x})$ in (14), $k_4(\boldsymbol{x})$ in (15) and

$$k_2(\boldsymbol{x})=(1-\delta_1)\eta_1\sum_{n_1\in\mathscr{N}}\frac{1}{\psi^2_{n_1}}\prod_{m\in\mathscr{N}^{m\neq n_1}}\times\left[1-\frac{(1-\delta_1)\eta_1}{\lambda_{n_1,0}-\lambda_{m,0}}\right]\phi_{n_1,0}(\boldsymbol{x}) \tag{16}$$

$$k_3(\boldsymbol{x})=(1-\delta_2)\eta_2\sum_{n_2\in\mathscr{N}^{n_2\neq 0}}\frac{1}{\psi^1_{n_2}}\prod_{m\in\mathscr{N}^{m\neq n_2,m\neq 0}}\times\left[1-\frac{(1-\delta_2)\eta_2}{\lambda_{0,n_2}-\lambda_{0,m}}\right]\phi_{0,n_2}(\boldsymbol{x}) \tag{17}$$

the shifted spectrum of the closed-loop system will be further shifted to $\{\lambda_{0,n_2}+\eta_2\}$ and $\{\lambda_{n_1,0}+\eta_1\}$ respectively.

Proof. Suppose that $k_1(\boldsymbol{x})$ depends only on x_2 and $k_2(\boldsymbol{x})$ depends only on x_1, i.e. $k_1(\boldsymbol{x})=k_1(x_2)$ and $k_2(\boldsymbol{x})=k_2(x_1)$. Also, suppose that $k_3(\boldsymbol{x})=0$ and $k_4(\boldsymbol{x})=0$. The corresponding closed-loop system is

$$\frac{\mathrm{d}z(t)}{\mathrm{d}t} = A_\alpha z(t) + B_\alpha \begin{bmatrix} \langle k_1(x_2), z(t)\rangle_{\mathscr{L}} \\ \langle k_2(x_1), z(t)\rangle_{\mathscr{L}} \\ 0 \\ 0 \end{bmatrix} \tag{18}$$

Using Corollary 1 and letting $k^1_{0,n_2} = \langle k_1(x_2), \phi_{0,n_2}\rangle_{\mathscr{L}}$ for $n_2 \in \mathscr{N}$, $n_2 \neq 0$, and $k^2_{n_1,0} = \langle k_2(x_1), \phi_{n_1,0}\rangle_{\mathscr{L}}$ for $n_1 \in \mathscr{N}$, gives

$$\begin{aligned}
A_\alpha z(t) + B_\alpha u = & \sum_{n_1,n_2 \in \mathscr{N}} \lambda_{n_1,n_2}\zeta_{n_1,n_2}\phi_{n_1,n_2} \\
& + \left(\sum_{n_2 \in \mathscr{N}} \psi^1_{n_2}\phi_{0,n_2}\right)\left(\sum_{n_2 \in \mathscr{N}} k^1_{0,n_2}\zeta_{0,n_2}\right) \\
& + \left(\sum_{n_1 \in \mathscr{N}} \psi^2_{n_1}\phi_{n_1,0}\right)\left(\sum_{n_1 \in \mathscr{N}} k^2_{n_1,0}\zeta_{n_1,0}\right) \\
= & \sum_{n_2 \in \mathscr{N} n_2 \neq 0} \lambda_{0,n_2}\zeta_{0,n_2}\phi_{0,n_2} \\
& + \left(\sum_{n_2 \in \mathscr{N} n_2 \neq 0} \psi^1_{n_2}\phi_{0,n_2}\right) \\
& \times \left(\sum_{n_2 \in \mathscr{N} n_2 \neq 0} k^1_{0,n_2}\zeta_{0,n_2}\right) \\
& + \sum_{n_1 \in \mathscr{N}} \lambda_{n_1,0}\zeta_{n_1,0}\phi_{n_1,0} \\
& + \left(\sum_{n_1 \in \mathscr{N}} \psi^2_{n_1}\phi_{n_1,0}\right)\left(\sum_{n_1 \in \mathscr{N}} k^2_{n_1,0}\zeta_{n_1,0}\right) \\
& + \sum_{n_1 \neq 0, n_2 \neq 0, n_1,n_2 \in \mathscr{N}} \lambda_{n_1,n_2}\zeta_{n_1,n_2}\phi_{n_1,n_2}
\end{aligned}$$

Note that the first two terms in the last expression reduce to the case with only one variable x_1, and the second two terms in the last expression reduce to the case with only one variable x_2. Hence from the results of references [**8**] and [**9**], the one-dimensional spectrum shift technique may be applied to shift the subsets $\{\lambda_{0,n_2}\}$ and $\{\lambda_{n_1,0}\}$ of the open-loop spectrum. The case with $k_1(\boldsymbol{x}) = k_2(\boldsymbol{x}) = 0$, $k_3(\boldsymbol{x})$ depending only on x_2 and $k_4(\boldsymbol{x})$ depending only on x_1 may be discussed in a similar manner.

It is seen that in (14) and (17) only eigenvectors ϕ_{0,n_2} are involved and in (15) and (16) only eigenvectors $\phi_{n_1,0}$ are involved. This is due to the decomposition of the eigenvalue shift problem of a planar system into two corresponding problems for systems defined over the one-dimensional domain, which enables the one-dimensional spectrum shift technique to be adopted. Though only the eigenvalues $\{\lambda_{n_1,0}\}$ and $\{\lambda_{0,n_2}\}$ can be shifted, the dominant eigenvalues are included, and relatively simple computations are needed to implement the feedback laws. It is also possible to shift certain other groups of eigenvalues with similar derivations, but it will involve all the eigenvectors, and the computations will be very complicated.

4 EXAMPLE

Suppose that $l_1 = 1$ and $l_2 = 1$, i.e.

$$\Omega = \{[x_1, x_2]^{\mathrm{T}} \mid 0 \leqslant x_1 \leqslant 1, 0 \leqslant x_2 \leqslant 1\}$$

Consider the distributed parameter system described by the following mathematical model:

$$\begin{aligned}
&\frac{\partial z(t,\boldsymbol{x})}{\partial t} = \frac{\partial^2}{\partial x_1^2} z(t,\boldsymbol{x}) + \frac{\partial^2}{\partial x_2^2} z(t,\boldsymbol{x}) \\
&-\frac{\partial}{\partial x_2} z(t,\boldsymbol{x})|_{x_2=0} = u_1(t) \\
&\frac{\partial}{\partial x_1} z(t,\boldsymbol{x})|_{x_1=1} = u_2(t) \\
&\frac{\partial}{\partial x_2} z(t,\boldsymbol{x})|_{x_2=1} = u_3(t) \\
&-\frac{\partial}{\partial x_1} z(t,\boldsymbol{x})|_{x_1=0} = u_4(t) \\
&z(0,\boldsymbol{x}) = 2, \qquad \forall \boldsymbol{x} \in \Omega
\end{aligned} \tag{19}$$

The above model may describe the temperature control system of a square plate in which there is only heat conduction. The state variable z represents the temperature distribution over the plate and the four control inputs u_i, $i = 1, 2, 3, 4$, deliver the required heat flux into or out of the plate from its four boundaries. For this system, ϕ_{n_1,n_2} in Theorem 1 may be written as

$$\begin{aligned}
&\phi_{0,0}(x_1,x_2) = 1 \\
&\phi_{0,n_2}(x_1,x_2) = \sqrt{2}\cos(n_2\pi x_2), \qquad n_2 \geqslant 1 \\
&\phi_{n_1,0}(x_1,x_2) = \sqrt{2}\cos(n_1\pi x_1), \qquad n_1 \geqslant 1 \\
&\phi_{n_1,n_2}(x_1,x_2) = \sqrt{2}\cos(n_1\pi x_1)\sqrt{2}\cos(n_2\pi x_2), \\
&\qquad\qquad n_1 \geqslant 1, n_2 \geqslant 1
\end{aligned} \tag{20}$$

Note that in the above the amplitude of ϕ_{n_1,n_2} is scaled for convenience, but it is easy to check that $A\phi_{n_1,n_2} = \lambda_{n_1,n_2}\phi_{n_1,n_2}$, where $\lambda_{n_1,n_2} = -(n_1^2 + n_2^2)\pi^2$. Also, for this system $\omega_0 = 0$. If $\omega = 1$ then G is

$$\begin{aligned}
&[G(1)u](\boldsymbol{x}) \\
&= \frac{\mathrm{e}^{x_2} + \mathrm{e}^{2-x_2}}{\mathrm{e}^2 - 1} u_1(t) + \frac{\mathrm{e}^{1+x_1} + \mathrm{e}^{1-x_1}}{\mathrm{e}^2 - 1} u_2(t) \\
&\quad + \frac{\mathrm{e}^{1+x_2} + \mathrm{e}^{1-x_2}}{\mathrm{e}^2 - 1} u_3(t) + \frac{\mathrm{e}^{x_1} + \mathrm{e}^{2-x_1}}{\mathrm{e}^2 - 1} u_4(t) \\
&= \left(\sum_{n_2 \in \mathscr{N}} g^1_{0,n_2}\phi_{0,n_2}\right) u_1(t) + \left(\sum_{n_1 \in \mathscr{N}} g^2_{n_1,0}\phi_{n_1,0}\right) u_2(t) \\
&\quad + \left(\sum_{n_2 \in \mathscr{N}} g^3_{0,n_2}\phi_{0,n_2}\right) u_3(t) + \left(\sum_{n_1 \in \mathscr{N}} g^4_{n_1,0}\phi_{n_1,0}\right) u_4(t)
\end{aligned}$$

where

$$g^1_{0,n_2} = \begin{cases} 1, & n_2 = 0 \\ \dfrac{\sqrt{2}}{1+n_2^2\pi^2}, & n_2 > 0 \end{cases} \tag{21}$$

$$g^2_{n_1,0} = \begin{cases} 1, & n_1 = 0 \\ \dfrac{(-1)^{n_1}\sqrt{2}}{1+n_1^2\pi^2}, & n_1 > 0 \end{cases} \tag{22}$$

$$g^3_{0,n_2} = \begin{cases} 1, & n_2 = 0 \\ \dfrac{(-1)^{n_2}\sqrt{2}}{1+n_2^2\pi^2}, & n_2 > 0 \end{cases} \tag{23}$$

$$g^4_{n_1,0} = \begin{cases} 1, & n_1 = 0 \\ \dfrac{\sqrt{2}}{1+n_1^2\pi^2}, & n_1 > 0 \end{cases} \tag{24}$$

For the first case, let $\delta_1 = \delta_2 = 1$, i.e. $u_2(t) = 0$ and $u_3(t) = 0$, and find $k_1(\boldsymbol{x})$ and $k_4(\boldsymbol{x})$ such that the spectrum of the closed-loop system becomes

$$\beta_{n_1,n_2} = \begin{cases} -1, & n_1 = 0,\ n_2 = 0 \\ \lambda_{0,n_2} - 1, & n_1 = 0,\ n_2 > 0 \\ \lambda_{n_1,0} - 1, & n_1 > 0,\ n_2 = 0 \\ \lambda_{n_1,n_2}, & n_1 > 0,\ n_2 > 0 \end{cases} \tag{25}$$

This modification of the eigenvalues changes the marginal stability of the open-loop system to the asymptotic stability of the closed-loop system. Figures 1 to 4, show the state $z(t, \boldsymbol{x})$ of the closed-loop system at four representative time instants, $t = 0.2$, 0.6, 1.0, 1.4 respectively. From the simulation results, it can be seen that the time response of the controlled system indeed decays with the desired rate. Also, since $u_2 = 0$ and $u_3 = 0$, the control effects come from u_1 (the down-side of the figures) and u_4 (the left-side of the figures). Therefore the state response at these two sides decays faster than that at the other two sides.

For the second case, let $\delta_1 = \delta_2 = 0.3$, and find $k_i(\boldsymbol{x})$, $i = 1, 2, 3, 4$, such that the spectrum of the closed-loop

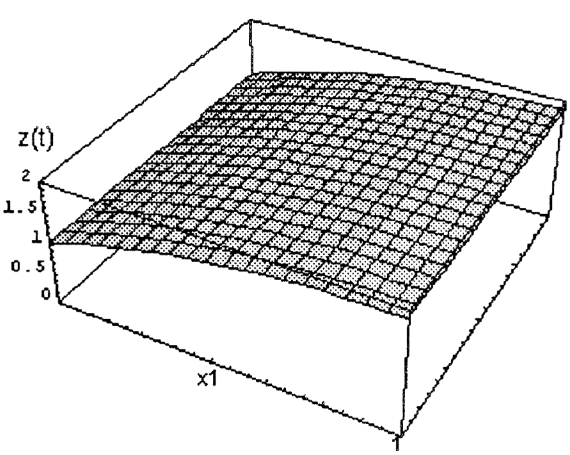

Fig. 1 First case at $t = 0.2$

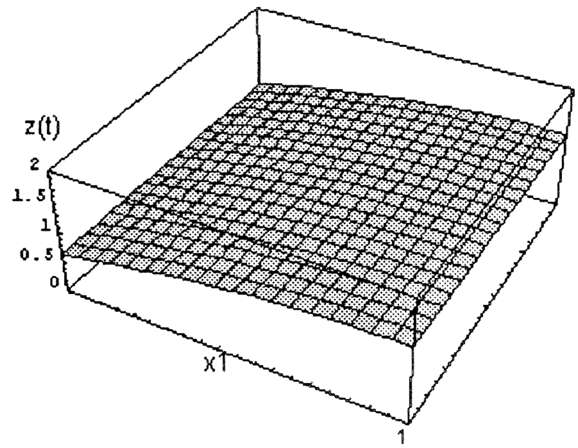

Fig. 2 First case at $t = 0.6$

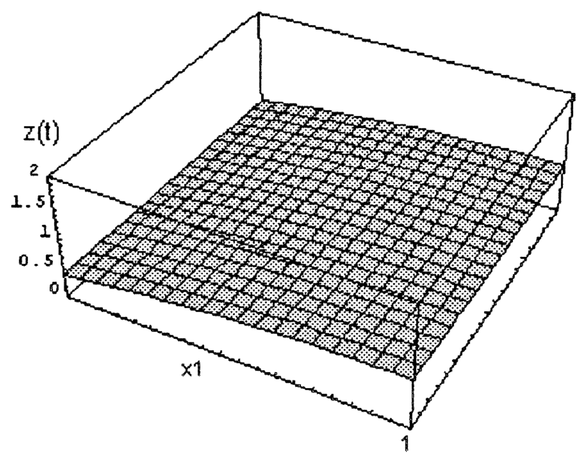

Fig. 3 First case at $t = 1.0$

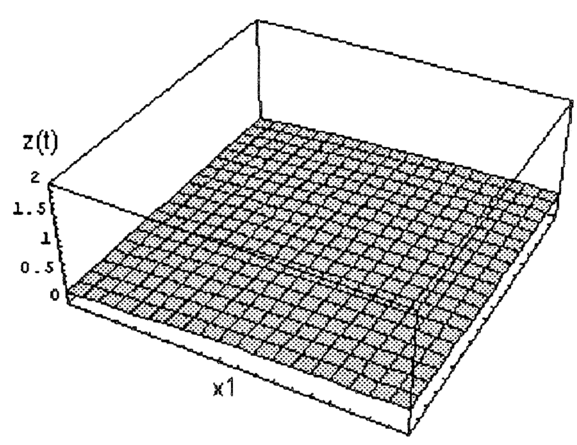

Fig. 4 First case at $t = 1.4$

system becomes

$$\beta_{n_1,n_2} = \begin{cases} -2, & n_1 = 0,\ n_2 = 0 \\ \lambda_{0,n_2} - 1, & n_1 = 0,\ n_2 > 0 \\ \lambda_{n_1,0} - 2, & n_1 > 0,\ n_2 = 0 \\ \lambda_{n_1,n_2}, & n_1 > 0,\ n_2 > 0 \end{cases} \tag{26}$$

Figures 5 to 8 show the state $z(t, \boldsymbol{x})$ at $t = 0.2$, 0.6, 1.0, 1.4 respectively. Compared with the first case, the second case shows a faster decay rate than the first case, which is desired. Also, due to the use of all four control inputs in this case, the temperature drop pattern is different from that in the first case. It is worth mentioning that the initial condition of this example is $z(0, \boldsymbol{x}) = 2$ for all $\boldsymbol{x} \in \Omega$, so if all of the control inputs are shut off, i.e. $u_i = 0$, $i = 1, 2, 3, 4$, then the system state will remain at the value of 2.

5 CONCLUSION

The spectrum shift problem of a special class of distributed parameter systems via the boundary control is discussed. The proposed procedure includes the transformation of system models and the synthesis of state feedback laws. The numerical example shows how the proposed method can be applied. With the transformed model (13), it is believed that more control schemes have the potential to be applied to the boundary control problem.

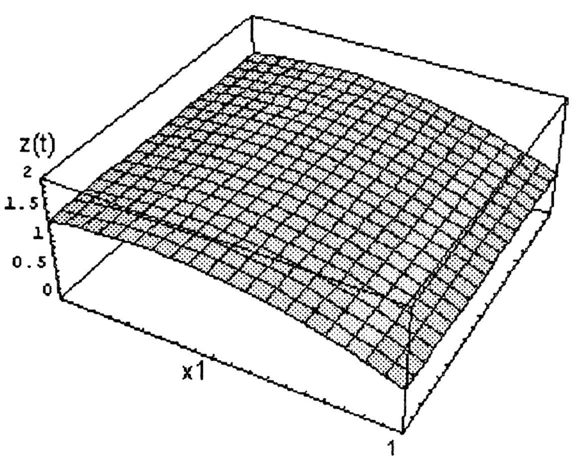

Fig. 5 Second case at $t = 0.2$

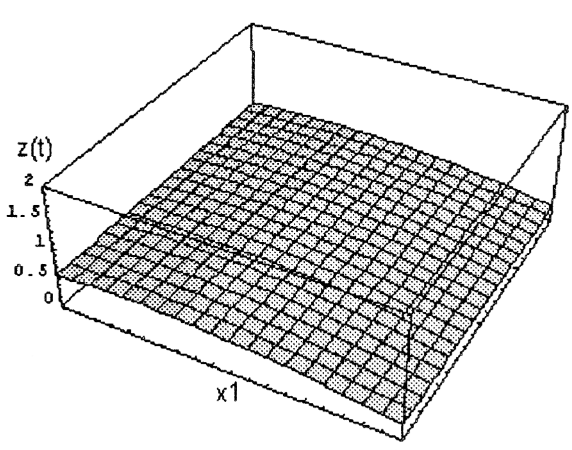

Fig. 6 Second case at $t = 0.6$

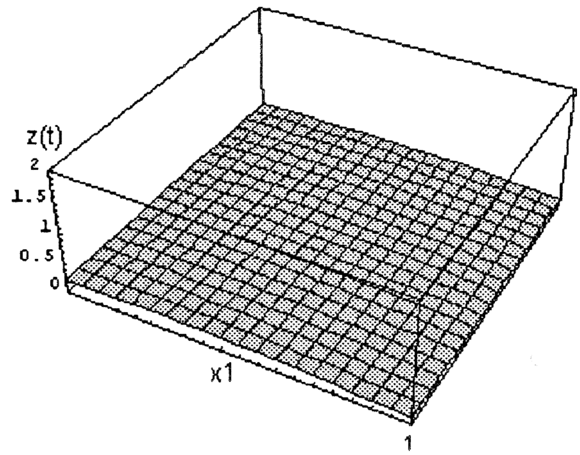

Fig. 7 Second case at $t = 1.0$

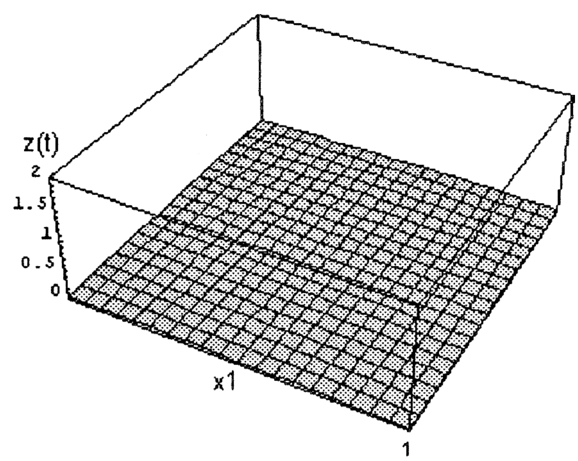

Fig. 8 Second case at $t = 1.4$

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